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1. Introduction and definitions

We shall consider the formulas constructed in the "permitted way" (Definition 2) from the Boolean variables and constants, from the symbols $=$, $\leqq$, $\cup$, $\cap$, $^-$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\equiv$ and parentheses, also the relations which are their realizations. We call them BP-formulas, BP-relations, respectively.

Section 2 contains some remarks and auxiliary results which concern the formulas without the propositional operators. In section 3 the question is solved, under what conditions a given BP-formula is a "BP-tautology" (Theorem 1). In section 4 the equivalence of the BP-formulas is studied and a simple system of representants of classes of the equivalence is presented. Every class is uniquely characterized by a set system (Theorem 2). In section 5 the BP-relations, their properties (Theorems 4, 6, 7, 8) and their number in connection with the number of elements of the corresponding Boolean algebra (Theorem 5) are studied.

Notations:

B always denotes a non-degenerate Boolean algebra containing at least two elements; $\boldsymbol{0}$, $\boldsymbol{1}$, $\leqq$ denote the smallest element, the largest element and the partial ordering in B, respectively; $D = \{0, 1\}$; x_i $(i = 1, 2, \ldots)$ are meta-symbols for the denotation of the free individual variables from a fixed well-ordered set X and x denotes the n-tuple $(x_1, \ldots, x_n)$ of pairwise different variables. $(\forall\, B, b)$ means "for every non-degenerate Boolean algebra B and for every n-tuple $b = (b_1, \ldots, b_n) \in B^n$" (the exponent at a symbol of a set or an algebraic structure always means the cartesian power). The terms (except the variables) and the formulas will be denoted by bold "basic letters", their realizations (in a given B) by the same ordinary letters (only $\boldsymbol{0}$ and $\boldsymbol{1}$ are considered as terms and, at the same time, as elements in a certain B). The sign $\doteq$ expresses the equality of terms or the equality of formulas as words. $\boldsymbol{F}(x)$, $\boldsymbol{R}_M(x)$ etc. will be formulas in which no other variables occur than $x_1, \ldots, x_n$. If $\boldsymbol{F}(x) \doteq \boldsymbol{G}(y)$, $x = (x_1, \ldots, x_n)$, $y = (x_1, \ldots, x_n, \ldots, x_m)$, then the relation F is n-ary and G is m-ary. The quantifiers and the symbols $\Rightarrow$, $\Leftrightarrow$ are used only outside the studied formulas.

Definition 1. *a) The symbols* $\boldsymbol{0}$, $\boldsymbol{1}$ *and the variables from* X *are B-terms.*

b) If $\boldsymbol{A}_1, \boldsymbol{A}_2$ *are B-terms, then* $\boldsymbol{A}_1, (\boldsymbol{A}_1 \cup \boldsymbol{A}_2), (\boldsymbol{A}_1 \cap \boldsymbol{A}_2)$ *(or* $\boldsymbol{A}_1\boldsymbol{A}_2$ *in the abbreviated form) are B-terms.*

Definition 2. *a) If* $\boldsymbol{A}_1, \boldsymbol{A}_2$ *are B-terms, then* $(\boldsymbol{A}_1 = \boldsymbol{A}_2)$, $(\boldsymbol{A}_1 \leqq \boldsymbol{A}_2)$ *are both B-formulas and BP-formulas.*

b) If $\boldsymbol{F}_1, \boldsymbol{F}_2$ *are BP-formulas, then* $\neg \boldsymbol{F}_1, (\boldsymbol{F}_1 \vee \boldsymbol{F}_2), (\boldsymbol{F}_1 \wedge \boldsymbol{F}_2), (\boldsymbol{F}_1 \rightarrow \boldsymbol{F}_2), (\boldsymbol{F}_1 \equiv \boldsymbol{F}_2)$ *are BP-formulas.*

In dealing with B-terms, B- and BP-formulas we shall omit the "outer parentheses" and also the parentheses "superfluous because of the associative law". Therefore we admit also the notations with the symbols $\bigvee, \bigwedge, \bigcup, \bigcap$ for disjunctions, conjunctions, joins and meets (of a finite number of formulas or terms), respectively. For the sake of uniqueness we assume that the terms in these expressions are lexicographically ordered. If $\boldsymbol{A}_\alpha, \boldsymbol{A}_i$ are B-terms, we put

$$\bigcup_{\alpha\in\varnothing} \boldsymbol{A}_\alpha \doteq \bigcup_{i=1}^{0} \boldsymbol{A}_i \doteq \boldsymbol{0}, \qquad \bigcap_{\alpha\in\varnothing} \boldsymbol{A}_\alpha \doteq \bigcap_{i=1}^{0} \boldsymbol{A}_i \doteq \boldsymbol{1} \tag{1}$$

and, similarly, for the "empty" join and meet of the elements of B (in this case we write $=$ instead of $\doteq$). The "empty" conjunction (or disjunction) of propositions, of propositional formulas, of BP-formulas means a fixed true proposition, a fixed tautology, the BP-formula $\boldsymbol{1} = \boldsymbol{1}$ (likewise a false proposition, a propositional contradiction, the BP-formula $\neg(\boldsymbol{1} = \boldsymbol{1})$), respectively.

If $\boldsymbol{F}(x)$ is a BP-formula and $\boldsymbol{A}$, $\boldsymbol{A}(x)$ are B-terms, then we put

$$\boldsymbol{F}^0(x) \doteq \neg \boldsymbol{F}(x), \quad \boldsymbol{F}^1(x) \doteq \boldsymbol{F}(x), \tag{2}$$

$$\boldsymbol{A}^0 \doteq \overline{\boldsymbol{A}}, \quad \boldsymbol{A}^0(x) \doteq \overline{\boldsymbol{A}(x)}, \quad \boldsymbol{A}^1 \doteq \boldsymbol{A}, \quad \boldsymbol{A}^1(x) \doteq \boldsymbol{A}(x) \tag{3}$$

and similarly for $A(b)$ and $F(b)$ $(b \in B^n)$.

Definition 3. *By a B-relation (BP-relation) we call the realization* F *of a B-formula (BP-formula)* $\boldsymbol{F}(x)$ *in a Boolean algebra* B.

Remark. From the definition of BP-formulas it follows that the system S of all the BP-relations in a given B is closed with respect to the set union, the set intersection and the complementation, so that S, considered with the mentioned operations, forms a Boolean algebra (a subalgebra of the Boolean algebra of all the n-ary relations in B). In the case of the B-relations the situation is changed: neither the complement nor the union of two B-relations need to be necessarily B-relations. It is easy to find out that in the two-element B there are no other BP-relations besides the B-relations; later we shall see that in every B containing more than two elements there exist much more BP-relations than B-relations.

Definition 4. *We say that the BP-formulas* $\boldsymbol{F}_1(x)$, $\boldsymbol{F}_2(x)$ *are equivalent and we write* $\boldsymbol{F}_1(x) \sim \boldsymbol{F}_2(x)$, *if*

$$(\forall B, b)(F_1(b) \Leftrightarrow F_2(b)).$$

If $\boldsymbol{F}_1(x) \sim \boldsymbol{F}_2(x)$, then F_1, F_2 are equal as n-ary BP-relations in any B. (For a "sufficiently large" B also the contrary holds: if $F = G$ in B, then $\boldsymbol{F}(x) \sim \boldsymbol{G}(x)$ — see the assertion b_2) of Theorem 4.)

Two arbitrary BP-formulas can be considered as formulas with the same free individual variables (see the agreement before Definition 1), and thus, Definition 4 introduces an equivalence relation on the set of all the BP-formulas. It is evident that the relation $\sim$ from Definition 4 is a congruence with respect to the propositional operations (used for formulas).

2. Characteristic sets of B-formulas and of n-tuples

For arbitrary B-terms $\boldsymbol{A}_1, \boldsymbol{A}_2$,

$$(\boldsymbol{A}_1 \leqq \boldsymbol{A}_2) \sim (\overline{\boldsymbol{A}_1} \cup \boldsymbol{A}_2 = \boldsymbol{1}), \tag{4}$$

$$(\boldsymbol{A}_1 = \boldsymbol{A}_2) \sim ((\overline{\boldsymbol{A}_1} \cup \boldsymbol{A}_2) \cap (\boldsymbol{A}_1 \cup \overline{\boldsymbol{A}_2}) = \boldsymbol{1}) \tag{5}$$

holds. Further it is known that to every B-term $\boldsymbol{A}(x)$ there exists a B-term

$$\boldsymbol{T}_M(x) \doteq \bigcup_{r \in M} \bigcap_{j=1}^{n} x_j^{r_j} \qquad (M \subset D^n, r = (r_1, \ldots, r_n)) \tag{6}$$

(the analogy of the complete disjunctive normal form of a Boolean function — see [2], Theorem 5.4, p. 215) such that $(\forall B, b)(A(b) = T_M(b))$. Hence, to every B-formula $\boldsymbol{F}(x)$, there exists a B-formula $\boldsymbol{R}_M(x) \doteq (\boldsymbol{T}_M(x) = \boldsymbol{1})$ (where $M \subset D^n$) equivalent to $\boldsymbol{F}(x)$. It is easy to show that, given a B-formula $\boldsymbol{F}(x)$, the set $M \subset D^n$ is determined uniquely.

Definition 5. *The B-formula* $\boldsymbol{R}_M(x) \doteq (\boldsymbol{T}_M(x) = \boldsymbol{1})$, *where* $M \subset D^n$ *and* $T_M(x)$ *is defined by* (6), *equivalent to a B-formula* $\boldsymbol{F}(x)$ *is called the normal form of the formula* $\boldsymbol{F}(x)$ *and the set* M *is called the characteristic set of the formula* $\boldsymbol{F}(x)$ *(both with respect to the variables* $x_1, \ldots, x_n$ *in this order).* M *will also be called the characteristic set of the B-relation* F *or* R_M *in an arbitrary* B.

If we denote a B-formula by a notation of type $\boldsymbol{F}(x)$ (see the text before Definition 1), it implicitly means that its normal form and its characteristic set are considered with respect to the variables $x_1, \ldots, x_n$ in this order.

The vector of the values of the characteristic function $\chi_M : D^n \to D$ ($\chi_M(r) = 1 \Leftrightarrow r \in M$) of the characteristic set $M (\subset D^n)$ of a B-formula $\boldsymbol{F}(x)$ can easily be obtained by a 0-1-method if we consider $\boldsymbol{F}(x)$ as the notation of the

Boolean function of the variables $x_1, \ldots, x_n$, where we replace $\cup, \cap, ^-, \leqq, =$ by the symbols of disjunction, conjunction, negation, implication and equivalence, respectively.

Lemma 1. *For the normal forms of B-formulas $\boldsymbol{R}_M(x)$, $\boldsymbol{R}_N(x)$, $\boldsymbol{R}_{Mi}(x)$ $(i = 1, \ldots, m)$, where M, N, $M_i \subset D^n$, there holds*

$$(7) \qquad (\forall B, b)\,(R_M(b) \Rightarrow R_N(b)), \quad \textit{if } M \subset N,$$

$$(8) \qquad \boldsymbol{R}_M(x) \wedge \boldsymbol{R}_N(x) \sim \boldsymbol{R}_{M \cap N}(x),$$

$$(9) \qquad \bigwedge_{i=1}^{m} \boldsymbol{R}_{M_i}(x) \sim \boldsymbol{R}_M(x), \quad \textit{if } M = \bigcap_{i=1}^{m} M_i.$$

(9) *holds also for $m = 0$, if we put $\bigcap_{i=1}^{0} M_i = D^n$ $(= \boldsymbol{1}$ in the Boolean algebra $(2^{D^n}, \cup, \cap, ^-)$ — cf. (1)).*

Proof. The assertion (7) is evident. Let us suppose that $R_M(b), R_N(b)$ hold for some $b \in B^n$. Then, for $r = (r_1, \ldots, r_n)$, $s = (s_1, \ldots, s_n)$, $r, s \in D^n$,

$$E_r = \bigcap_{j=1}^{n} b_j^{r_j}, \qquad E_s = \bigcap_{j=1}^{n} b_j^{s_j},$$

there holds $E_r \cap E_s = \boldsymbol{0}$ if $r \neq s$, and $E_r \cap E_s = E_r$ if $r = s$. Therefore

$$\bigcup_{r \in M \cap N} \bigcap_{j=1}^{n} b_j^{r_j} = \bigcup_{r \in M \cap N} E_r = \bigcup_{r \in M} E_r \cap \bigcup_{s \in N} E_s = \boldsymbol{1} \cap \boldsymbol{1} = \boldsymbol{1}.$$

Thus $R_{M \cap N}(b)$ is valid. The converse implication in the proof of (8) follows from (7). Finally, (9) can be verified by induction.

Now we introduce an auxiliary notion and we prove some lemmas. In a fixed B we choose some $b \in B^n$ and we consider the system of all the sets $M_i \subset D^n$ $(i = 1, \ldots, m)$ for which $R_{M_i}(b)$ holds. According to (9), also $R_M(b)$ holds, where $M = \bigcap_{i=1}^{m} M_i$. Hence we can formulate the following definition (the set M_b introduced in the definition always exists).

Definition 6. *If $b \in B^n$, then the smallest (in the sense of the set inclusion) set $M \subset D^n$ for which $R_M(b)$ holds is called the characteristic set of the n-tuple b (in the given B) and is denoted by M_b.*

Remark. In any B, every set M_b is evidently non-empty.

Using relation (7) we get the following

Lemma 2. *a) Let $M \subset D^n$ and M_b be the characteristic set of $b \in B^n$. Then*

$$\text{(a)} \qquad M = M_b \textit{ iff } R_M(b) \wedge (\forall t \in M) \neg R_{M-\{t\}}(b),$$

$$\text{(b)} \qquad R_M(b) \textit{ iff } M_b \subset M.$$

Lemma 3. *Let M_b be the characteristic set of $b \in B^n$, $M = \{s \in D^n \mid \neg R_{D^n-\{s\}}(b)\}$ $M_1 = \{s \in D^n \mid \bigcap_{i=1}^{n} b_i^{s_i} \neq \mathbf{0},\ s = (s_1, \ldots, s_n)\}$. Then $M_b = M = M_1$.*

Proof. Note that $R_{D^n}(b)$ holds for any $b \in B^n$. Since

$$(\forall s \in M) \neg R_{M-\{s\}}(b)$$

is valid, then, using (7), we get

$$(\forall s \in M) \neg R_{M-\{s\}}(b).$$

From $R_{D^n-\{s\}}(b)$ which, according to the definition of the set M, holds for every $s \in D^n - M$, from the equality

$$M = \bigcap_{s \in D^n - M} (D^n - \{s\})$$

and according to (9), we obtain also $R_M(b)$. Therefore, according to (a) from Lemma 2, $M_b = M$ holds. The equality $M = M_1$ follows from

$$a \neq \mathbf{1} \Leftrightarrow \bar{a} \neq \mathbf{0} \quad (\text{for every } a \in B).$$

Definition 7. *A set $K \subset D^n$ is called solvable (unsolvable) in B if there exists (if there does not exist) an n-tuple $b \in B^n$ such that $M_b = K$.*

Remark. According to the remark following Definition 6 the solvable sets are non-empty.

We describe another way for the determination of the characteristic set of the n-tuple $b = (b_1, \ldots, b_n)$ of the elements of the 2^k element B (k is a positive integer). Every such B is isomorphic to the Boolean algebra $(D^k, \cup, \cap, ^-)$ with the operations "componentwise" defined, where $0 \cup 0 = 0$, $0 \cup 1 = 1 \cup 0 = 1 \cup 1 = 1$ and similarly the meet and the complement. Further (cf. (3)): $0^0 = \bar{0} = 1^1 = 1$, $1^0 = \bar{1} = 0^1 = 0$, i. e. for $u, v \in D^n$, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ there holds

$$\bigcap_{i=1}^{n} u_i^{v_i} = 1 \quad \Leftrightarrow \quad u = v. \tag{10}$$

Let $\varphi\colon B \to D^k$ be a fixed isomorphism which we extend to the n-tuples by the equality $b\varphi = (b_1\varphi, \ldots, b_n\varphi)$, where we put $b_i\varphi = (b_{i1}, \ldots, b_{ik})$ $(\in D^k)$ for $i = 1, \ldots, n$. Let $A_{b\varphi}$ be a matrix of the type $n \times k$ with the row vectors* $b_i\varphi$ $(i = 1, \ldots, n)$ and the column vectors $s_h = (b_{1h}, \ldots, b_{nh})$ $(h = 1, \ldots, k)$. Schematically:

* Here and in the following we use the description row vector or column vector only for placing them in a particular matrix. We do not distinguish them otherwise.

$$\begin{array}{c c c}
 & \begin{array}{ccc} s_1 & \dots & s_k \\ \downarrow & & \downarrow \end{array} & \\
A_{b\varphi} = & \begin{pmatrix} b_{11} & \dots & b_{1k} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nk} \end{pmatrix} & \begin{array}{l} \leftarrow \quad b_1\varphi \\ \qquad \dots \\ \leftarrow \quad b_n\varphi \end{array}
\end{array}$$

Let us denote $N_{b\varphi} = \{s_1, \dots, s_k\}$ (s_h evidently need not be pairwise different for $h = 1, \dots, k$).

Lemma 4. *Under the notation just introduced the following holds in every 2^k element algebra B ($k \geqq 1$):*

a) *$M_b = N_{b\varphi}$, that is M_b is the set of all the column vectors of the matrix $A_{b\varphi}$.*

b) *A set $K \subset D^n$ is solvable in B if and only if $K \neq \emptyset$ and* card $K \leqq k$.

Proof. Since φ is an isomorphism, then $M_b = M_{b\varphi}$. Thus, it is sufficient to show that $M_{b\varphi} = N_{b\varphi}$. Let $s \in N_{b\varphi}$, i. e. $s = s_h = (b_{1h}, \dots, b_{nh})$ for some $h \leqq k$. Then

$$\bigcap_{i=1}^{n} (b_i\varphi)^{b_{ih}} = (\dots, \bigcap_{i=1}^{n} b_{ih}^{b_{ih}}, \dots) = (\dots, 1, \dots) \neq \mathbf{0},$$

i. e., according to Lemma 3, $s \in M_{b\varphi}$. If $t = (t_1, \dots, t_n) \notin N_{b\varphi}$, then $t \neq s_h$ ($h = 1, \dots, k$) and

$$\bigcap_{i=1}^{n} b_{ih}^{t_i} = 0 \ (h = 1, \dots, k), \qquad \bigcap_{i=1}^{n} (b_i\varphi)^{t_i} = \mathbf{0}.$$

Hence, according to Lemma 3, $t \notin M_{b\varphi}$.

b) The assertion follows from a) if we recall the following: For every K consisting of m column vectors there exists a k column matrix $A_{b\varphi}$ which contains all of them (and contains no others) if and only if $m \leqq k$.

Remark. The solutions of the equation $M_b = K$ in a 2^k element B, where $K = \{s_1, \dots, s_r\} \subset D^n$ is given ($r \leqq k$), are thus exactly those n-tuples b for which the matrix $A_{b\varphi}$ contains all the column vectors $s_1, \dots, s_r$ and no others.

Lemma 5. *In a Boolean algebra B with at least 2^{2^n} elements every non-empty set $N \subset D^n$ is solvable.*

Proof. For a finite B with 2^k elements, where $k \geqq 2^n$, the assertion follows from b) of the last lemma. In the case of B being infinite, it is sufficient to recall that, for every positive integer p, B contains a finite subalgebra having at least p elements.

3. BP-tautologies

Definition 8. *A BP-formula $\boldsymbol{F}(x)$ is said to be a BP-tautology if* $(\forall B, b)F(b)$.

Our next aim is to determine a necessary and sufficient condition for a BP-formula to be a BP-tautology.

Let $p_1, p_2, \ldots$ be propositional variables, $p = (p_1, \ldots, p_m)$ and $\approx$ be an equivalence between propositional formulas defined by $\boldsymbol{A} \approx \boldsymbol{B}$ if and only if $(\boldsymbol{A} \equiv \boldsymbol{B})$ is a propositional tautology. To every propositional formula $\boldsymbol{A}(p)$ (containing no other variables than $p_1, \ldots, p_m$) there exists an equivalent formula of the form

$$\bigwedge_{s \in M} \bigvee_{i=1}^{m} p_i^{s_i}, \quad \text{where} \quad M \subset D^m,\ s = (s_b, \ldots, s_m)$$

(the analogy of the complete conjunctive normal form of a Boolean function — see [2], p. 215). It is clear from Definition 2 that every BP-formula arises from some propositional formula by substituting the propositional variables by B-formulas. The equivalence $\approx$ of propositional formulas yields the equivalence $\sim$ (Definition 4) of BP-formulas obtained after the substitution (see [1], 2.1, p. 283). Hence (see also Definition 5 and the text preceding it), every BP-formula $\boldsymbol{F}(x)$ is equivalent to a BP-formula of the form

$$\bigwedge_{s \in M} \bigvee_{i=1}^{m} \boldsymbol{R}_{M_i}^{s_i}(x), \tag{11}$$

where $M \subset D^m$, $s = (s_1, \ldots, s_m)$, $M_i \subset D^n$ $(i = 1, \ldots, m)$.

Theorem 1. *Let* (11) *be a formula equivalent to the BP-formula* $\boldsymbol{F}(x)$. *Then* $\boldsymbol{F}(x)$ *is a BP-tautology if and only if, for arbitrary* $s = (s_1, \ldots, s_m) \in M$, there *holds*

a) $s_1 = \ldots = s_m = 1 \;\Rightarrow\; (\exists j \leqq m)\,(M_j = D^n)$;

b) $s_1 = \ldots = s_m = \;\Rightarrow\; \bigcap_{i=1}^{m} M_i = \emptyset$;

c) *if there are* $i, j \leqq m$ *such that* $s_i = 0$, $s_j = 1$, *then*

$$(\exists j \leqq m)\,(s_j = 1,\ \bigcap_{s_i = 0} M_i \subset M_j). \tag{12}$$

Proof. According to definitions 4, 8, $\boldsymbol{F}(x)$ is a BP-tautology if and only if also formula (11) is a BP-tautology, i. e. iff

$$(\forall s \in M)\,(\forall B, b) \bigvee_{i=1}^{m} \boldsymbol{R}_{M_i}^{s_i}(b), \tag{13}$$

where $b = (b_1, \ldots, b_n) \in B^n$.

Let us suppose that (13) holds and choose $s = (s_1, \ldots, s_m) \in M$.

a) Let $s_1 = \ldots = s_m = 1$, i. e. we can omit the exponents in (13). In virtue of Lemma 5, in an algebra B of 2^{2^n} elements there exists an n-tuple b of its elements such that $M_b = D^n$. Following (13), there is a $j \leqq m$ such that $R_{M_j}(b)$ holds, i. e. $M_b \subset M_j$, according to Lemma 2. Hence $D^n = M_b \subset M_j$ $(\subset D^n)$, $M_j = D^n$.

b) Let $s_1 = \ldots = s_m = 0$. Let us denote $N = \bigcap_{i=1}^{m} M_i \ (\subset D^n)$ and suppose that $N \neq \emptyset$. Consider again the mentioned B and an n-tuple $b \in B^n$ such that $M_b = N$. Then $M_b \subset M_i$, $R_{M_i}(b)$ $(i = 1, \ldots, m)$. Thus, we get

$$\bigwedge_{i=1}^{m} R_{M_i}(b), \quad \neg \bigvee_{i=1}^{m} \neg R_{M_i}(b), \quad \neg \bigvee_{i=1}^{m} R_{M_i}^{s_i}(b)$$

which is a contradiction to the assumption (13).

c) If there are $i, j \leqq m$ such that $s_i = 0$, $s_j = 1$, let us suppose that, for $P = \bigcap_{s_i=0} M_i$,

$$s_j = 1 \ \Rightarrow \ P - M_j \neq \emptyset \qquad (1 \leqq j \leqq m). \tag{14}$$

To the set $N = \bigcup_{s_j=1} (P - M_j) \neq \emptyset$, again according to Lemma 5, there exists an n-tuple $b \in B^n$ such that $M_b = N$. Following (14), for every $j \leqq m$, where $s_j = 1$, there is an element $b_j \in P - M_j \subset N = M_b$, that is $b_j \in M_b$, $b_j \in M_j$. Therefore, according to Lemma 2, $\neg R_{M_j}^{s_j}(b)$ for $s_j = 1$. Moreover, for $s_i = 0$, we have $M_b = N \subset P \subset M_i$, $R_{M_i}(b)$, $\neg R_{M_i}^{s_i}(b)$. This is a contradiction to (13).

Let us suppose that our criterion holds. The disjunctions in (11), where $s_1 = \ldots = s_m = 1$ or $s_1 = \ldots = s_m = 0$, are evidently BP-tautologies (in the second case we use (9)). For an m-tuple $s \in M$ with different components and for $b \in B^n$, according to c), either $(\exists i \leqq m)\ (s_i = 0, \neg R_{M_i}(b))$ (i. e. (13) holds) or, for every $i \leqq m$, where $s_i = 0$, there holds $R_{M_i}(b)$ (i. e. $M_b \subset M_i$). Then $M_b \subset \bigcap_{s_i=0} M_i \subset M_j$, $R_{Mj}(b)$ for some j, where $s_j = 1$. Therefore (13) holds.

Remarks. 1. Since, according to definitions 4 and 8, the BP-formulas $\boldsymbol{F}_1(x)$, $\boldsymbol{F}_2(x)$ are equivalent if and only if the formula $\boldsymbol{F}_1(x) \equiv F_2(x)$ is a BP-tautology, it is possible to verify the equivalence of the given BP-formulas with help of Theorem 1.

2. Theorem 1 can be used also for the generalization of the form (11), where m, n, $x = (x_1, \ldots, x_n)$ depend on s (but not on i) and M_i also depends on s:

$$\bigwedge_{s \in M} \bigvee_{i=1}^{m(s)} R_{M_i(s)}^{s_i}(x(s)), \tag{15}$$

where $M \subset D^1 \cup D^2 \cup \ldots$, $M_i(s) \subset D^{n(s)}$ $\quad (1 \leqq i \leqq m(s))$,
$s = (s_1 \ldots, s_{m(s)})$, $x(s) = (x_1(s), \ldots, x_{n(s)}(s))$,
$x_i(s) \in \{x_1, \ldots, x_n\}$ for $s \in M$, $1 \leqq i \leqq n(s)$,
$x_i(s) \neq x_j(s)$ for $i \neq j$, $s \in M$.

The form (15) is more complicated than (11) but in the case of a particular given BP-formula it is usually shorter than (11) and the conditions a), b), c) from Theorem 1 can be verified easier than from (11). (In order to save the unity of notation it is necessary to put $M_i(s), M_j(s), m(s), n(s)$ instead of M_i, M_j, m, n, respectively.)

3. We shall describe the 0—1-method for testing of the criterion from Theorem 1. First we find out (by the method described previous to Lemma 1) the characteristic sets of B-formulas from $\boldsymbol{F}(x)$ (with respect to the variables $x_1, \ldots, x_n$). Then we substitute for the B-formulas propositional variables (instead of formulas with the same characteristic set we substitute the same variable). The obtained propositional formula can be performed, e. g., to the minimal conjunctive normal form. The disjunctions of the formula can be ordered in such a way that the negations precede the other terms. Finally, we obtain the form (15) if, instead of propositional variables, we substitute the normal forms of the corresponding B-formulas with respect to the minimal number of Boolean variables but in such a way that in the same disjunction there are only normal forms with respect to the same variables in the same order.

For every $s = (s_1, \ldots, s_m) \in M$ we write s as an "exponents column" and in the corresponding rows the vectors of the values of the characteristic functions χ_{M_i} of the characteristic sets M_i of the B-formulas. We obtain a "B-formulas matrix" B_s of the type $m \times 2^n$ (where every column corresponds to a certain n-tuple $r \in D^n$). In the exponents column zeros (if they exist there) precede the units. Thus, B_s can be devided into a submatrix with 0-exponents" B_{s0} and a „submatrix with 1-exponents" B_{s1}. Theorem 1 implies that a BP-formula is a BP-tautology if and only if, for any $s \in M$, there holds:

a) If the exponents column contains only units, then B_s contains a row with only units.

b) If the exponents column contains only zeros, then B_s contains no column with units only.

c) If s contains both components, then B_{s1} contains a row with a unit in any column whose part in B_{s0} consists of units only.

4. Equivalence of the BP-formulas

In the entire section 4 we put $m = 2^{2^n}$ and the domain of the variable M will be 2^{D^n}. Further we shall choose a fixed (e. g. lexicographical) order $M_1, \ldots, M_m$ of all the sets $M \subset D^n$.

If, for $s = (s_{M_1}, \ldots, s_{M_m}) \in D^m$, we put

$$\text{(16)} \qquad \boldsymbol{E}_s(x) \doteq \bigvee_{M \subset D^n} \boldsymbol{R}_M^{s_M}(x)$$

(we omit the indices of M), then every BP-formula $\boldsymbol{F}(x)$ is equivalent to some formula of the type

$$\bigwedge_{s \in N} \boldsymbol{E}_s(x) \qquad (N \subset D^m) \tag{17}$$

(which is a particular case of (11) for $m = 2^{2^n}$). First we find out which of the formulas $\boldsymbol{E}_s(x)$ are not BP-tautologies.

Let $K \subset D^n$. We put $s_M(K) = 0$ for $K \subset M$; $s_M(K) = 1$ for $K \not\subset M$ and $s(K) = (s_{M_1}(K), \ldots, s_{M_m}(K))$. We note that for $K_1 \neq K_2$ also $s(K_1) \neq s(K_2)$. Thus, for $\boldsymbol{E}_K(x) \doteq \boldsymbol{E}_{s(K)}(x)$ there holds

$$\boldsymbol{E}_K(x) \doteq \bigvee_{M \subset D^n} \boldsymbol{R}_M^{s_M(K)}(x), \quad \text{where} \quad s_M(K) = 0 \Leftrightarrow K \subset M. \tag{18}$$

Lemma 6. *In the notation just introduced there holds: $\boldsymbol{E}_s(x)$ is not a BP-tautology if and only if there exists a non-empty $K \subset D^n$ such that $s = s(K)$, i. e. $\boldsymbol{E}_s(x) \doteq \doteq \boldsymbol{E}_K(x)$.*

Proof. Let $s = s(K)$, $\emptyset \neq K \subset D^n$. Then $(0, \ldots, 0) \neq s \neq (1, \ldots, 1)$ and, according to c) from Theorem 1, $\boldsymbol{E}_s(x)$ is not a BP-tautology because there is no $M_j \supset K = \bigcap\{M \mid K \subset M\} = \bigcap\{M \mid s_M(K) = 0\}$ such that $s_{M_j}(K) = 1$.

Conversely, assume that $\boldsymbol{E}_s(x)$ is not a BP-tautology. Since $M = D^n$ for some M and $\bigcap 2^{D^n} = \emptyset$, according to Theorem 1, neither $s_1 = \ldots = s_m = 1$ nor $s_1 = \ldots = s_m = 0$ can be valid. Thus, only the case c) remains. We put

$$K = \bigcap\{M \mid s_M = 0\} \qquad (\subset D^n).$$

Then $K \subset M$ for $s_M = 0$, but for no M such that $s_M = 1$, $K \subset M$ is valid, because, according to Theorem 1, $\boldsymbol{E}_s(x)$ would be a BP-tautology (thus, $K = \emptyset$ is also impossible because then $K \subset M$ would hold for every M). Therefore, in all cases $s_M = 0 \Leftrightarrow K \subset M$ so that $(\forall M)(s_M = s_M(K))$, $s = s(K)$.

In the following lemma we find to the formula $\boldsymbol{E}_K(x)$ $(\emptyset \neq K \subset D^n)$ an equivalent but shorter formula.

Lemma 7. *Let*

$$\boldsymbol{Q}_K(x) \doteq \neg \boldsymbol{R}_K(x) \vee \bigvee_{s \in K} \left(\bigcap_{j=1}^{n} x_j^{s_j} = \boldsymbol{0}\right) \qquad (\emptyset \neq K \subset D^n).$$

Then $\boldsymbol{Q}_K(x) \sim \boldsymbol{E}_K(x)$.

Proof. According to (18) we have

$$\boldsymbol{E}_K(x) \sim \bigvee_{M \supset K} \neg \boldsymbol{R}_M(x) \vee \bigvee_{K \not\subset M} \boldsymbol{R}_M(x). \tag{19}$$

According to de Morgan's rule, following (9) and using the equality $K = \bigcap\{M \mid M \subset K\}$, we have $\bigvee_{M \supset K} \neg \boldsymbol{R}_M(x) \sim \neg \boldsymbol{R}_K(x)$. Thus, it is sufficient to show that

$$(20) \qquad \bigvee_{K \not\subset M} R_M(b) \Leftrightarrow \bigvee_{s \in K} (\bigcap_{j=1}^{n} b_j^{s_j} = \mathbf{0})$$

for every n-tuple $b \in B^n$.

Let $\bigvee_{K \not\subset M} R_M(b)$. Then there exists a set $M \subset D^n$ and an n-tuple $s = (s_1, \ldots, s_n) \in \in K - M$ such that $R_M(b)$, $R_{M-\{s\}}(b)$ hold (because $M = M - \{s\}$). Thus, $R_{D^n-\{s\}}(b)$(by (7)), $\bigcap_{j=1}^{n} b_j^{s_j} = \mathbf{0}$, which gives the right-hand side of (20) (because $s \in K$). Conversely, let the right-hand side be valid. Then for some $s \in K$, $M = D^n - \{s\}$ ($\not\supset K$) we have $R_M(b)$, which gives the left-hand side of the equivalence (20).

Lemma 8. *Let* $\boldsymbol{E}_K(x)$ (*where* $K \subset D^n$) *be defined by* (18), $W = 2^{D^n} - \{\emptyset\}$ *and* $S_1, S_2 \subset W$. *Then*

a) $(\forall B, b)((\bigvee_{K \in S_1} E_K(b) \Leftrightarrow \bigwedge_{K \in S_2} E_K(b)) \Leftrightarrow \bigwedge_{K \in S_1 \dot{-} S_2} E_K(b))$;

b) $\bigwedge_{K \in S_1} \boldsymbol{E}_K(x) \sim \bigwedge_{K \in S_2} \boldsymbol{E}_K(x)$ iff $S_1 = S_2$.

Proof. We put $T = D^m - \{s(K) \mid K \in W\}$, $T_i = \{s(K) \mid K \in S_i\}$, $\overline{T}_i = \{s(K) \mid K \in W - S_i\}$ $(i = 1, 2)$. Then

$$\text{(i)} \qquad T \cup T_i \cup \overline{T}_i = D^m, \quad T \cap T_i = T \cap \overline{T}_i = T_i \cap \overline{T}_i = \emptyset \quad (i = 1,2).$$

For every $K \in W$, $K \in S_i$ iff $s(K) \in T_i$ $(i = 1, 2)$ and, it is easy to see, $K \in S_1 \dot{-} S_2$ iff $s(K) \in T_1 \dot{-} T_2$. Thus, we have to show that, for any B and $b \in B^n$,

$$\text{(ii)} \qquad \bigwedge_{s \in T_1} E_s(b) \Leftrightarrow \bigwedge_{s \in T_2} E_s(b)$$

holds iff $\bigwedge_{s \in T_1 \dot{-} T_2} E_s(b)$. Instead of $\neg \bigwedge_{s \in T_i} E_s(b)$, according to Lemma 6, we can write $\neg \bigwedge_{s \in T \cup T_i} E_s(b)$ and this, according to (i), holds iff $\bigwedge_{s \in \overline{T}_i} E_s(b)$ $(i = 1, 2)$. Therefore (ii) is equivalent to

$$\text{(iii)} \qquad (\bigwedge_{s \in T_1} E_s(b) \vee \bigwedge_{s \in \overline{T}_2} E_s(b)) \wedge (\bigwedge_{s \in \overline{T}_1} E_s(b) \vee \bigwedge_{s \in T_2} E_s(b)).$$

Using the distributive law and the fact that $E_{s_1}(b) \vee E_{s_2}(b)$ holds for $s_1 \neq s_2$ we see that (iii) holds iff

$$\bigwedge_{s \in T_1 \cap \overline{T}_2} E_s(b) \wedge \bigwedge_{s \in \overline{T}_1 \cap T_2} E_s(b),$$

that is $\bigwedge_{s \in T_1 \dot{-} T_2} E_s(b)$.

The assertion b) follows from a), according to Lemma 6, if we recall that $S_1 \dot{-} S_2 = \emptyset \Leftrightarrow S_1 = S_2$.

Theorem 2. *For every BP-formula* $\boldsymbol{F}(x)$ *there exists exactly one system* $S_F \subset \subset S(n) = 2^{D^n} - \{\emptyset\}$ *such that*

$$\boldsymbol{F}(x) \sim \bigwedge_{K \in S_F} \boldsymbol{E}_K(x),$$

where $\boldsymbol{E}_K(x)$ *is defined by* (18). *Moreover*

$$\text{(i)} \qquad \boldsymbol{F}(x) \sim G(x) \Leftrightarrow S_F = S_G$$

holds. Finally, $\boldsymbol{F}(x)$ *is a BP-tautology if and only if* $S_F = \emptyset$.

Proof. $\boldsymbol{F}(x)$ is equivalent to a formula of the type (17). If we omit all the factors from the conjunction $\bigwedge_{s \in N} \boldsymbol{E}_s(x)$ which are BP-tautologies, we obtain an equivalent formula $\bigwedge_{s \in N_1} \boldsymbol{E}_s(x)$ which, according to Lemma 6, can be written in the form $\bigwedge_{K \in S_F} \boldsymbol{E}_K(x)$, where $S_F \subset S(n)$ (according to the agreement made after relations (1), there can also $N_1 = S_F = \emptyset$ hold). The uniqueness of S_F and the equivalence (i) follow now from b) in Lemma 8; the assertion giving a condition for $\boldsymbol{F}(x)$ to be a BP-tautology follows from Lemma 6.

Remarks. The formulas $\bigwedge_K \boldsymbol{E}_K(x)$ form a "natural" system of representatives of the classes of equivalence of the BP-formulas — they arise from the complete conjunctive normal forms (see [2], p. 215) with the propositional variables $p_1, \ldots, p_m$ ($m = 2^{2^n}$) if we replace p_i by the normal form $\boldsymbol{R}_{M_i}(x)$ ($i = 1, \ldots, m$). Due to Lemma 7, $\boldsymbol{E}_K(x)$ can be replaced by $\boldsymbol{Q}_K(x)$. The formulas $\bigwedge_K \boldsymbol{Q}_K(x)$ form some reduced system of representatives — they are substantially shorter: $\boldsymbol{Q}_K(x)$ is approximately twofold but $\boldsymbol{E}_K(x)$ 2^{2^n} — fold longer than $\boldsymbol{R}_K(x)$.

For a one element set $K \subset D^n$ instead of $\boldsymbol{Q}_K(x)$ it is possible to use an equivalent but shorter formula $\neg\, \boldsymbol{R}_K(x)$. Moreover

$$\boldsymbol{Q}_K(x) \sim \neg\,(\bigcup_{s \notin K} \bigcap_{j=1}^{n} x_j^{s_j} = \boldsymbol{0}) \vee \bigvee_{s \in K} (\bigcap_{j=1}^{n} x_j^{s_j} = \boldsymbol{0}),$$

where, for card $K < 2^{n-1}$, the formula on the right-hand side is shorter than $\boldsymbol{Q}_K(x)$ (if we consider $\bar{x}_j$ as one symbol).

Definition 9. *The system* S_F *from Theorem 2 is called the characteristic system of the BP-formula* $\boldsymbol{F}(x)$ (*with respect to the variables* $x_1, \ldots, x_n$ *in this order*).

For the variables for which we determine S_F a similar remark as the one following Definition 5 holds. S_F depends on the order $x_1, \ldots, x_n$ but, according to b) from Lemma 8, it does not depend on the order $M_1, \ldots, M_m$ of the sets $M \subset D^n$ (because another order changes $\boldsymbol{E}_K(x)$ from (18) only in the order of its terms).

Lemma 9. *The characteristic system S_F of a BP-formula $\boldsymbol{F}(x)$ equivalent to the formula*

$$(21) \qquad \bigvee_{j=1}^{k} \boldsymbol{R}_{N_j}^{s_j}(x) \qquad (N_j \subset D^n;\ j = 1, \ldots, k)$$

contains a non-empty set $K \subset D^n$ if and only if

$$(22) \qquad K \subset N_j \Leftrightarrow s_j = 0 \qquad (j = 1, \ldots, k).$$

Proof. It is sufficient to consider pairwise different N_j. Hence, if

$$(23) \qquad N_i = N_j \Rightarrow s_i = s_j \qquad (i, j = 1, \ldots, k),$$

then we can omit the terms of the disjunction (21) corresponding to the "repeated" N_j and in (22) leave out the corresponding i, j; if (23) is false, then $\boldsymbol{F}(x)$ is a BP-tautology, $S_F = \emptyset$ and (22) holds for no K. Thus we suppose further that N_j are pairwise different and that $N_j = M_{i_j}$ ($j = 1, \ldots, k$), where $M_1, \ldots, M_m$ ($k \leqq m = 2^{2^n}$) is the order mentioned in the introduction to section 4. We put

$$\boldsymbol{E}_t(x) \doteq \bigvee_{i=1}^{m} \boldsymbol{R}_{M_i}^{t_i}(x), \qquad \boldsymbol{H}(x) \doteq \bigwedge_{t} \boldsymbol{E}_t(x),$$

where t runs over the m-tuples $(t_1, \ldots, t_m) \in D^m$ such that $(\forall j \leqq k)(t_{i_j} = s_j)$. Using the well-known theorems of the propositional calculus we have $\boldsymbol{H}(x) \sim \sim \boldsymbol{F}(x)$. If we omit the terms of $\boldsymbol{H}(x)$, which are BP-tautologies (see Lemma 6), then, with respect to the uniqueness of the characteristic system, it is sufficient to prove that from the disjunctions $\boldsymbol{E}_K(x)$ (see (18)), where $\emptyset \neq K \subset D^n$, there belong to the terms of $\boldsymbol{H}(x)$ exactly those which satisfy (22). But, according to (18), this follows from the fact that (22) is equivalent to the condition $(\forall j \leqq k)(s_{N_j}(K) = s_j)$.

Theorem 3. *a) Let $\boldsymbol{F}(x)$ be the BP-formula from Lemma 9 and $S(n) = 2^{D^n} - \{\emptyset\}$. Then*

$$S_F = \Big(\bigcap_{s_j=0} 2^{N_j} - \{\emptyset\}\Big) \cap \Big(S(n) - \bigcup_{s_j=1} 2^{N_j}\Big) \qquad (1 \leqq j \leqq k).$$

b) *If S_i is the characteristic system of the BP-formula $\boldsymbol{F}_i(x)$ ($i = 1, \ldots, k$), then the characteristic system of the formula $\bigwedge_{i=1}^{k} \boldsymbol{F}_i(x)$ is $S = \bigcup_{i=1}^{k} S_i$.*

Proof. a) is a consequence of Lemma 9 and b) follows from Theorem 2.

Remark. Since every BP-formula $\boldsymbol{F}(x)$ is equivalent to a conjunction of BP-formulas of the type (21) (cf. (11)), Theorem 3 gives an effective method for the determination of the system S_F to a given $\boldsymbol{F}(x)$. (The determination of S_F by the method from the proof of Theorem 2 is, for particular formulas, very slow: conjunctions (17) of disjunctions (16) with 2^{2^n} terms are used.)

5. BP-relations in a given Boolean algebra

In this section we present some theorems concerning the BP relations.

Lemma 10. *Let* $\emptyset \neq K \subset D^n$. *Then* $(\forall B, b)(Q_K(b) \Leftrightarrow M_b \neq K)$.

Proof. Choose $B, b \in B^n$ and $K \subset D^n$. According to (18) and Lemma 7,

$$Q_K(b) \Leftrightarrow \bigvee_{K \not\subset M} R_M(b) \vee \bigvee_{M \supset K} \neg R_M(b), \tag{24}$$

where the index M runs over 2^{D^n}. Further, according to Lemma 2,

$$\bigvee_{K \not\subset M} R_M(b) \Leftrightarrow (\exists M \subset D^n)(K \not\subset M, M_b \subset M), \tag{25}$$

$$\bigvee_{M \supset K} \neg R_M(b) \Leftrightarrow \neg \bigwedge_{M \supset K} R_M(b) \Leftrightarrow \neg R_K(b) \Leftrightarrow M_b \not\subset K, \tag{26}$$

where the second equivalence in (26) follows from (9) and from the equality $\bigcap \{M \mid K \subset M \subset D^n\} = K$.

Now suppose $Q_K(b)$. If $M_b \not\subset K$, then $M_b \neq K$. If $M_b \subset K$, then, according to (24)—(26), $K \not\subset M$, $M_b \subset M$ for some $M \subset D^n$. Thus, $M_b \neq K$.

Let $M_b \neq K$. If $M_b \not\subset K$, then, according to (26), (24), $Q_K(b)$ holds. If $M_b \subset K$, then, for $M =' M_b$, there is $M_b \subset M$, $M \subset K$ and $K \not\subset M$ (in the other case $K = M = M_b \neq K$). By (25), (24), $K \not\subset M$, $M_b \subset M$ imply $Q_K(b)$.

Remark. According to Lemma 9 and the remark following Lemma 4, it is possible, for small n, to find all the n-tuples $b \in B^n$ (if card $B = 2^k$) for which $\neg Q_K(b)$ (i. e. $M_b = K$), where the set $K = \{s_1, \ldots, s_r\} \subset D^n$ is given ($r \leq k$). It is sufficient to choose $\varphi : B \to D^k$ preceding Lemma 4 and then gradually construct from the column vectors $s_1, \ldots, s_r$ all the matrices $A_{b\varphi}$, and with it the n-tuples $b\varphi$ and also b. If we know the characteristic system of the BP-formula $\boldsymbol{F}(x)$, according to Theorem 3 the above construction gives a constructive method for the determination of all the n-tuples b belonging to the BP-relation F in a finite B.

According to Lemmas 10 and 4, we get the following

Lemma 11. *Let* Q_K *be the realization of a BP-formula* $\boldsymbol{Q}_K(x)$ *(see Lemma 7) in a* 2^k *element* B. *Then* $Q_K = B^n$ *if and only if* card $K > k$.

Theorem 4. *Let* k *be a positive integer and* $S_k(n)$ *(or* $S(n)$*) the system of all the non-empty and at most* k *element (or of all the non-empty) subsets of the set* D^n. *Let* $\boldsymbol{F}(x)$, $\boldsymbol{G}(x)$ *be BP-formulas with characteristic systems* S_F, S_G *and let* $F, G(\subset B^n)$ *be the realizations of these formulas in a given* B. *Then there holds*:

If B has exactly 2^k elements, then

a_1) $F = B^n$ if and only if $S_F \cap S_k(n) = \emptyset$;

a_2) $F = G$ if and only if $S_F \cap S_k(n) = S_G \cap S_k(n)$;

a_3) (to a given F) there exists a system $S = S_{kF} \subset S_k(n)$ such that

$$F = \bigcap_{K \in S} Q_K.$$

The system S_{kF} is determined uniquely, that is

$$F = G \Leftrightarrow S_{kF} = S_{kG}.$$

If B has at least 2^{2^n} elements (thus, it can also be infinite), then

b_1) $F = B^n$ if and only if $S_F = \emptyset$, i. e. iff $\boldsymbol{F}(x)$ is a BP-tautology;

b_2) $F = G$ if and only if $S_F = S_G$, i. e. iff $\boldsymbol{F}(x) \sim \boldsymbol{G}(x)$.

Proof. Let $\operatorname{card} B = 2^k$.

a_1) According to Lemma 11,

$$F = \bigcap_{K \in S_F} Q_K = \bigcap_{K \in S_F \cap S_k(n)} Q_K, \tag{27}$$

i. e. $F = B^n$ iff $S_F \cap S_k(n) = \emptyset$ ($\bigcap_{K \in \emptyset} Q_K = B^n = \boldsymbol{1}$ in $(B^n, \cup, \cap, ^-)$ — cf. (1)).

a$_2$) Let us denote $S_1 = S_F \cap S_k(n)$, $S_2 = S_G \cap S_k(n)$. In virtue of (27) and an analogous equality for $G, F = G$ if and only if, for every $b \in B^n$, $\bigwedge_{K \in S_1} Q_K(b) \Leftrightarrow \bigwedge_{K \in S_2} Q_K(b)$. According to Lemma 8, the last condition is equivalent to $(\forall\, b \in B^n) \bigwedge_{K \in S_1 \dot{-} S_2} Q_K(b)$, i. e. $(\forall\, K \in S_1 \dot{-} S_2)(Q_K = B^n)$. This holds iff $S_1 \dot{-} S_2 = \emptyset$ (see (Lemma 11), i. e. iff $S_1 = S_2$.

a$_3$) According to (27), $S = S_F \cap S_k(n)$. The uniqueness of the system S can be obtained in the same way as $S_1 = S_2$ in a$_2$).

Let $\operatorname{card} B \geqq 2^{2^n}$.

b$_1$), b$_2$). If $k \geqq 2^n$, $\operatorname{card} B = 2^k$, then both assertions are special cases of a$_1$), a$_2$) (because $S_F \cap S_k(n) = S_F$ and similarly S_G). If B is infinite, it contains a 2^k element Boolean subalgebra B_0, where $k \geqq 2^n$. Then $F_0 = F \cap B_0^n$ is the realization of $\boldsymbol{F}(x)$ in B_0. Thus, the condition $F = B^n$ implies $F_0 = B_0^n$, $S_F = \emptyset$. Similarly, if $F = G$, then, for $F_0 = F \cap B_0^n$, $G_0 = G \cap B_0^n$, we have $F_0 = G_0$, $S_F = S_G$. The converse implications in b$_1$), b$_2$) are trivial.

As a consequence of Theorem 4 we have

Theorem 5. *The number of all the n-ary BP-relations in a 2^k element B is $2^{t_k(n)}$ where*

$$t_k(n) = \sum_{i=1}^{k} \binom{2^n}{i}.$$

The number of all the n-ary BP-relations in an at least 2^{2^n} element B is $2^{2^{2^n}-1}$ (i. e. the non-equivalent BP-formulas have different realizations).

Theorem 6. *Let $S(n)$, $S_k(n)$ be the systems from Theorem 4 and let S_F be the characteristic system of the BP-formula $\boldsymbol{F}(x)$ with the realization F in an at least 2^{2^n} element (or an exactly 2^k element) B. Then F is a B-relation if and only if the eqality $S_F = S(n) - 2^M$ (or $S_F \cap S_k(n) = S_k(n) - 2^M$) holds for some $M \subset D^n$. (If the condition holds, then $F = R_M$.)*

Proof. According to Theorem 3, $S(n) - 2^M$ is the characteristic system of the B-formula $\boldsymbol{R}_M(x)$ ($M \subset D^n$). Now we use b$_2$) and a$_2$) from Theorem 4.

Remark. In $B = (D, \cup, \cap, ^-)$ there holds $R_M = M$ for every $M \subset D^n$. Thus, every n-ary relation (i. e. also every n-ary BP-relation) in a 2 element B is a B-relation.

The last two theorems concern the inclusions between two BP-relations.

Theorem 7. *Let S_F, S_G be the characteristic systems of the BP-formulas $\boldsymbol{F}(x)$, $\boldsymbol{G}(x)$ with the realizations F_B, G_B in a given 2^k element B, respectively, and let $S_k(n)$ has the same meaning as in Theorem 4. Then $F_B \subset G_B$ holds if and only if $S_F \cap S_k(n) \supset S_G \cap S_k(n)$.*

Proof. The realization of the BP-formula $\boldsymbol{F}(x) \wedge \boldsymbol{G}(x)$ with the characteristic system $S_{F \wedge G} = S_F \cup S_G$ (see Theorem 3) in B is $F_B \cap G_B$. Thus, $F_B \subset G_B$ (i. e. $F_B \cap G_B = F_B$) holds iff $(S_F \cup S_G) \cap S_k(n) = S_F \cap S_k(n)$ (see a$_2$) from Theorem 4), i. e. iff $S_F \cap S_k(n) \supset S_G \cap S_k(n)$.

Theorem 8. *In the notation from Theorem 7 the following assertions are equivalent*

(a) $F_B \subset G_B$ *for every* B;

(b) $F_B \subset G_B$ *for some at least* 2^{2^n} *element* B;

(c) $S_F \supset S_G$.

Proof. (a) $\Rightarrow$ (b) is trivial. Suppose (b). Then, according to Theorem 7, $S_F \cap S(n) \supset S_G \cap S(n)$ (because $S_k(n) = S(n)$ for $k = 2^n$), i. e. (c) holds. Finally, (c) $\Rightarrow$ (a) follows directly from the definitions.

Finally we shall apply the last theory to the case $n = 2$. According to Theorems 2, 4 (we keep the notation), we shall determine all the binary BP-relations in an arbitrary B. (Instead of (0, 1) we shall write only 01 etc.) There holds
$S(2) = 2^{D^2} - \{\emptyset\} = \{\{00\}, \{01\}, \{10\}, \{11\}, \{00, 01\}, \{00, 10\}, \{00, 11\}, \{01, 10\}, \{01, 11\}, \{10, 11\}, \{00, 01, 10\}, \{00, 01, 11\}, \{00, 10, 11\}, \{01, 10, 11\}, D^2\}$.
(The first four elements form the system $S_1(2)$, the first 10 form $S_2(2)$; $S_3(2)$ is composed of the first 14 elements and, ultimately, $S_k(2) = S(2)$ for $k \geqq 4$.) Now for the fifteen $K \in S(2)$ in the given order we can determine the formulas $\boldsymbol{Q}_K(x)$ and, according to the remarks following Theorem 2, also the equivalent

shorter formulas $\boldsymbol{F}_1(x), \ldots, \boldsymbol{F}_{15}(x)$, where $x = (x_1, x_2)$ (instead of $\neg(\ldots = \ldots)$ we write only $(\ldots \neq \ldots)$ and we omit the sign of meet):
$\boldsymbol{Q}_{\{00\}}(x) \sim \boldsymbol{F}_1(x) \doteq (\bar{x}_1\bar{x}_2 \neq \boldsymbol{1}), \ldots, \boldsymbol{Q}_{\{00,01\}}(x) \sim \boldsymbol{F}_5(x) \doteq ((\bar{x}_1\bar{x}_2 \cup \bar{x}_1x_2 \neq \boldsymbol{1}) \vee \vee (\bar{x}_1\bar{x}_2 = \boldsymbol{0}) \vee (\bar{x}_1x_2 = \boldsymbol{0})), \ldots, \boldsymbol{Q}_{\{01,10,11\}}(x) \sim \boldsymbol{F}_{14}(x) \doteq ((\bar{x}_1\bar{x}_2 \neq \boldsymbol{0}) \vee \vee (\bar{x}_1x_2 = \boldsymbol{0}) \vee (x_1\bar{x}_2 = \boldsymbol{0}) \vee (x_1x_2 = \boldsymbol{0})), \boldsymbol{Q}_{D^2}(x) \sim \boldsymbol{F}_{15}(x) \doteq ((\bar{x}_1\bar{x}_2 = \boldsymbol{0}) \vee \vee (\bar{x}_1x_2 = \boldsymbol{0}) \vee (x_1\bar{x}_2 = \boldsymbol{0}) \vee (x_1x_2 = \boldsymbol{0}))$.
All the binary BP-relations in a 2, 4, 8, at least 16 element B are given by all the conjuctions which can be constructed of the first four, ten, fourteen, all the formulas $\boldsymbol{F}_1(x), \ldots, \boldsymbol{F}_{15}(x)$, respectively. The obtained relations are pairwise different.

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