Matematický časopis

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Matematický časopis, Vol. 20 (1970), No. 1, 62--71

Persistent URL: http://dml.cz/dmlcz/126957

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REMARKS ON A NONLINEAR THEORY OF THIN ELASTIC PLATES

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In this paper we shall discuss a certain two-dimensional boundary value problem which arises when we investigate the equilibrium of a built-in plate lying in the plane xy and subjected to a load q perpendicular to the plane xy and to forces g_1, g_2 acting in the plane xy. When denoting the axes in a three dimensional Euclidean space by x, y, z then the u, v, w will denote the displacements parallel to them, respectively. In the paper we shall work in real spaces and with real functions.

Terminology and notation. For simplicity let Ω be a bounded region ig the plane xy with a Lipschitz boundary. Let us denote by $\varepsilon(\Omega)$ the space of infinitely many times differentiable functions on Ω which are continuously prolongable with all their derivatives to $\bar{\Omega}$. $\mathscr{D}(\Omega)$ are functions of $\varepsilon(\Omega)$ with a compact support in Ω. Let $W_p^{(k)}(\Omega)$ be a system of functions having all generalized derivatives up to the k-th order integrable with the p-th power in Ω. $W_p^{(k)}(\Omega)$ with the norm $\|u\|_{W_p^{(k)}(\Omega)} = (\sum_{i=0}^{k} \|u^{(i)}\|^p_{Lp(\Omega)})^{1/p}$ (addition through all derivatives) is a Banach space. Let the closure of $\mathscr{D}(\Omega)$ in the $W_p^{(k)}(\Omega)$ norm be denoted by $W_p^{(k)}(\Omega)$. In the following we shall write $W_p^{(k)}$ instead of $W_p^{(k)}(\Omega)$.

Let $W = W_2^{(2)} \times W_2^{(1)} \times W_2^{(1)}$ (a Cartesian product of spaces) and let us define for $\vec{u} = (w, u, v) \in W$ (where $w \in W_2^{(2)}$, $u \in W_2^{(1)}$, $v \in W_2^{(1)}$) the norm by

$$\|\vec{u}\|^2_W = \|w\|^2_{W_2^{(2)}} + \|u\|^2_{W_2^{(1)}} + \|v\|^2_{W_2^{(1)}}.$$

Put $V = W_2^{(2)} \times W_2^{(1)} \times W_2^{(1)}$. Let P_1 be the space of all polynomials of the order $\leqslant 1$ and $P \subset P_1 \times P_1 \times P_1$, P generated by the vectors $(0, 1, 0)$, $(0, 0, 1)$, $(0, y, -x)$. That means the polynomials in question are of the type $\vec{p} = (0, a + \lambda y, b - \lambda x)$. Let us denote by V/P the space of classes $\tilde{\vec{u}}$ of functions $\vec{u} \in V$; $\vec{u}, \vec{v} \in \tilde{\vec{u}} \Leftrightarrow \vec{u} - \vec{v} \in P$. The norm in V/P we define as usual

$$\|\tilde{\vec{u}}\|_{V/P} = \inf_{\vec{u} \in \tilde{\vec{u}}} \|\vec{u}\|_V.$$

Statement 1. *V/P with this norm is a Hilbert space (hence V/P is reflexive)*

Proof. Let $V = P \dotplus R$ (direct sum). If $\tilde{\vec{u}} \in V/P$, there is only one element

$\vec{u}_r \in R$ such that for any $\vec{u} \in \tilde{\vec{u}}$ there is $\vec{u} = \vec{u}_p + \vec{u}_r$. In particular, $\vec{u}_r \in \tilde{\vec{u}}$ (because $\vec{u} - \vec{u}_r = \vec{u}_p \in P$ for $\vec{u} \in \tilde{\vec{u}}$).

Now it is clear that the scalar product in $V|P$ may be defined in the following way

$$(\tilde{\vec{u}}, \tilde{\vec{v}})_{V|P} = (\vec{u}_r, \vec{v}_r)_V$$

and we have $(\tilde{\vec{u}}, \tilde{\vec{u}})_{V|P} = \|\tilde{\vec{u}}\|^2_{V|P} = \inf\limits_{\vec{u} \in \tilde{\vec{u}}} \|\vec{u}\|^2_V = \inf\limits_{\vec{u} \in \tilde{\vec{u}}} (\|\vec{u}_p\|^2 + \|\vec{u}_r\|^2) = \|\vec{u}_r\|^2_V$.
Now, let $q \in L_2(\Omega)$, $g_1 \in L_2(\dot{\Omega})$, $g_2 \in L_2(\dot{\Omega})$ where by $\dot{\Omega}$ we denote the boundary of Ω.

We shall study the existence of a weak solution of the following system of equations (system which describes the physical problem mentioned at the beginning)

$$\frac{D}{h}\Delta^2 w = \frac{\partial^2 w}{\partial x^2}\sigma_x + \frac{\partial^2 w}{\partial y^2}\sigma_y + 2\frac{\partial^2 w}{\partial x\,\partial y}\tau + \frac{q}{h},$$

(1)
$$\frac{\partial \sigma}{\partial x} + \frac{\partial \tau}{\partial y} = 0,$$

$$\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,$$

where

$$\sigma_x = \frac{E}{1-\mu^2}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \mu\left(\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right)\right],$$

$$\sigma_y = \frac{E}{1-\mu^2}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \mu\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\right],$$

$$\tau = \frac{E}{2(1+\mu)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right],$$

h the plate thickness
E the compression modulus of elasticity
μ the Poisson number
D the plate stiffness
under the boundary conditions

$$w = \frac{\partial w}{\partial n} = 0, \quad \text{on } \dot{\Omega},$$

$$\left.\begin{aligned} \sigma_x n_x + \tau n_y &= g_1 \\ \tau n_x + \sigma_y n_y &= g_2 \end{aligned}\right\} \text{ on } \dot{\Omega} ,$$

n_x, n_y are the components of a normal to $\dot{\Omega}$.

Remark. The equations (1) are to be satisfied in the sense of distributions.

The vector $(w, u, v) \in V$ is a weak solution of the given boundary value problem if for any vector $(\tilde{w}, \tilde{u}, \tilde{v}) \in V$ there is

$$\int\limits_{\Omega} \frac{D}{h} \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \tilde{w}}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \tilde{w}}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \tilde{w}}{\partial y^2} \right) \mathrm{d}x\mathrm{d}y$$

$$\int\limits_{\Omega} \left(\frac{\partial^2 w}{\partial x^2} \sigma_x + \frac{\partial^2 w}{\partial y^2} \sigma_y + 2 \frac{\partial^2 w}{\partial x \partial y} \tau \right) \tilde{w} \, \mathrm{d}x\mathrm{d}y +$$

$$+ \int\limits_{\Omega} \left(\sigma_x \frac{\partial \tilde{u}}{\partial x} + \tau \frac{\partial \tilde{u}}{\partial y} + \tau \frac{\partial \tilde{v}}{\partial x} + \sigma_y \frac{\partial \tilde{v}}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

$$\frac{1}{h} \int\limits_{\Omega} q \tilde{w} \, \mathrm{d}x\mathrm{d}y - \int\limits_{\dot{\Omega}} g_1 \tilde{u} \, \mathrm{d}s \quad \int\limits_{\dot{\Omega}} g_2 \tilde{v} \, \mathrm{d}s \quad 0 .$$

(In general, for the definition of a weak solution see e. g. [1]).

Rearrangeing the second integral (using integration by parts) we obtain that the vector $\vec{\alpha} = (w, u, v) \in V$ is a weak solution of the given problem if the following equation folds for any $\vec{\beta} = (\tilde{w}, \tilde{u}, \tilde{v}) \in V$

$$F(\vec{\alpha})\vec{\beta} = \int\limits_{\Omega} \frac{D}{h} \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \tilde{w}}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \tilde{w}}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \tilde{w}}{\partial y^2} \right) \mathrm{d}x\mathrm{d}y + \tag{2}$$

$$+ \int\limits_{\Omega} \left(\frac{\partial w}{\partial x} \frac{\partial \tilde{w}}{\partial x} \sigma_x + \frac{\partial w}{\partial y} \frac{\partial \tilde{w}}{\partial y} \sigma_y + \frac{\partial w}{\partial x} \frac{\partial \tilde{w}}{\partial y} \tau + \frac{\partial w}{\partial y} \frac{\partial \tilde{w}}{\partial x} \tau \right) \mathrm{d}x\mathrm{d}y +$$

$$+ \int\limits_{\Omega} \left(\sigma_x \frac{\partial \tilde{u}}{\partial x} + \tau \frac{\partial \tilde{u}}{\partial y} + \tau \frac{\partial \tilde{v}}{\partial x} + \sigma_y \frac{\partial \tilde{v}}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

$$\frac{1}{h} \int\limits_{\Omega} q \tilde{w} \, \mathrm{d}x\mathrm{d}y - \int\limits_{\dot{\Omega}} g_1 \tilde{u} \, \mathrm{d}s \quad \int\limits_{\dot{\Omega}} g_2 \tilde{v} \, \mathrm{d}s = 0 .$$

It is easy to verify that the operator $F(\vec{\alpha}) \in [V \to V^*]$ defined by this equation is the potential (see [2]). Hence there exists a functional $g(\vec{\alpha})$ for which the following condition must be satisfied

$$\operatorname{grad} g(\vec{\alpha}) = F(\vec{\alpha}) \, .$$

The equation $F(\vec{\alpha})\,\vec{\beta} = 0, \forall \vec{\beta} \in V$ now implies that we can investigate critical points of $g(\vec{\alpha})$ instead of solving (2). By a calculation it is found that

$$(3) \qquad g(\vec{\alpha}) = \int_{\Omega} \frac{D}{2h} \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] \mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\Omega} \frac{E}{2(1-\mu^2)} \left[\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + (1-\mu) \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \right.$$

$$\left. + \frac{1-\mu}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right] \mathrm{d}x\mathrm{d}y + \int_{\Omega} \frac{E}{8(1-\mu^2)} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right]^2 \mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\Omega} \frac{E}{2(1-\mu^2)} \left[\frac{\partial u}{\partial x} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial v}{\partial y} \left(\frac{\partial w}{\partial y} \right)^2 + \mu \frac{\partial v}{\partial y} \left(\frac{\partial w}{\partial x} \right)^2 + \mu \frac{\partial u}{\partial x} \left(\frac{\partial w}{\partial y} \right)^2 \right] \mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\Omega} \frac{E}{2(1+\mu)} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \mathrm{d}x\mathrm{d}y - \frac{1}{h} \int_{\Omega} q w \, \mathrm{d}x\mathrm{d}y - \int_{\dot{\Omega}} g_1 u \, \mathrm{d}\dot{\Omega} - \int_{\dot{\Omega}} g_2 v \, \mathrm{d}\dot{\Omega} \, .$$

Let us denote the integrals on the right-hand side of (3) by $J_1, \ldots, J_8$ respectively so that $g(\vec{\alpha}) = \sum_{j=1}^{5} J_j - \sum_{j=6}^{8} J_j$ and let us consider the functional $f(\vec{\alpha}) = g(\alpha) + \sum_{j=6}^{8} J_j$.

In [2] it is shown that $f(\vec{\alpha})$, $g(\vec{\alpha})$ are weakly lower semicontinuous on V. The functional $g(\vec{\alpha})$ may further be written in the form

$$(4) \qquad g(\vec{\alpha}) = \int_{\Omega} \frac{D}{2h} \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] \mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\Omega} \frac{E_u}{2(1-\mu^2)} \left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right]^2 \mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\Omega} \frac{E}{2(1+\mu)} \left[\left(\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right)^2 + \left(\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right)^2 \right] \mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\Omega} \frac{E}{4(1 \quad \mu)} \left[\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right]^2 \mathrm{d}x\mathrm{d}y$$

$$\frac{1}{h} \int_{\Omega} qw \,\mathrm{d}x\mathrm{d}y - \int_{\dot{\Omega}} g_1 u \,\mathrm{d}\dot{\Omega} - \int_{\dot{\Omega}} g_2 v \,\mathrm{d}\dot{\Omega} .$$

In the following, let g_1, g_2 be such elements of $L_2(\dot{\Omega})$ that

$$\int_{\dot{\Omega}} g_1(a + \lambda y) \,\mathrm{d}s \quad 0 , \qquad \int_{\dot{\Omega}} g_2(b \quad \lambda x) \,\mathrm{d}s \quad 0$$

Now let us define the functional $G(\tilde{\vec{\alpha}})$ in $V \; P$ as follows for $\tilde{\vec{\alpha}} \in V \; P$ putting $G(\tilde{\vec{\alpha}}) = f(\vec{\alpha})$, where $\vec{\alpha} \in V$, $\vec{\alpha} \in \tilde{\vec{\alpha}}$ is arbitrary. One can see from the form of the functional $f(\vec{\alpha})$ that the definition is meaningful.

Statement 2. $G(\tilde{\vec{\alpha}})$ *is weakly lower semicontinuous on* $V \; P$.

Proof. Let $\tilde{\vec{\alpha}}_n \rightharpoonup \tilde{\vec{\alpha}}_0$ in $V \; P$ (where the symbol $\rightharpoonup$ denotes a weak convergence), i. e. for any $\tilde{\vec{u}} \in V \; P$

$$(\tilde{\vec{\alpha}}_n, \tilde{\vec{u}})_{V \; P} \quad \to \quad (\tilde{\vec{\alpha}}_0, \tilde{\vec{u}})_{V \; P} .$$

According to the definition

$$(*) \qquad (\tilde{\vec{\alpha}}_n, \tilde{\vec{u}})_{V \; P} \quad (\vec{\alpha}_{n,r} \, \vec{u}_r)_V ; \qquad (\tilde{\vec{\alpha}}_0, \tilde{\vec{u}})_{V \; P} \quad (\vec{\alpha}_{0.r}, \vec{u}_r)_V .$$

Therefore it is sufficient to show

$$(\vec{\alpha}_{n,r}, \vec{u}) \quad \to \quad (\vec{\alpha}_{0.r}, \vec{u}) \text{ for any } \vec{u} \in V .$$

(Namely, using the weak lower semicontinuity of $f(\vec{\alpha})$ we obtain the desired result.)

This, however, follows by (*) and by the fact that

$$\vec{u} - \vec{u}_p \quad \vec{u}_r , \quad \vec{u}_p \in P , \quad \vec{u}_r \in R , \quad P \quad R .$$

Remark. Let us use the notation $U - (u, v)$; by the space $V \; P$ we may understand the space $\mathring{W}_2^{(2)} \times (W_2^{(1)})^2/P'$, where P' is the space of polynomials of the type $\{a + \lambda y, b - \lambda x\}$. Now, the integral J_1 is equivalent to the norm of the element w in $W_2^{(2)}$ (see e. g. [1]); J_2 on the other hand is equivalent to the norm of the class $\tilde{U} \quad (\tilde{u}, \tilde{v})$ in $(W_2^{(1)})^2/P$. Here, the inequality $J \leqslant c\|\tilde{U}\|^2$ is evident and the inequality $J_2 \geqslant c' \|\tilde{U}\|^2$ can be obtained using Korn's inequality (see [3]). We shall therefore write $|w|^2_{W_2}$ instead of J and $\|\tilde{U}\|^2$ instead of J_2.

Theorem 1. *There is*

$$\liminf_{\tilde{\vec{u}}\,|\to+\infty} \frac{G(\tilde{\vec{u}})}{|\tilde{\vec{u}}\,|} \quad c > 0\,.$$

Proof. From formula (3) we obtain

$$(5) \qquad G(\tilde{\vec{u}}) \quad w|^2_{W_2^{(2}} \quad \tilde{U}|^2_{VI} \quad R(\tilde{\vec{u}})\,,$$

where $R(\tilde{\vec{u}}) > J_4 \quad J_5$ (because $J_3 \geqslant 0$). From formula (4) we obtain

$$(6) \qquad G(\tilde{\vec{u}}) \quad w\,^2_{W_2^{(2}} \quad k(\tilde{\vec{u}})\,,$$

where $k(\tilde{\vec{u}}) \quad 0$. Let us estimate $I_4 + I_5$ using Schwartz's inequality. (Note that $w \in L_4$, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y} \in L_4$ and that $w_{W^{(1}} \quad c\, w\|_{W_2^{(2}}$ where c does not depend on w: these facts follow from the Sobolev imbedding theorems.) Let $\vec{u} \in \tilde{u}$ be arbitrary. Then

$$J_4 \leqslant c_1(\,u\,^2_{W_2^{(1)}} + v\,^2_{W_2^{(1)}})^{1}\,|w\,^2_{W_4^{(1}} < c_1'\|U\,|\,\|w\,|^2_{W_2^{2}}\,,$$

$$J_5 \leqslant c_2(\,u|^2_{W_2^{(1}} + v\,^2_{W_2^{(1)}})^{1\,2}\|w\|^2_{W_4^{(1)}} \leqslant c_2'\|U\|\,\|w\|^2_{W_2^{2}}\,,$$

so that we have

$$J_4 + J_5 \leqslant c\,\,U| \quad w\,|^2_{\mathring{W}_2^{(2)}} \text{ for any } \vec{u} \in \tilde{\vec{u}}\,,$$

i. e.,

$$J_4 \quad J_5 \leqslant c\,|\tilde{U}\|_{V/P}\|w\|^2_{\mathring{W}^{(}}\,.$$

Let now $\tilde{\vec{u}}\,_{VP} \quad r > 0$ (we can consider $r \quad 1$).

$$G(\tilde{\vec{u}}) \geqslant r^2 \quad c\,|\tilde{U}|'_{V/P}\,|w|\,^2_{\mathring{W}_2^{(2)}} \geqslant r^2 \quad cr\,|w\,^2_{\mathring{W}_2} \quad r(r \quad c\|w\|^2_{\mathring{W}_2^{(2)}}) \geqslant r\alpha$$

for those $\tilde{\vec{u}}$ satisfying $|\,w\,|^2_{W_2^{(2)}} \leqslant (r \quad \alpha)/c$ (we can choose a convenient α), e. g. $\alpha \quad 1$). In this case we can see that

$$\frac{G(\tilde{\vec{u}})}{\tilde{\vec{u}}|} > \alpha\,.$$

If $w_{W} > \frac{r \quad \alpha}{c}$, using formula (6) we obtain an estimate

$$G(\tilde{\vec{u}}) \geqslant \frac{r \quad \alpha}{c} \quad k(\vec{u}) \geqslant \frac{r - \alpha}{c}\,,$$

so that

$$\frac{G(\tilde{\vec{u}})}{|\tilde{\vec{u}}\,|} \geqslant \frac{1}{c} \quad \frac{\alpha}{cr} > \frac{1}{c} - \frac{\alpha}{c} \quad \frac{1-\alpha}{c}$$

In any case we have

$$\frac{G(\tilde{\tilde{u}})}{\|\tilde{\tilde{u}}\|} \geqslant \min\left(\alpha, \frac{1-\alpha}{c}\right).$$

Theorem 2. *If* g_1 $g_2 = 0$, *then for any* $q \in L_2(\Omega)$ *there exists a solution of the problem in question.*

Proof. When writing $\frac{1}{h}\int_\Omega qw \, \mathrm{d}\Omega = \langle w, q\rangle$ it is sufficient to prove

$$(\overset{*}{*}) \qquad \liminf_{\|u\|\to\infty} (G(\tilde{\tilde{u}}) - \langle w, q\rangle) = +\infty$$

because $G(\tilde{\tilde{u}}) - \langle w, q\rangle$ is a lower weakly semicontinuous functional in a reflexive Banach space $V|P$, thus by $(\overset{*}{*})$ it has an absolute minimum on $V P$ and the point that minimizes $G(\tilde{\tilde{u}}) - \langle w, q\rangle$ is a solution of the given boundary value problem with $g_1 = g_2$ 0 (see e. g. [4]). Let us prove $(\overset{*}{*})$.
For any $K > 0$ we shall find $R > 0$ such that for $\|\tilde{\tilde{u}}\| \geqslant R$

$$(7) \qquad G(\tilde{\tilde{u}}) - \langle w, q\rangle \geqslant K .$$

We have $\|\tilde{\tilde{u}}\|^2 = \|w\|^2 + \|\tilde{U}\|^2$; let $r_1 \geqslant \max\left(2\|q\|, \frac{K}{|q\|}\right)$.
For $\|w\|_{\mathring{W}_2^{(2)}} \geqslant r_1$ using formula (6) we obtain

$$G(\tilde{\tilde{u}}) - \langle w, q\rangle \geqslant \|w\|^2_{\mathring{W}_2^{(2)}} + k(\tilde{\tilde{u}}) - \|w\|_{\mathring{W}_2^{(2)}} \|q\|_{L_2(\Omega)} \geqslant |w (|w - q|) \geqslant$$
$$\geqslant r_1(r_1 - \|q\|) \geqslant r_1\|q\| \geqslant K .$$

If $\|w\| \leqslant r_1$, then using (5) we obtain

$$G(\tilde{\tilde{u}}) - \langle w, q\rangle \geqslant \|\tilde{\tilde{u}}\|^2 - \|w\| \|q\| - c\|\tilde{U}\| \|w\|^2 \geqslant \|\tilde{U}\|^2 - r_1 |q\| - c|\tilde{U} r_1^2 \geqslant$$
$$\geqslant \|\tilde{U}\|(\|\tilde{U}\| - cr_1^2) - r_1 |q\| .$$

If we now choose $r_2 > 0$ such that

$$r_2(r_2 - Cr_1^2) - r_1\|q\| \geqslant K ,$$

then for $\|\tilde{U}\| \geqslant r_2$ we have $G(\tilde{\tilde{u}}) - \langle w, q\rangle \geqslant K$.
Finally put $R^2 = r_1^2 + r_2^2$; then for $|\tilde{\tilde{u}}\| \geqslant R$ there is $\|\tilde{U}\|^2 \geqslant R^2 - w|^2$ and for $\|w\|^2 \geqslant r_1^2$ the relation (7) is true; for $\|w\|^2 \leqslant r_1^2$ we have $\|\tilde{U}\|^2 \geqslant r_1^2 + r_2^2 - r_1^2$ r_2^2, so that (7) is true again, what was to be proved.

In the following considerations we shall study a wider problem:
Let H be a Hilbert space and such that $V|P$ is a subspace of H . Let F be a bounded linear functional on H. Instead of the symbol $\tilde{\tilde{u}}$ we shall simply

write u and instead of $G(\tilde{\vec{u}})$ we shall write $f(\vec{u})$ (according to the definition of $G(\tilde{\vec{u}})$).

We shall put

$$M_s \quad \{F \in H^*; \liminf_{|\vec{u}|_{V/P}\to\infty} (f(\vec{u}) \quad (\vec{u}, F)) = +\infty\};$$

$$M_i - \{F \in H^*; \liminf_{\vec{u}|_{V/P}\to\infty} (f(\vec{u}) \quad (\vec{u}, F)) - c \neq \pm\infty\};$$

$$M_l \quad \{F \in H^*; \liminf_{\vec{u}|_{V/P}\to\infty} (f(\vec{u}) \quad (\vec{u}, F)) = -\infty\}.$$

We shall show that $M_s \neq \emptyset$; for $F \in M_s$ there exists an absolute minimum of the functional $f(\vec{u}) - (\vec{u}, F)$, hence a solution of certain boundary value problem as it follows from the foregoing considerations (especially from the proof of Theorem 2.)

Theorem 3. *The set M is convex.*

Proof. Let $F_1, F_2 \in M_s$; then for $F \quad (1-\lambda)F_1 + \lambda F_2 (0 < \lambda < 1)$ we have

$$f(\vec{u}) \quad (\vec{u}, F) = f(\vec{u}) - (1 \quad \lambda)(\vec{u}, F_1) - \lambda(\vec{u}, F_2) =$$
$$(1 \quad \lambda)(f(\vec{u}) - (\vec{u}, F_1)) + \lambda(f(\vec{u}) \quad (\vec{u}, F_2)),$$

hence $F \in M_s$.

For $F \in H^*$, $|F| \quad 1$ we define a real-valued function corresponding to the chosen H in the following way:

$$\lambda_H(F) \quad \sup\{\sigma; \sigma F \in M_s\}.$$

Then $\lambda_H(F) > 0$ (from this it is clear that $M_s \neq \emptyset$). Namely, by Theorem 1 the existence of such $R > 0$ follows that for $\|\vec{u}|_{V/P} > R$ we have $f(\vec{u}) \geqslant \alpha \; \vec{u}_{\;V\;P}$ (for some $\alpha > 0$) so that all right-hand sides with a sufficiently small norm belong to the M_s. From this there also follows an existence of a neighbourhood of zero at H^*, the whole belonging to the M_s.

$$\left(\text{Really, } f(\vec{u}) \quad (\vec{u}, F) \geqslant \alpha\, \vec{u}|_{V\,P} \quad c|\vec{u}|_{V/P}\, \|F|_{H^*};\; \left\{F, \|F\|_{H^*} \leqslant \frac{1}{2}\frac{\alpha}{c}\right\} \subset M.\right).$$

We shall prove several theorems concerning $\lambda_H(F)$.

Theorem 4. *If $\sigma > \lambda_H(F)$, then $\sigma F \in M_l$ (so that for $\sigma > \lambda_H(F)$ there is no absolute minimum of $f(\vec{a}) - \sigma(\vec{u}, F)$)*

Proof. Let $\sigma > \lambda_H(F)$. If $\liminf_{|\vec{u}\to+\infty} (f(\vec{u}) \quad \sigma(\vec{u}, F)) \quad c \neq \pm\infty$, then for $\sigma > \sigma_1 > \lambda_H(F)$ there should be $\liminf_{|\vec{u}|\to+\infty} (f(\vec{u}) - \sigma_1(\vec{u}, F)) \quad K \neq \pm\infty$. Really, $\liminf_{\vec{u}\leftarrow+\infty} (f(\vec{u}) - \sigma_1(\vec{u}, F)) \quad +\infty$ cannot hold because $\sigma_1 > \lambda_H(F)$ and if $\liminf_{\vec{u}\to\infty} (f(\vec{u}) - \sigma_1(\vec{u}, F)) \quad -\infty$ then

$$\infty = \liminf_{\|\vec{u}\| \to \infty} (f(\vec{u}) \quad \sigma_1(\vec{u}, F)) \geqslant \liminf_{\vec{u} \to +\infty} (f(\vec{u}) \quad \sigma(\vec{u}, F)) \quad c + \quad \infty$$

Moreover, we have

$$f(\vec{u}) \quad \sigma_1(\vec{u}, F) \quad (f(\vec{u}) - \sigma(\vec{u}, F)) \frac{\sigma_1}{\sigma} + f(\vec{u}) \left(1 \quad \frac{\sigma_1}{\sigma}\right)$$

and

$$\liminf_{\vec{u} \to \infty} (f(\vec{u}) \quad \sigma_1(\vec{u}, F)) \geqslant \frac{\sigma_1}{\sigma} \liminf_{\|\vec{u}\| \to \infty} \left[(f(\vec{u}) \quad \sigma(\vec{u}, F)) \quad \frac{\sigma}{\sigma_1} f(\vec{u}) \left(1 \quad \frac{\sigma_1}{\sigma}\right)\right]$$

which means that

$$\pm \infty + K \geqslant \frac{\sigma_1}{\sigma} C + \infty \left(0 \in M_s \to \liminf_{\|\vec{u}\| \to \infty} f(\vec{u}) \left(1 \quad \frac{\sigma_1}{\sigma}\right) \quad \infty\right)$$

which is a contradiction.

Theorem 5. *Function $\lambda_H(F)$ is continuous.*

Proof. At first let F_0 be such that $\lambda_H(F_0) < + \infty$. Let $\varepsilon > 0$ be arbitrary (but fixed). By the preceding there is $r : 0 < r < \lambda_H(F_0)$ such that $D \quad \{F; \ F_H < \leqslant r\} \subset M_s$. Let us take a cone $C \quad \{aF; a \geqslant 0, F \in D - \lambda_H(F_0)F_0\} \quad \lambda_H(F_0)F_0$ so that C is a convex cone with a vertex at the point $\lambda_H(F_0)F_0$ and containing all the points of D. From the convexity of M_s it follows that the points of the type $\alpha\lambda_H(F_0)F_0 + (1 \quad \alpha)F$, $F \in D$, $\alpha \in (0, 1)$ belong to M_s. Furthermore, let K be another cone, $K \quad \{aF; \ a \geqslant 0, \ F \in D \quad \lambda_H(F_0)F_0\} + \lambda_H(F_0)F_0$. One can easily see that $V \quad \{aF; \ a \geqslant 0, \ F \in K \cap \{F; \ F \quad \lambda_H(F_0)F_0 \quad \varepsilon\}$ is a convex cone with a vertex at the origin and $F_0 \in \operatorname{Int} V$. Now, the set $\operatorname{Int} A \quad \operatorname{Int} (V \cap \{F; \ \|F\| \quad 1\}$ is the neigbourhood of F_0 we were looking for

Certainly, let $F \in \operatorname{Int} A$; $\lambda_H(F)F$ lies on the ray aF, $a > 0$. We must show that $\lambda_H(F)F$ lies in the ε -neighbourhood of $\lambda_H(F_0)F_0$. But it is clear that $\lambda_H(F) < \lambda_H(F_0) \quad \varepsilon$ is impossible (by the definition of $\lambda_H(F)$ and for all the interior points of $M \quad \{aF; \ 1 \geqslant a \geqslant 0, \ F \in D - \lambda_H(F_0)F_0\} + \lambda_H(F_0)F_0$ belong to M_s and in the case of $\lambda_H(F) \quad \lambda_H(F_0) \quad \varepsilon$ the point $\lambda_H(F)F$ would lay in $\operatorname{Int} M$) and having $\lambda_H(F) > \lambda_H(F_0) + \varepsilon$ we can easily find that on the ray aF_0, $a \geqslant 0$ there is a point $\sigma F_0 \in M_s$ with $\sigma > \lambda_H(F_0)$, which is a contradiction.

Now let $\lambda_H(F_0) \quad \infty$. Choose $R > 0$ and consider a „cone" $K \quad \{aF; 1 \geqslant a \geqslant 0, \ F \in D \quad 2RF_0\} + 2RF_0$. It is clear that $\operatorname{Int} K \subset M_s$. Now for all $F \in V \cap \{F; \ \|F\|_{H^*} \quad 1\}$ we have $\lambda_H(F) > R$, where $V \quad \{aF; \ a \quad 0, F \in \{x; \ \|x\| \quad R\} \cap K\}$ and $V \cap \{F; \|F\|_{H^*} \quad 1\}$ is the neighbourhood we were looking for.

Finally we shall mention another property of the function $\lambda_H(F)$. Let $B \supset H$ and let us suppose that the identical imbedding $H \to B$ is totally continuous. Let $B_n \subset B$, B_n closed subspaces of B $(n = 1, 2, \ldots)$ such that $\lim_{n\to\infty} B_n = B$ (i. e. $(\forall\ v \in B)\ (\exists\, v_n \in B_n)\ [\lim_{n\to\infty} \| v - v_n \|_B = 0]$).

The following theorem is true

Theorem 6. *If we denote by* $D_n = \{F \in B^*;\ v \in B_n \Rightarrow Fv = 0\}$ *then*

$$\lim_{n\to\infty} \Big(\inf_{\|F\|_B = 1,\ F \in D_n} \lambda_{B^*}(F) \Big) = \infty$$

Proof. It is sufficient to prove that

$$\lim_{n\to\infty} \Big(\sup_{\|F\|_{B^*} = 1,\ F \in D_n} \|F\|_{H^*} \Big) = 0\,.$$

Let us suppose that this does not hold. Then there exists such an $F_n \in D_n$ that $\|F_n\|_{H^*} \geq \varepsilon$ for some $\varepsilon > 0$. Let $v \in B$ be arbitrary; then $F_n v = F_n v - F_n v_n = F_n(v - v_n) \to 0$ so that $F_n \rightharpoonup 0$ in B^*. But the identical imbedding $B^* \to H^*$ is totally continuous hence $F_n \to 0$ in H^* so that $\|F_n\|_{H^*} \to 0$, which is a contradiction.

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Received January 18, 1968

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