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ON THE GEOMETRY OF SUBMANIFOLDS IN HOMOGENEOUS SPACES

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For a submanifold of a homogeneous space G/H we show how to calculate the groups of left movements of the space G/H preserving the element of the first and second order of the given manifold. Thus the differential geometry of the second order of any submanifold is known. For the sake of simplicity I suppose that G is a subgroup of a full linear group, this being always the case in classical differential geometry.

1. AUXILIARY RESULTS

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If $g \in G$ and $A, B \in \mathfrak{g}$, we have

$$[A, B] = AB - BA, \quad \mathrm{ad}(g)A = gAg^{-1}. \tag{1}$$

The following is known (or it is easy to verify): Let $g \in G$ and $A, B \in \mathfrak{g}$. Then

$$\mathrm{ad}(g^{-1})\,[A, \mathrm{ad}(g)B] = [\mathrm{ad}(g^{-1})A, B], \tag{2}$$

$$\mathrm{ad}(g^{-1})\,[A, [A, \mathrm{ad}(g)B]] = [\mathrm{ad}(g^{-1})A, [\mathrm{ad}(g^{-1})A, B]]. \tag{3}$$

Recall the fundamental existence theorem; for the proof see [1].

Theorem 1. *Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, (a, b) an interval of real numbers, $c \in (a, b)$. Let there be given a mapping $A : (a, b) \to \mathfrak{g}$. Then there is exactly one mapping $g : (a, b) \to G$ such that*

$$g(s)^{-1} \cdot \frac{\mathrm{d}g(s)}{\mathrm{d}s} = A(s) \quad \textit{for each} \quad s \in (a, b) \tag{4}$$

and

$$g(c) = e, \tag{5}$$

e being the identity of G.

Applying Theorem 1 to the case $(a, b) = (-\infty, \infty)$ and A constant, we get the existence of a uniquely determined mapping

$$\exp A : (-\infty, \infty) \to G \tag{6}$$

such that

$$\frac{\mathrm{d}(\exp As)}{\mathrm{d}s} = \exp As \,.\, A, \quad \exp 0 = e. \tag{7}$$

It is easy to prove that

$$\exp A(s_1 + s_2) = \exp As_1 \,.\, \exp As_2 \tag{8}$$

and

$$\frac{\mathrm{d}(\exp(-As))}{\mathrm{d}s} = -A\exp(-As). \tag{9}$$

In what follows, let G be a Lie group and H its fixed Lie subgroup; let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebras of these groups. Suppose that $[v, \mathfrak{h}] \subset \mathfrak{h}$ implies $v \in \mathfrak{h}$.

Theorem 2. *Let $A \in \mathfrak{g}$, $B \in \mathfrak{h}$. Then the following two conditions are equivalent:* 1.

$$[A, B] \in \mathfrak{h}; \tag{10}$$

2. *we have*

$$\mathrm{ad}(\exp(-Bt))A - A \in \mathfrak{h} \quad \textit{for each} \quad t \in (-\infty, \infty). \tag{11}$$

Proof. Let us write

$$v(t) = \mathrm{ad}(\exp(-Bt))A - A. \tag{12}$$

Using (7) and (9), we get

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = [v(t) + A, B] \tag{13}$$

and

$$v(0) = 0, \quad \frac{\mathrm{d}v(0)}{\mathrm{d}t} = [A, B]. \tag{14}$$

If $v(t) \in \mathfrak{h}$ for each t, we have $\mathrm{d}v(0)/\mathrm{d}t \in \mathfrak{h}$, and (10) is valid. Let us now suppose (10). It is easy to see that

$$\frac{\mathrm{d}^n v(0)}{\mathrm{d}t^n} = \left[\frac{\mathrm{d}^{n-1}v(0)}{\mathrm{d}t^{n-1}}, B\right]; \quad n = 2, 3, \ldots \tag{15}$$

Our condition yields $\mathrm{d}v(0)/\mathrm{d}t \in \mathfrak{h}$, and — according to (15) —

$$(16) \qquad \frac{\mathrm{d}^n v(0)}{\mathrm{d}t^n} \in \mathfrak{h}; \quad n = 0, 1, 2, \ldots$$

The curve $v(t)$ being analytic, we get $v(t) \in \mathfrak{h}$ from (16). Q.E.D.

Theorem 3. *Let us write*

$$(17) \qquad K(A) = \{h \in H \mid (18)\}$$

where $A \in \mathfrak{g}$ *and*

$$(18) \qquad [A - \mathrm{ad}(h^{-1})A, v] \in \mathfrak{h} \quad \textit{for each} \quad v \in \mathfrak{h}.$$

Then $K(A)$ *is a Lie group.*

Proof. It is obviously sufficient to show that $K(A)$ is a group. Let $h_1, h_2 \in K(A)$, i.e.,

$$[A - \mathrm{ad}(h_1^{-1})A, v_1] \in \mathfrak{h} \quad \text{for each} \quad v_1 \in \mathfrak{h},$$

$$[A - \mathrm{ad}(h_2^{-1})A, v_2] \in \mathfrak{h} \quad \text{for each} \quad v_2 \in \mathfrak{h}.$$

Let $v \in \mathfrak{h}$ be an arbitrary element. Recall that $\mathrm{ad}(h)w \in \mathfrak{h}$ for each $h \in H$, $w \in \mathfrak{h}$. Now let us choose $v_1 = \mathrm{ad}(h_1^{-1}h_2)v \in \mathfrak{h}$; we have

$$[A - \mathrm{ad}(h_1^{-1})A, \mathrm{ad}(h_1^{-1}h_2)v] \in \mathfrak{h}$$

and — see (2) —

$$w_1 \equiv \mathrm{ad}(h_2^{-1}h_1)\,[A - \mathrm{ad}(h_1^{-1})A, \mathrm{ad}(h_1^{-1}h_2)v] =$$
$$= [\mathrm{ad}(h_2^{-1}h_1)A - \mathrm{ad}(h_2^{-1})A, v] \in \mathfrak{h}.$$

Further, choosing $v_2 = v$, we get

$$w_2 \equiv [A - \mathrm{ad}(h_2^{-1})A, v] \in \mathfrak{h}$$

and $w_2 - w_1 \in \mathfrak{h}$, i.e.,

$$[A - \mathrm{ad}(h_2^{-1}h_1)A, v] \in \mathfrak{h}$$

and $h_1^{-1}h_2 \in K(A)$. Q.E.D.

Theorem 4. *Let us write*

$$(19) \qquad \mathfrak{k}(A) = \{v \in \mathfrak{h} \mid (20)\}$$

where $A \in \mathfrak{g}$ *and*

$$(20) \qquad [A, v] \in \mathfrak{h}.$$

Then $\mathfrak{k}(A)$ *is a Lie algebra.*

Proof. Let $v_1, v_2 \in \mathfrak{k}(A)$. We have $[A, v_1] \in \mathfrak{h}$, $[A, v_2] \in \mathfrak{h}$ and the Jacobi identity yields

$$[A, [v_1, v_2]] = -[v_1, [v_2, A]] - [v_2, [A, v_1]] \in \mathfrak{h},$$

the right hand members being in $\mathfrak{h}$. Q.E.D.

Theorem 5. *$\mathfrak{k}(A)$ is the Lie algebra of the Lie group $K(A)$.*

Proof. Let us restrict ourselves to a neighbourhood of the identity in the group H such that each element $\gamma \in H$ may be written as

$$\gamma = \exp B, \quad B \in \mathfrak{h}. \tag{21}$$

Let us consider, for a given element (21), the one-parametric subgroup

$$\gamma(t) = \exp Bt, \quad t \in (-\infty, \infty). \tag{22}$$

$K(A)$ being a subgroup of H, we have $\gamma \in K(A)$ if and only if $\gamma(t) \in K(A)$ for each $t \in (-\infty, \infty)$. Further, it is obvious that the condition (18) is, according to the assumption $[v, \mathfrak{h}] \subset \mathfrak{h} \Rightarrow v \in \mathfrak{h}$, equivalent to the condition

$$A - \mathrm{ad}(h^{-1})A \in \mathfrak{h}. \tag{23}$$

In our case, this condition is

$$A - \mathrm{ad}(\gamma(t)^{-1})A = A - \mathrm{ad}(\exp(-Bt))A \in \mathfrak{h}, \tag{24}$$

and we get from Theorem 3 that (24) is equivalent to (10). Q.E.D.

Theorem 6. *Let us write*

$$K(A, B) = \{h \in K(A) \mid (26)\} \tag{25}$$

where $A, B \in \mathfrak{g}$ and

$$\begin{aligned} &[B - \mathrm{ad}(h^{-1})B, v] + [A - \mathrm{ad}(h^{-1})A, [A, v]] - \\ &- [\mathrm{ad}(h^{-1})A, [A - \mathrm{ad}(h^{-1})A, v]] \in \mathfrak{h} \quad \textit{for each } v \in \mathfrak{h}. \end{aligned} \tag{26}$$

Then $K(A, B)$ is a Lie group.

Proof. Let $h_1, h_2 \in K(A, B)$, i.e.,

$$\begin{aligned} &[B - \mathrm{ad}(h_1^{-1})B, v_1] + [A - \mathrm{ad}(h_1^{-1})A, [A, v_1]] - \\ &- [\mathrm{ad}(h_1^{-1})A, [A - \mathrm{ad}(h_1^{-1})A, v_1]] \in \mathfrak{h} \quad \text{for each } v_1 \in \mathfrak{h}, \\ &A - \mathrm{ad}(h_1^{-1})A \in \mathfrak{h}; \end{aligned} \tag{27}$$

$$\begin{aligned} &[B - \mathrm{ad}(h_2^{-1})B, v_2] + A - \mathrm{ad}(h_2^{-1})A, [A, v_2]] - \\ &- [\mathrm{ad}(h_2^{-1})A, [A - \mathrm{ad}(h_2^{-1})A, v_2]] \in \mathfrak{h} \quad \text{for each } v_2 \in \mathfrak{h}, \\ &A - \mathrm{ad}(h_2^{-1})A \in \mathfrak{h}. \end{aligned} \tag{28}$$

Let us choose an arbitrary vector $v \in \mathfrak{h}$. Putting $v_1 = \mathrm{ad}(h_1^{-1}h_2)v \in \mathfrak{h}$, we get

$$w_1 \equiv [B - \mathrm{ad}(h_1^{-1})B, \mathrm{ad}(h_1^{-1}h_2)v] + [A - \mathrm{ad}(h_1^{-1})A, [A, \mathrm{ad}(h_1^{-1}h_2)v]] -$$
$$- [\mathrm{ad}(h_1^{-1})A, [A - \mathrm{ad}(h_1^{-1})A, \mathrm{ad}(h_1^{-1}h_2)v]] \in \mathfrak{h},$$

and we get $w_2 \equiv \mathrm{ad}(h_2^{-1}h_1)w_1 \in \mathfrak{h}$ where

$$w_2 = [\mathrm{ad}(h_2^{-1}h_1)B - \mathrm{ad}(h_2^{-1})B, v] +$$
$$+ [\mathrm{ad}(h_2^{-1}h_1)A - \mathrm{ad}(h_2^{-1})A, [\mathrm{ad}(h_2^{-1}h_1)A, v] -$$
$$- [\mathrm{ad}(h_2^{-1})A, [\mathrm{ad}(h_2^{-1}h_1)A - \mathrm{ad}(h_2^{-1})A, v]] \in \mathfrak{h}.$$

Further, write $v_2 = v$; from (28), we get

$$w_3 \equiv [B - \mathrm{ad}(h_2^{-1})B, v] + [A - \mathrm{ad}(h_2^{-1})A, [A, v]] -$$
$$- [\mathrm{ad}(h_2^{-1})A, [A - \mathrm{ad}(h_2^{-1})A, v]] \in \mathfrak{h}.$$

(27_2) yields

$$\mathrm{ad}(h_2^{-1}h_1)\,(A - \mathrm{ad}(h_1^{-1})A) \in \mathfrak{h};$$

$K(A)$ being a group we have $h_2^{-1}h_1 \in K(A)$ and

$$\mathrm{ad}(h_2^{-1}h_1)A - A \in \mathfrak{h}.$$

Thus we get

$$w_4 \equiv [\mathrm{ad}(h_2^{-1})A - \mathrm{ad}(h_2^{-1}h_1)A, [\mathrm{ad}(h_2^{-1}h_1)A - A, v]] \in \mathfrak{h}.$$

We have $w_5 \equiv w_3 - w_2 - 2w_4 \in \mathfrak{h}$; a simple calculation yields

$$w_5 = [B - \mathrm{ad}(h_2^{-1}h_1)B, v] + [A - \mathrm{ad}(h_2^{-1}h_1)A, [A, v]] -$$
$$- [\mathrm{ad}(h_2^{-1}h_1)A, [A - \mathrm{ad}(h_2^{-1}h_1)A, v]],$$

i.e., $h_1^{-1}h_2 \in K(A, B)$. Q.E.D.

Theorem 7. *Let us write*

$$\mathfrak{k}(A, B) = \{v \in \mathfrak{k}(A) \mid (30)\} \tag{29}$$

where $A, B \in \mathfrak{g}$ *and*

$$[B, v] - [A, [A, v]] \in \mathfrak{h}. \tag{30}$$

Then $\mathfrak{k}(A, B)$ *is a Lie algebra.*

Proof. Let $v_1, v_2 \in \mathfrak{k}(A, B)$. Evidently, it is sufficient to show that $[v_1, v_2] \in \in \mathfrak{k}(A, B)$. Applying the Jacobi identity, we get

$$[A, [A, [v_1, v_2]]] = - [A, [v_1, [v_2, A]]] - [A, [v_2, [A, v_1]]] =$$
$$= [v_1, [[v_2, A], A]] + [[v_2, A], [A, v_1]] + [v_2, [A, v_1], A] + [[A, v_1], [A, v_2]] =$$
$$= [v_1, [A, [A, v_2]]] - [v_2, [A, [A, v_1]]] + 2\,[A, v_1], [A, v_2]].$$

Further

$$[B, [v_1, v_2] = - v_1, [v_2, B]] - [v_2, [B, v_1]]$$

and

$$w = [B, [v_1, v_2] - [A, [A, [v_1, v_2]]] = [v_1, [B, v_2] - [A, [A, [A, v_2]]] -$$

$$- [v_2, [B, v_1] - [A, [A, v_1]]] - 2 [A, v_1], [A, v_2]].$$

Now it is easy to see that $w \in \mathfrak{h}$ and $[v_1, v_2] \in \mathfrak{k}(A, B)$. Q.E.D.

Theorem 8. *$\mathfrak{k}(A, B)$ is the Lie algebra of the Lie group $K(A, B)$.*

Proof. Let us restrict ourselves to a neighbourhood of the identity in the group $K(A)$ such that each element of this neighbourhood may be written as $\gamma = \exp x$ with $x \in \mathfrak{k}(A)$. Let $\gamma \in K(A)$. $K(A)$ being a group, we have

(31) $$\gamma(t) = \exp xt \in K(A) \quad \text{for each } t \in (-\infty, \infty)$$

and

(32) $$[x, A] \in \mathfrak{h}.$$

Let $v \in \mathfrak{h}$ be an arbitrary vector. Define

(33) $$w(t) = [B - \mathrm{ad}(\exp(-xt))B, v] + [A - \mathrm{ad}(\exp(-xt))A, [A, v]] -$$
$$- [\mathrm{ad}(\exp(-xt))A, [A - \mathrm{ad}(\exp(-xt))A, v]].$$

From Theorem 2 and (32), we get the existence of vectors

(34) $$A - \mathrm{ad}(\exp(-xt))A = \nu(t) \in \mathfrak{h},$$

and we may write

(35) $$w(t) = [B - \mathrm{ad}(\exp(-xt))\, B, v] + [\nu(t), [A, v]] - [A - \nu(t), [\nu(t), v]].$$

By a direct calculation, we get

(36) $$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \left[[x,\ \mathrm{ad}(\exp(-xt))B] - \left[A, \frac{\mathrm{d}\nu(t)}{\mathrm{d}t}\right],\ v\right] +$$
$$+ \left[\frac{\mathrm{d}\nu(t)}{\mathrm{d}t,}, [\nu(t), v]\right] + \left[\nu(t), \left[\frac{\mathrm{d}\nu(t)}{\mathrm{d}t}, v\right]\right].$$

From (14), $\mathrm{d}\nu(0)/\mathrm{d}t = [x, A]$ and

(37) $$\frac{\mathrm{d}w(0)}{\mathrm{d}t} = [-[B, x] + [A, [A, x]], v].$$

If $\gamma = \exp x \in K(A, B)$, we have $\mathrm{d}w(0)/\mathrm{d}t \in \mathfrak{h}$ for each vector v, i.e., $x \in \mathfrak{k}(A, B)$. A very complicated calculation leads to a quite clear result according to which the Lie algebra of the group $K(A, B)$ is not less than $\mathfrak{k}(A, B)$. Let us describe the first step of this. Our aim is to show that

$$(38) \qquad [A, x] \in \mathfrak{h}, \; [B, x] - [A, [A, x]] \in \mathfrak{h}$$

implies

$$(39) \qquad \frac{\mathrm{d}^n w(0)}{\mathrm{d}t^n} \in \mathfrak{h} \quad \text{for } n = 0, 1, 2, \ldots$$

Since $\nu(t) \in \mathfrak{h}$, $\mathrm{d}\nu(t)/\mathrm{d}t \in \mathfrak{h}$, we are not interested in the terms

$$\left[\frac{\mathrm{d}\nu(t)}{\mathrm{d}t}, [\nu(t), v]\right] \in \mathfrak{h}, \left[x(t), \left[\frac{\mathrm{d}\nu(t)}{\mathrm{d}t}, v\right]\right] \in \mathfrak{h}.$$

According to (15), we have

$$\frac{\mathrm{d}^2\nu(0)}{\mathrm{d}t^2} = \left[\frac{\mathrm{d}\nu(0)}{\mathrm{d}t}, x\right] = [[x, A], x].$$

Derivating (36), we get mod $\mathfrak{h}$

$$(40) \qquad \frac{\mathrm{d}^2 w(t)}{\mathrm{d}t^2} \equiv \left[[x, [\mathrm{ad}(\exp(-xt))B, x]] - \left[A, \frac{\mathrm{d}^2\nu(t)}{\mathrm{d}t^2}\right], v\right],$$

$$\frac{\mathrm{d}^2 w(0)}{\mathrm{d}t} \equiv [[x, [B, x]] - [A, [[x, A], x]], v].$$

On the other hand, we have

$$-[A, [[x, A], x]] = [[x, A], [x, A]] + [x, [A, [x, A]]],$$

hence

$$\frac{\mathrm{d}^2 w(0)}{\mathrm{d}t^2} \equiv [x, [B, x] - [A, [A, x]]], v]$$

and $\mathrm{d}^2 w(0)\mathrm{d}t^2 \in \mathfrak{h}$ according to (38_2). Derivating successively (40) and applying the just described procedure we would get (39). Q.E.D.

2. CURVES IN HOMOGENEOUS SPACES

Let there be given a Lie group G and its closed subgroup H subject to the above conditions. The set of the left classes gH may be endowed by a structure of a differentiable manifold, this manifold being the homogeneous space G/H. Denote by $\pi : G \to G/H$ the natural correspondence. The group G operates on G/H to the left: $(\gamma, gH) \in (G, G/H) \to (\gamma g)H \in G/H$. L$x$

Let there be given a curve in our homogeneous space, i.e., a mapping

$$\varphi : (-1, 1) \to G/H. \tag{41}$$

This mapping may be determined by its lift, i.e. a mapping

$$f : (-1,1) \to G \tag{42}$$

such that the diagram

$$\begin{tikzcd} & G \arrow[dd, "\pi"] \\ (-1, 1) \arrow[ur, "f"] \arrow[dr, "\varphi"'] & \\ & G/H \end{tikzcd} \tag{43}$$

is commutative. f being a lift of φ, we get each other lift f^* as follows: choose a mapping $h : (-1, 1) \to H$ and set

$$f^*(t) = f(t)h(t) \quad \text{for} \quad t \in (-1,1). \tag{44}$$

To each lift f, let us associate the mapping $A : (-1, 1) \to \mathfrak{g}$ defined by

$$A(t) = f(t)^{-1} \frac{\mathrm{d}f(t)}{\mathrm{d}t} \quad \text{for } t \in (-1, 1). \tag{45}$$

Let A^* be associated to the lift f^* (44). Then

$$f^*(t)A^*(t) = \frac{\mathrm{d}f^*(t)}{\mathrm{d}t}$$

and — according to (44) —

$$f(t)h(t)A^*(t) = f(t)A(t)h(t) + f(t)\frac{\mathrm{d}h(t)}{\mathrm{d}t}\,;$$

i.e.,

$$A^*(t) = \mathrm{ad}(h(t)^{-1})A(t) + h(t)^{-1}\frac{\mathrm{d}h(t)}{\mathrm{d}t}\,. \tag{46}$$

Thus we get

Theorem 9. *If the lifts f, f^* are related by* (44), *we have* (46) *for the associated mappings A, A^*.*

Let us choose a fixed lift f to the given curve φ, and let us consider the point $\varphi(0)$ of φ. First of all, let us construct the mapping $g : (-1, 1) \to G$ defined by the relation

$$g(t) = f(0)^{-1}f(t), \tag{47}$$

further, consider the one-parametric system of subalgebras

$$\mathfrak{h}(t) = \mathrm{ad}(g(t))\mathfrak{h}. \tag{48}$$

Let $\gamma \in G$ be an arbitrary element. Consider the one-parametric system of subalgebras

$$\mathfrak{h}_\gamma(t) = \mathrm{ad}(\gamma g(t))\mathfrak{h}. \tag{49}$$

Our task is to find all elements γ such that the systems $\mathfrak{h}(t)$, $\mathfrak{h}_\gamma(t)$ have, for $t = 0$, the contact of order 0, 1 or 2, resp. Recall the definition of the contact: Let W be a vector space, and $U(t)$, $V(t)$ two one-parametric systems of subspaces; $\dim U(t) = \dim V(t) = \text{const}$. The systems $U(t)$, $V(t)$ have, for $t = t_0$, the contact of order (at least) k if there are bases $u_\alpha(t)$, $v_\alpha(t)$ of the spaces $U(t)$, $V(t)$ resp. such that

$$\frac{\mathrm{d}^l u_\alpha(t_0)}{\mathrm{d}t^l} = \frac{\mathrm{d}^l v_\alpha(t_0)}{\mathrm{d}t^l} \tag{50}$$

$$\text{for } l = 0, 1, \ldots, k \quad \text{and for all } \alpha\text{'s}.$$

The contact of order 0 of the systems $\mathfrak{h}(t)$ and $\mathfrak{h}_\gamma(t)$ for $t = 0$ means $\mathfrak{h}(0) = \mathfrak{h}_\gamma(0)$, i.e., $\mathfrak{h} = \mathrm{ad}(\gamma)\mathfrak{h}$, and it is equivalent to $\gamma \in H$. Therefore, let us consider the contact of order 1 and 2. In $\mathfrak{h}$, let us choose a fixed basis

$$\mathscr{B} = \{u_1, \ldots, u_n\}; \quad n = \dim \mathfrak{h} \quad . \tag{51}$$

In what follows, use the obvious notation

$$\mathrm{ad}(g)\mathscr{B} = \{\mathrm{ad}(g)u_1, \ldots\}, \quad [v, \mathscr{B}] = \{[v, u_1], \ldots\},$$

etc. In $\mathfrak{h}(t)$,

$$\mathscr{B}(t) = \mathrm{ad}(g(t))\mathscr{B} \tag{52}$$

is a basis,

$$\mathscr{B}_\gamma(t) = \mathrm{ad}(\gamma g(t))\mathscr{B} \tag{53}$$

being a basis of $\mathfrak{h}_\gamma(t)$. The most general bases in the spaces $\mathfrak{h}(t)$ are given by the relations

$$\mathscr{B}^*(t) = \mathscr{B}(t)S(t) \quad \text{where} \quad S : (-1, 1) \to GL(n). \tag{54}$$

The condition of the contact of order 1 or 2 at $t = 0$ is equivalent to the existence of a mapping S such that

$$\mathscr{B}^*(0) = \mathscr{B}_\gamma(0), \quad \frac{d\mathscr{B}^*(0)}{dt} = \frac{d\mathscr{B}_\gamma(0)}{dt} \tag{55}$$

or (55) and

$$\frac{d^2\mathscr{B}^*(0)}{dt^2} = \frac{d^2\mathscr{B}_\gamma(0)}{dt^2}, \tag{56}$$

resp. From (45) and (47), we get

$$\frac{df(0)}{dt} = f(0)A(0)$$

and

$$g(0) = e, \quad \frac{dg(0)}{dt} = A(0), \tag{57}$$

e being the identity of G. Further,

$$\frac{d^2f(t)}{dt^2} = \frac{df(t)}{dt}A(t) + f(t)\frac{dA(t)}{dt}$$

and

$$\frac{d^2g(0)}{dt^2} = A(0)A(0) + \frac{dA(0)}{dt}. \tag{58}$$

From (52), we get

$$\mathscr{B}(t)g(t) = g(t)\mathscr{B},$$

$$\frac{d\mathscr{B}(t)}{dt}g(t) + \mathscr{B}(t)\frac{dg(t)}{dt} = \frac{dg(t)}{dt}\mathscr{B} \tag{59}$$

and

$$\frac{d\mathscr{B}(0)}{dt} = [A(0), \mathscr{B}]. \tag{60}$$

A further derivation of (59) yields

$$\frac{d^2\mathscr{B}(t)}{dt^2}g(t) + 2\frac{d\mathscr{B}(t)}{dt}\frac{dg(t)}{dt} + \mathscr{B}(t)\frac{d^2g(t)}{dt^2} = \frac{d^2g(t)}{dt^2}\mathscr{B},$$

$$\frac{d^2\mathscr{B}(0)}{dt^2} + 2[A(0), \mathscr{B}]A(0) + \mathscr{B}A(0)A(0) + \mathscr{B}\frac{dA(0)}{dt} = A(0)A(0)\mathscr{B} + $$

$$+ \frac{dA(0)}{dt}\mathscr{B},$$

and, finally,

$$\frac{d^2\mathscr{B}(0)}{dt^2} = \left[\frac{dA(0)}{dt}, \mathscr{B}\right] + [A(0), [A(0), \mathscr{B}]]. \tag{61}$$

Now, it is easy to see that

$$\mathscr{B}^*(0) = \mathscr{B}S(0), \tag{62}$$

$$\frac{d\mathscr{B}^*(0)}{dt} = \mathscr{B}\frac{dS(0)}{dt} + [A(0), \mathscr{B}]S(0), \tag{63}$$

$$\frac{d^2\mathscr{B}^*(0)}{dt^2} = \mathscr{B}\frac{d^2S(0)}{dt^2} + 2\,[A(0), \mathscr{B}]\,\frac{dS(0)}{dt} + \tag{64}$$

$$+ \left[\frac{dA(0)}{dt}, \mathscr{B}\right] S(0) + [A(0), [A(0), \mathscr{B}]]S(0).$$

From (53), we get

$$\mathscr{B}_\gamma(0) = \mathrm{ad}(\gamma)\mathscr{B}, \tag{65}$$

$$\frac{d\mathscr{B}_\gamma(0)}{dt} = \mathrm{ad}(\gamma)[A(0), \mathscr{B}], \tag{66}$$

$$\frac{d^2\mathscr{B}_\gamma(0)}{dt^2} = \mathrm{ad}(\gamma)\left[\frac{dA(0)}{dt}, \mathscr{B}\right] + \mathrm{ad}(\gamma)[A(0), [A(0), \mathscr{B}]]. \tag{67}$$

From (55_1), we get

$$\mathscr{B}S(0) = \mathrm{ad}(\gamma)\mathscr{B}. \tag{68}$$

The relation (55_2) yields

$$\mathscr{B}\frac{dS(0)}{dt} + [A(0), \mathscr{B}]S(0) = \mathrm{ad}(\gamma)[A(0), \mathscr{B}]$$

and

$$\mathscr{B}\frac{dS(0)}{dt} = \mathrm{ad}(\gamma)[A(0), \mathscr{B}] - [A(0), \mathrm{ad}(\gamma)\mathscr{B})]. \tag{69}$$

The systems $\mathfrak{h}(t)$ and $\mathfrak{h}_\gamma(t)$ have the contact of order 1 at $t = 0$ if and only if $\gamma \in H$ and there is a matrix $dS(0)/dt$ such that (69) is valid. But this condition is equivalent to

$$\operatorname{ad}(\gamma)[A(0), v] - [A(0), \operatorname{ad}(\gamma)v] \in \mathfrak{h} \tag{70}$$

for each $v \in \mathfrak{h}$. The final result is given by

Theorem 10. *The systems $\mathfrak{h}(t)$ and $\mathfrak{h}_\gamma(t)$ have the contact of order 1 at $t = 0$ if and only if $\gamma \in K(A(0))$.*

Let us now study the contact of order 2. Using (56), (68) and (69), we get

$$\mathscr{B}\frac{\mathrm{d}^2S(0)}{\mathrm{d}t^2} + 2[A(0), \operatorname{ad}(\gamma)\,[A(0),\mathscr{B}]] - 2[A(0), [A(0), \operatorname{ad}(\gamma)\mathscr{B}]] +$$

$$+ \left[\frac{\mathrm{d}A(0)}{\mathrm{d}t}, \operatorname{ad}(\gamma)\mathscr{B}\right] + [A(0), [A(0), \operatorname{ad}(\gamma)\mathscr{B}]] =$$

$$= \operatorname{ad}(\gamma)\left[\frac{[\mathrm{d}A(0)}{\mathrm{d}t}, \mathscr{B}\right] + \operatorname{ad}(\gamma)[A(0), [A(0), \mathscr{B}]]$$

and

$$\operatorname{ad}(\gamma^{-1})\mathscr{B}\frac{\mathrm{d}^2S(0)}{\mathrm{d}t^2} = \left[\frac{\mathrm{d}A(0)}{\mathrm{d}t} - \operatorname{ad}(\gamma^{-1})\frac{\mathrm{d}A(0)}{\mathrm{d}t}, \mathscr{B}\right] + \tag{71}$$

$$+ [A(0), [A(0), \mathscr{B}]] - [\operatorname{ad}(\gamma^{-1})A(0), [\operatorname{ad}(\gamma^{-1})A(0), \mathscr{B}]] -$$

$$- 2[\operatorname{ad}(\gamma^{-1})A(0), [A(0), \mathscr{B}]].$$

It is now easy to prove

Theorem 11. *The systems $\mathfrak{h}(t)$ and $\mathfrak{h}_\gamma(t)$ have the contact of order 2 at $t = 0$ if and only if $\gamma \in K(A(0), \mathrm{d}A(0)/\mathrm{d}t)$.*

Let us *summarize*: We have a curve $\varphi : (-1, 1) \to G/H$, and we have chosen its lift $f : (-1, 1) \to G$. We construct the mapping $A : (-1, 1) \to \mathfrak{g}$. To the point $0 \in (-1, 1)$, we associate the Lie subalgebras

$$\mathfrak{k}(A(0)), \quad \mathfrak{k}\left(A(0), \frac{\mathrm{d}A(0)}{\mathrm{d}t}\right)$$

of $\mathfrak{h}$; we have described above their geometrical signification. Now, we are interested in the manner on which they depend on the lift f. First of all, let us prove that they are independent on the parametrization of φ.

Theorem 12. *Let there be given the curves $\varphi : (-1, 1) \to G/H$ and $\varphi_1 : (-1, 1) \to G/H$ such that there is a mapping $T : (-1, 1) \to (-1, 1)$ such that*

(72) $$T(0)=0,\ \varphi_1(T(t))=\varphi(t)\ \textit{for}\ t\in(-1,1);\quad \frac{\mathrm{d}T(0)}{\mathrm{d}t}\neq 0.$$

Let there be chosen a lift $f_1:(-1,1)\to G$ *of* φ_1, *and let us determine the lift* $\boldsymbol{f}$ *of* φ *by*

(73) $$f(t)=f_1(T(t)).$$

Then

(74) $$\mathfrak{k}(A(0))=\mathfrak{k}(A_1(0)),\ \mathfrak{k}\left(A(0),\frac{\mathrm{d}A(0)}{\mathrm{d}t}\right)=\mathfrak{k}\left(A_1(0),\frac{\mathrm{d}A_1(0)}{\mathrm{d}t}\right).$$

Proof. From (73) and (72_1), we get

(75) $$f(0)=f_1(0).$$

Further,

(76) $$\frac{\mathrm{d}f(t)}{\mathrm{d}t}=\frac{\mathrm{d}f_1(T(t))}{\mathrm{d}T}\frac{\mathrm{d}T(t)}{\mathrm{d}t},\qquad \frac{\mathrm{d}f(0)}{\mathrm{d}t}=\frac{\mathrm{d}f_1(0)}{\mathrm{d}T}\frac{\mathrm{d}T(0)}{\mathrm{d}t}.$$

From

(77) $$A(0)=f(0)^{-1}\frac{\mathrm{d}f(0)}{\mathrm{d}t},\quad A_1(0)=f_1(0)^{-1}\frac{\mathrm{d}f_1(0)}{\mathrm{d}t}$$

and (76_2), (75), we get

(78) $$A(0)=A_1(0)\frac{\mathrm{d}T(0)}{\mathrm{d}t},$$

this proving (74_1). Further,

(79) $$\frac{\mathrm{d}^2f(0)}{\mathrm{d}t^2}=\frac{\mathrm{d}^2f_1(0)}{\mathrm{d}T^2}\left(\frac{\mathrm{d}T(0)}{\mathrm{d}t}\right)^2+\frac{\mathrm{d}f_1(0)}{\mathrm{d}T}\frac{\mathrm{d}^2T(0)}{\mathrm{d}t^2},$$

(80) $$\frac{\mathrm{d}A(0)}{\mathrm{d}t}=f(0)^{-1}\frac{\mathrm{d}^2f(0)}{\mathrm{d}t^2}-A(0)A(0),$$

$$\frac{\mathrm{d}A_1(0)}{\mathrm{d}t}=f_1(0)^{-1}\frac{\mathrm{d}^2f_1(0)}{\mathrm{d}t^2}-A_1(0)A_1(0).$$

From (79), we have

$$\frac{\mathrm{d}A(0)}{\mathrm{d}t}=f_1(0)^{-1}\frac{\mathrm{d}^2f_1(0)}{\mathrm{d}T^2}\left(\frac{\mathrm{d}T(0)}{\mathrm{d}t}\right)^2+f_1(0)^{-1}\frac{\mathrm{d}f_1(0)}{\mathrm{d}T}\frac{\mathrm{d}^2T(0)}{\mathrm{d}t^2}-A_1(0)\,A_1(0)\left(\frac{\mathrm{d}T(0)}{\mathrm{d}t}\right)^2,$$

i.e.,

(81) $$\frac{\mathrm{d}A(0)}{\mathrm{d}t} = \frac{\mathrm{d}A_1(0)}{\mathrm{d}T}\left(\frac{\mathrm{d}T(0)}{\mathrm{d}t}\right)^2 + A_1(0)\,\frac{\mathrm{d}^2T(0)}{\mathrm{d}t^2}.$$

Now,

(82) $$\left[\frac{\mathrm{d}A(0)}{\mathrm{d}t}, v\right] - [A(0), [A(0), v]] =$$

$$= \left(\left[\frac{\mathrm{d}A_1(0)}{\mathrm{d}T}, v\right] - [A_1(0), [A_1(0), v]]\right)\left(\frac{\mathrm{d}T(0)}{\mathrm{d}t}\right)^2 + [A_1(0), v]\frac{\mathrm{d}^2T(0)}{\mathrm{d}t^2},$$

and we have (74_2). Q.E.D.

Let us now study the changes caused by the change of the lift. First of all, we have

Theorem 13. *Let there be given a curve $\varphi : (-1, 1) \to G/H$. Let us choose two lifts $f, f_1 : (-1, 1) \to G$ such that*

(83) $$f(0) = f_1(0),$$

and let us construct the associated mappings $A, A_1 : (-1, 1) \to \mathfrak{g}$. Then

(84) $$\mathfrak{k}(A(0)) = \mathfrak{k}(A_1(0)),\ \mathfrak{k}\left(A(0), \frac{\mathrm{d}A(0)}{\mathrm{d}t}\right) = \mathfrak{k}\left(A_1(0), \frac{\mathrm{d}A_1(0)}{\mathrm{d}t}\right).$$

Proof. According to (44), we have

(85) $$f_1(t) = f(t)h(t),\ h(0) = e; \quad h(t) \in H.$$

Define the mapping $R : (-1, 1) \to \mathfrak{h}$ by the equation

(86) $$R(t) = h(t)^{-1}\frac{\mathrm{d}h(t)}{\mathrm{d}t}.$$

From (46), we get

(87) $$h(t)A_1(t) = A(t)h(t) + \frac{\mathrm{d}h(t)}{\mathrm{d}t}$$

and

(88) $$A_1(0) = A(0) + R(0).$$

Derivating (87), we have

(89) $$\frac{\mathrm{d}A_1(0)}{\mathrm{d}t} = \frac{\mathrm{d}A(0)}{\mathrm{d}t} + \frac{\mathrm{d}R(0)}{\mathrm{d}t} + [A(0), R(0)].$$

The equation (84_1) is now obvious. Further,

$$(90)\quad \left[\frac{\mathrm{d}A_1(0)}{\mathrm{d}t}, v\right] - [A_1(0), [A_1(0), v]] = \left[\frac{\mathrm{d}A(0)}{\mathrm{d}t}, v\right] - [A(0), [A(0), v]] + S$$

where

$$(91)\qquad S = \left[\frac{\mathrm{d}R(0)}{\mathrm{d}t}, v\right] - [R(0), [R(0), v]] - 2[R(0), [A(0), v]];$$

$S \in \mathfrak{h}$ if $v \in \mathfrak{k}(A(0))$, i.e., $[A(0), v] \in \mathfrak{h}$. Q.E.D.

Theorem 14. *Let there be given a curve $\varphi : (-1, 1) \to G/H$. Let us choose its two lifts $f, f_1 : (-1, 1) \to G$ such that*

$$(92)\qquad f_1(t) = f(t)h,$$

$h \in H$ being a fixed element. Let $A, A_1 : (-1, 1) \to \mathfrak{g}$ be the associated mappings. Then

$$(93)\quad \mathfrak{k}(A_1(0)) = \mathrm{ad}(h^{-1})\mathfrak{k}(A(0)),\ \mathfrak{k}\left(A_1(0), \frac{\mathrm{d}A_1(0)}{\mathrm{d}t}\right) = \mathrm{ad}(h^{-1})\mathfrak{k}\left(A(0), \frac{\mathrm{d}A(0)}{\mathrm{d}t}\right).$$

Proof. According to (46), we have

$$(94)\qquad A_1(t) = \mathrm{ad}(h^{-1})A(t)$$

and

$$(95)\qquad A_1(0) = \mathrm{ad}(h^{-1})A(0), \frac{\mathrm{d}A_1(0)}{\mathrm{d}t} = \mathrm{ad}(h^{-1}) \frac{\mathrm{d}A(0)}{\mathrm{d}t}.$$

The Lie algebra $\mathfrak{k}(A_1(0))$ consists of all vectors $v \in \mathfrak{h}$ such that

$$(96)\qquad [A_1(0), v] = [\mathrm{ad}(h^{-1})A(0), v] \in \mathfrak{h}.$$

The relation (96) is equivalent to

$$(97)\qquad \mathrm{ad}(h)[\mathrm{ad}(h^{-1})A(0), v] = [A(0), \mathrm{ad}(h)v] \in \mathfrak{h}.$$

Thus the vector $v \in \mathfrak{h}$ is situated in $\mathfrak{k}(A_1(0))$ if and only if the vector $\mathrm{ad}(h)v$ is situated in $\mathfrak{k}(A(0))$, and (93_1) is proved. (93_2) follows analoguously from

$$\left[\frac{\mathrm{d}A_1(0)}{\mathrm{d}t}, v\right] - [A_1(0), [A_1(0), v]] =$$

$$= \mathrm{ad}(h^{-1})\left(\left[\frac{\mathrm{d}A(0)}{\mathrm{d}t}, \mathrm{ad}(h)v\right] - [A(0), [A(0), \mathrm{ad}(h)v]]\right].$$

Q.E.D.

Summary. Let us summarize all we know up to this moment.

Let there be given a homogeneous space G/H and a fixed curve $\varphi : (a, b) \to \to G/H$. Let $\Phi \subset G$ be the manifold of all points which are above the points of φ:

$$\Phi = \pi^{-1}(\varphi(a, b)). \tag{98}$$

The space G/H being regarded as the principal fibre bundle $G(G/H, H)$, Φ is the principal fibre bundle constructed from $G(G/H, H)$ by the restriction of the base space G/H to $\varphi(a, b)$.

To each point $q \in \Phi$, we associate two subsets $\mathfrak{n}_q \subset \mathfrak{m}_q \subset \mathfrak{h}$ as follows. Let $\pi(q) = p$, and let us choose an arbitrary mapping $f : (-1, 1) \to G$ such that

$$f(0) = q, \ \pi(f(-1, 1)) \subset \varphi(a, b). \tag{99}$$

By means of f, we construct the mapping $A : (-1, 1) \to \mathfrak{g}$ defined by

$$A(t) = f(t)^{-1} \frac{\mathrm{d}f(t)}{\mathrm{d}t}. \tag{100}$$

Then we set

$$\mathfrak{m}_q = \mathfrak{k}(A(0)), \quad \mathfrak{n}_q = \mathfrak{k}\left(A(0), \frac{\mathrm{d}A(0)}{\mathrm{d}t}\right), \tag{101}$$

where

$$\mathfrak{k}(A_0) = \{v \in \mathfrak{h} \mid [A_0, v] \in \mathfrak{h}\}, \tag{102}$$

$$\mathfrak{k}(A_0, A_1) = \{v \in \mathfrak{k}(A_0) \mid [A_1, v] - [A_0, [A_0, v]] \in \mathfrak{h}\} \tag{103}$$

for $A_0, A_1 \in \mathfrak{g}$. The sets $\mathfrak{m}_q$, $\mathfrak{n}_q$ are Lie algebras, and they depend only on $q \in G$ (being independent on f). Further, we have

$$\mathfrak{m}_{qh} = \mathrm{ad}(h^{-1})\mathfrak{m}_q, \mathfrak{n}_{qh} = \mathrm{ad}(h^{-1})\mathfrak{n}_q \text{ for } h \in H.. \tag{104}$$

The geometrical signification of $\mathfrak{m}_q$, $\mathfrak{n}_q$ has been given above, let us restrict ourselves to the following description. Let $\varphi : (a,b) \to G/H$ be a curve such that for a certain $c \in (a, b)$ we have $\varphi(c) = e$, e being the identity of G. Let $\gamma \in G$ be an arbitrary element. Let φ_γ be the curve in G/H defined by $\varphi_\gamma(t) = = \gamma\varphi(t)$ for $t \in (a,b)$, let M (N) be the set of all γ's such that the curves φ and φ_γ have the contact of order 1 (2) at $H \in G/H$. Then M, N are Lie groups with the Lie algebras $\mathfrak{m}_e$ and $\mathfrak{n}_e$ resp. The signification of the algebras $\mathfrak{m}_h$, $\mathfrak{n}_h$ for $h \in H$ is given by (104); if $\varphi(c) \neq e$, we replace φ by a left translation in G/H into a curve φ_1 such that $\varphi_1(c) = e$.

3. MANIFOLDS IN HOMOGENEOUS SPACES

The theory of curves may be easily extended to arbitrary submanifolds; here, we present only some remarks on this subject.

Let G/H be the given homogeneus space, M a domain of $\Re^n$ (with the coordinates $u^1, \ldots, u^n$), and let $\varphi : M \to G/H$ be an embedding. Let $f : M \to G$ be a lift of φ. To f, we construct the $\mathfrak{g}$-valued 1-form ω on M by

$$(105) \qquad f(u)^{-1}\mathrm{d}f(u) = \omega .$$

The mappings $A_i : M \to \mathfrak{g}$; $i = 1, \ldots, n$; being defined by

$$(106) \qquad A_i(u) = f(u)^{-1}\frac{\partial f(u)}{\partial u},$$

we have

$$(107) \qquad \omega = \sum_{i=1}^{n} A_i(u)\mathrm{d}u_i.$$

The form ω satisfies the so-called structure equation. From (106), we get

$$\frac{\partial^2 f(u)}{\partial u^j \partial u^i} = f(u)A_jA_i + f(u)\frac{\partial A_i}{\partial u^i}$$

and

$$(108) \qquad \frac{\partial A_i}{\partial u^j} - \frac{\partial A_j}{\partial u^i} = [A_i, A_j]; \quad i, j = 1, \ldots, n.$$

The structure equation (108) may be written as

$$(109) \qquad \mathrm{d}\omega = -\omega \wedge \omega,$$

$\mathrm{d}\omega$ being the exterior differential of ω and $\omega \wedge \omega$ being obtained as the product of matrices where we replace each term $\omega_i^k \omega_k^l$ by $\omega_i^k \wedge \omega_k^l$.

Let $m \in M$ be a fixed point, write $p = \varphi(m)$ and $q = f(m)$. Further, let $\varrho : (-1, 1) \to M$ be a curve such that $\varrho(0) = m$; ϱ be given by the equations

$$(110) \qquad u^i = u^i(t); \quad i = 1, \ldots, n.$$

Now, consider the curve $\varphi\varrho : (-1, 1) \to G/H$ and its lift $f\varrho : (-1, 1) \to G$. To the curve $f\varrho$ there is associated the mapping $A^\varrho : (-1, 1) \to \mathfrak{g}$ given by

$$(111) \qquad A^\varrho(t) = \sum_{i=1}^{n} A_i(u(t))\frac{\mathrm{d}u^i(t)}{\mathrm{d}t}.$$

Thus we may construct the Lie groups

$$(112) \qquad M_q^\varrho = K(A^\varrho(0), \quad N_q^\varrho = K\left(A^\varrho(0), \frac{\mathrm{d}A(0)}{\mathrm{d}t}\right)$$

and the corresponding Lie algebras. The Lie algebra of the group M_q^ϱ is the set of all vectors $v \in \mathfrak{h}$ such that

$$(113) \qquad \sum_{i=1}^{n} [A_i(m), v] \frac{du^i(0)}{dt} \in \mathfrak{h}.$$

Write $M_\varrho = \bigcap_{\varrho} M_q^\varrho$, ϱ being an arbitrary curve $\varrho : (-1, 1) \to M$ with $\varrho(0) = m$. The Lie algebra of M_q is the set of all vectors $v \in \mathfrak{h}$ satisfying

$$(114) \qquad [A_i(m), v] \in \mathfrak{h}; \quad i = 1, \ldots, n.$$

The Lie algebra of N_q^ϱ is the set of all vectors $v \in \mathfrak{h}$ satisfying (113) and

$$(115) \qquad \sum_{i,j=1}^{n} \left(\left[\frac{\partial A_i(m)}{\partial u^j}, v \right] - [A_i(m), [A_j(m), v]] \right) \frac{du^i(0)}{dt} \frac{du^j(0)}{dt} + \\ + \sum_{i=1}^{n} [A_i(m), v] \frac{d^2u^i(0)}{dt^2} \in \mathfrak{h}.$$

Now, the following is easy to see: Let there be given a manifold $\varphi : M \to G/H$, let $m \in M$ be a fixed point. Let $q \in G$ be an arbitrary point with $\varphi(m) = \pi(q)$, and let t be a tangent of M at m. To q, we may associate the Lie algebras $\mathfrak{m}_q^t$, $\mathfrak{m}_q$, $\bar{\mathfrak{n}}_q^t$, $\mathfrak{n}_q$ (with obvious geometrical significations) defined as follows. Let $f : M \to G$ be an arbitrary lift of φ such that $f(m) = q$; let u^i be local coordinates on M, and t be given by the vector

$$(116) \qquad T = x^i \frac{\partial}{\partial u^i}\bigg|_m .$$

To f, we construct the mappings $A_i : M \to \mathfrak{g}$ by means of (106). Then 1. $\mathfrak{m}_q^t$. 2. $\mathfrak{m}_q$, 3. $\bar{\mathfrak{n}}_q^t$, 4. $\mathfrak{n}_q$ is the Lie algebra of all vectors $v \in \mathfrak{h}$ such that 1. we have

$$(117) \qquad \sum_{i=1}^{n} [A_i(m), v] x^i \in \mathfrak{h},$$

2. we have

$$(118) \qquad [A_i(m), v] \in \mathfrak{h} \quad \text{for } i = 1, \ldots, n,$$

3. we have (118) and

$$(119) \qquad \sum_{i,j=1}^{n} \left(\left[\frac{\partial A_i(m)}{\partial u^j}, v \right] - [A_i(m), [A_j(m), v]] \right) x^i x^j \in \mathfrak{h},$$

4. we have (118) and

$$(120)\qquad \left[\frac{\partial A_i(m)}{\partial u^j}, v\right] - [A_i(m), [A_j(m),]] \in \mathfrak{h} \quad \text{for } i, j = 1, \ldots, n,$$

resp. Further, $\mathfrak{m}^t_{gh} = \mathrm{ad}(h^{-1})\mathfrak{m}^t_g$, etc.

4. MANIFOLDS IN AFFINE SPACES

Let A^n be the n-dimensional affine space with a fixed basis

$$\mathscr{F} = \{F; f_1, \ldots, f_n\} = \{F, f\}.$$

Each basis $E = \{E, e\}$ of A^n is given by

$$(121)\qquad \{E, e\} = \{F, f\} \begin{pmatrix} 1 & 0 \\ a & \alpha \end{pmatrix},$$

a being an $(n \times 1)$-matrix and $\alpha \in GL(n)$. All matrices

$$(122)\qquad A = \begin{pmatrix} 1 & 0 \\ a & \alpha \end{pmatrix}$$

of the just described type form the so-called affine group $GA(n)$; the bases of A^n are thus in a $1-1$ correspondence with the elements of $GA(n)$. The Lie algebra $\mathfrak{ga}(n)$ consists of all matrices of the form

$$(123)\qquad R = \begin{pmatrix} 0 & 0 \\ r & \varrho \end{pmatrix}$$

where r is an $(n \times 1)$-matrix and ϱ an $(n \times n)$-matrix. Denote by $GA_0(n)$ the subgroup of $GA(n)$ consisting of the elements of the form

$$(124)\qquad A = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix};$$

of course, $GA_0(n)$ is isomorphic to $GL(n)$, and we have $A^n = GA(n)/GA_0(n)$.

Let there be given a submanifold $\varphi : M \to A^n$. Let us choose its lift $A : M \to GA(n)$. To A, we associate the $\mathfrak{ga}(n)$-valued 1-form ω on M defined by

$$(125)\qquad \mathrm{d}A = A\omega.$$

(121) may be written as

$$(126)\qquad \{E, e\} = \{F, f\}A;$$

we have

$$\mathrm{d}\{E, e\} = \{F, f\}\mathrm{d}A = \{F, f\})A\omega$$

and

$$(127)\qquad \mathrm{d}\{E, e\} = \{E, e\}\omega.$$

These are just the equations known in the classical differential geometry. Let us write

(128) $$\boldsymbol{\omega} = \begin{pmatrix} 0 & 0 \\ \omega & \Omega \end{pmatrix},$$

i.e.,

(129) $$\mathrm{d}E = e\omega, \quad \mathrm{d}e = e\Omega$$

and

(130) $$\mathrm{d}E = \sum_{i=1}^{n} \omega^i e_i, \quad \mathrm{d}e_i = \sum_{j=1}^{n} \omega_i^j e_j$$

where $\omega = (\omega_i)$, $\Omega = (\omega_i^j)$. The structure equation (109) is

$$\begin{pmatrix} 0 & 0 \\ \mathrm{d}\omega & \mathrm{d}\Omega \end{pmatrix} = - \begin{pmatrix} 0 & 0 \\ \omega & \Omega \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ \omega & \Omega \end{pmatrix},$$

i. e.,

(131) $$\mathrm{d}\omega = - \Omega \wedge \omega, \quad \mathrm{d}\Omega = - \Omega \wedge \Omega.$$

Writing (131) component-wise, we get

(132) $$\mathrm{d}\omega^i = - \sum_{j=1}^{n} \omega_j^i \wedge \omega^j, \quad \mathrm{d}\omega_i^j = - \sum_{k=1}^{n} \omega_k^j \wedge \omega_i^k,$$

these being the well known formulas.

According to (107), let us write

(133) $$\boldsymbol{\omega} = \sum_{\alpha=1}^{m} R_\alpha(u)\mathrm{d}u^\alpha, \quad R_\alpha = \begin{pmatrix} 0 & 0 \\ r_\alpha & \varrho_\alpha \end{pmatrix},$$

$u^1, \ldots, u^m$ being local coordinates on M. From the structure equations, we get

(134) $$\frac{\partial r_\alpha}{\partial u^\beta} - \frac{\partial r_\beta}{\partial u^\alpha} = \varrho_\alpha r_\beta - \varrho_\beta r_\alpha.$$

Using the notation

(135) $$s_{\alpha\beta} = \frac{\partial r_\alpha}{\partial u^\beta} + \varrho_\beta r_\alpha,$$

we get

(136) $$s_{\alpha\beta} = s_{\beta\alpha}.$$

from (134). The matrices $s_{\alpha\beta}$ are known as well. Indeed, let us write

(137) $$r_\alpha = (r_\alpha^i), \quad \varrho_\alpha = (\varrho_{i\alpha}^j), \quad s_{\alpha\beta} = (s_{\alpha\beta}^i).$$

Then we have

$$\omega^i = \sum_{\alpha=1}^{m} r^i_\alpha \mathrm{d}u^\alpha. \tag{138}$$

The exterior differentiation and (132) yield

$$\sum_{\alpha=1}^{m} (\mathrm{d}r^i_\alpha + \sum_{j=1}^{m} r^j_\alpha \omega^i_j) \wedge \mathrm{d}u^\alpha = 0; \tag{139}$$

using Cartan's lemma, we see the existence of functions $s^i_{\alpha\beta}$ such that

$$\mathrm{d}r^i_\alpha + \sum_{j=1}^{n} r^j_\alpha \omega^i_j = s^i_{\alpha\beta}, \quad \text{i. e.} \quad \frac{\partial r^i_\alpha}{\partial u^\beta} + \sum_{j=1}^{n} \varrho^i_{j\beta} r^j_\alpha = s^i_{\alpha\beta}, \tag{140}$$

this being just the equation (135).

The matrices r_α and $s_{\alpha\beta}$ play the fundamental role in the determination of the spaces $\mathfrak{m}_q, \mathfrak{n}_q$; $q = \{E, e\}$. Each vector $v \in \mathfrak{ga}_0(n)$ may be written as

$$v = \begin{pmatrix} 0 & 0 \\ 0 & V \end{pmatrix}. \tag{141}$$

It is easy to see that

$$[R_\alpha, v] = \begin{pmatrix} 0, & 0 \\ -Vr_\alpha, & [\varrho_\alpha, V] \end{pmatrix}, \tag{142}$$

$$\left[\frac{\partial R_\alpha}{\partial u^\beta}, v\right] - [R_\alpha, [R_\beta, v]] = \tag{143}$$

$$= \begin{pmatrix} 0 & , 0 \\ -Vs_{\alpha\beta} + \varrho_\beta V r_\alpha - \varrho_\alpha V r_\beta, & \left[\frac{\partial \varrho_\alpha}{\partial u^\beta}, V\right] - [\varrho_\beta, [\varrho_\beta, V]], \end{pmatrix},$$

and we get: The Lie algebra $\mathfrak{m}_q$ is the set of all vectors (141) such that

$$Vr_\alpha = 0; \quad \alpha = 1, \ldots, m. \tag{144}$$

The Lie algebra n_q is the set of all vectors v (141) such that (144) and

$$Vs_{\alpha\beta} = 0; \quad \alpha, \beta = 1, \ldots, m. \tag{145}$$

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