Cyril Lenárt

On an Estimate of the Remainder in the Central Limit Theorem

Matematický časopis, Vol. 25 (1975), No. 4, 319--331

Persistent URL: `http://dml.cz/dmlcz/126679`

ON AN ESTIMATE OF THE REMAINDER IN THE CENTRAL LIMIT THEOREM

CYRIL LENÁRT

Let $X_1, \ldots, X_n$ be independent random variables. Let $F_k(x)$, α_k and σ_k^2, $k = 1, \ldots, n$ be their distribution functions, mean values and variances. For $k = 1, \ldots, n$ let $\alpha_k = E(X_k) = 0$, $\sigma_k^2 = E(X_k^2) < \infty$, $\sigma^2 = \sum_{k=1}^{n} \sigma_k^2 > 0$. Let $F(x)$ be the distribution function of the sum

$$X = \sum_{k=1}^{n} X_k \,. \tag{1}$$

Further, for each $k = 1, \ldots, n$ an interval $(-t_k, t_k')$, $0 < t_k \leqslant \infty$, $0 < t_k' \leqslant \infty$ let be given. Define the random variables $\bar{X}_k$ and $\bar{\bar{X}}_k$, $k = 1, \ldots, n$ as follows:

$$\bar{X}_k = \begin{cases} X_k & \text{if} \quad X_k \in (-t_k, t_k') \\ 0 & \text{if} \quad X_k \notin (-t_k, t_k') \end{cases},$$

$$\bar{\bar{X}}_k = X_k - \bar{X}_k \,, \tag{2}$$

where X_k are the independent random variables defined above.

Let us denote

$$\bar{\alpha}_k = E(\bar{X}_k)\,, \qquad \bar{\beta}_k = E(\bar{X}_k^2)\,, \qquad \bar{\gamma}_k = E(|\bar{X}_k|^3)\,,$$

$$\bar{\alpha} = \sum_{k=1}^{n} \bar{\alpha}_k\,, \qquad \bar{\beta} = \sum_{k=1}^{n} \bar{\beta}_k \qquad \bar{\gamma} = \sum_{k=1}^{n} \bar{\gamma}_k\,,$$

$$\bar{\bar{\beta}}_k = E(\bar{\bar{X}}_k^2)\,, \qquad \bar{\bar{\beta}} = \sum_{k=1}^{n} \bar{\bar{\beta}}_k \,. \tag{3}$$

Let $\Phi_k(t)$, $k = 1, \ldots, n$, be the characteristic functions of the independent random variables X_k and $\Phi(t)$ the characteristic function of the random variable X.

Put

$$\Delta = \sup_x |F(x\sigma) - G(x)|\,, \quad \text{where}$$

$$(4) \qquad G(x) = (2\pi)^{\frac{1}{2}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t\,.$$

Many upper estimates are known for the quantity Δ defined in (4). The well-known Esseen's inequality (cf. e.g. [3] 20.3A) uses an expression which is a linear combination of the functions

$$U_1(T) = \frac{1}{T}\,,$$

$$(5) \qquad U_2(T) = \int_{-T}^{T} \left| \Phi\left(\frac{u}{\sigma}\right) - \exp\left(-\frac{u^2}{2}\right) \right| |u|^{-1}\, \mathrm{d}u\,.$$

Evidently the upper estimate for Δ can be improved if at least one of the multipliers in the combination is reduced.

In [5] Zolotarev proved an inequality for an upper estimate of Δ, from which we obtain the Esseen's inequality if we choose a certain class of functions which are densities of symmetric distributions.

In [1] Berry gave an upper estimate for Δ using the product of an upper estimate of an absolute constant and the well-known Liapounov ratio depending on the third absolute moments and the second moments of random variables X_k, assuming the finiteness of their third absolute moments. The upper estimate of this absolute constant has been improved by many authors.

In [2] Feller obtained an upper estimate for Δ as a product of an upper estimate of an absolute constant (the existence of such a constant has been proved by Osipov in [4]) and an expression depending only on the second moments of the random variables X_k and their absolute second and third truncated moments. To obtain this estimate it is therefore not necessary to assume the existence of the third absolute moments of the random variables X_k and such an estimate does in fact hold even when these moments do not exist. To obtain this estimate, Feller used the well-known Esseen's inequality.

Using Feller's method to compute an upper estimate for Δ it is possible to improve the results in [2] in two ways: first, by using the Zolotarev's inequality which — as we shall demonstrate — is a refinement of the Esseen's inequality, and second, by improving other estimates used in the method; this is just what the present paper proposes to do.

We have the following

Lemma 1. *Let $R(x)$ be a distribution function and $S(x)$ a function with a bounded variation and the following properties:*

$$q = \sup_x |S'(x)| < \infty, \qquad S(-\infty) = 1 - S(\infty) = 0. \tag{6}$$

Let $r(t), s(t)$ be the Fourier-Stieltjes transforms corresponding to the functions $R(x)$ and $S(x)$.

Put

$$\bar{\Delta} = \sup_x |R(x) - S(x)|, \tag{7}$$

$$\delta(t) = r(t) - s(t). \tag{8}$$

Then for every $T > 0$

$$\bar{\Delta} \leqslant \frac{2qA}{T} + B\int_0^1 (1-t)\,|\delta(tT)|\frac{\mathrm{d}t}{t}, \tag{9}$$

where $A = 2.689388$ and $B = 0.409999$.

Proof. Let

$$p(x) = \frac{1-\cos x}{\pi x^2} \quad \text{for} \quad x \neq 0,\ p(0) = \frac{1}{2\pi}. \tag{10}$$

The function (10) is the well-known density function of the symmetric distribution with the characteristic function

$$\omega(t) = \begin{cases} 1 - |t| & \text{for} \quad |t| \leqslant 1, \\ 0 & \text{for} \quad |t| > 1. \end{cases} \tag{11}$$

Further, we use Zolotarev's inequality (cf. [5], Lemma 3), which in our case states that for all $T > 0$, $x > x_0$

$$\bar{\Delta} \leqslant 2q\frac{x[K(x) + Q(T)]}{T[4K(x) - x]}, \tag{12}$$

where

$$Q(T) = \frac{T}{2\pi q}\int_0^\infty |\omega(t)\delta(tT)|\frac{\mathrm{d}t}{t}, \tag{13}$$

$$K(x) = x\int_0^x p(u)\,\mathrm{d}u \tag{14}$$

and x_0 is a positive solution of the equation

$$4\,K(x) = x\,. \tag{15}$$

Using the Taylor series expansion for the function $p(u)$ of (10) and (14) we get

$$K(x) = x \int_0^x \frac{1-\cos u}{\pi u^2}\,du = \frac{x}{\pi} \int_0^x \sum_{k=0}^{\infty} (-1)^k \frac{u^{2k}}{[2(k+1)]!}\,du\,. \tag{16}$$

The integrand in (16) is a probability density function and evidently a positive solution of (15) exists.

For $u \in \left\langle 0, \frac{47}{10} \right\rangle$ and for the integer $k > 5$ we have

$$\frac{u^{2k}}{[2(k+1)]!} \geqslant \frac{u^{2(k+1)}}{[2(k+2)]!}\,. \tag{17}$$

From the Taylor series expansion for $\pi p(u)$, using (17), we get the estimate

$$0 \leqslant \sum_{k=0}^{5} (-1)^k \frac{u^{2k}}{[2(k+1)]!} \leqslant \frac{1-\cos u}{u^2} \leqslant \tag{18}$$

$$\leqslant \sum_{k=0} (-1)^k \frac{u^{2k}}{[2(k+1)]!}\,,$$

which is valid for $u \in \left\langle 0, \frac{47}{10} \right\rangle$.

Now let x_1 be a positive solution of the equation

$$\int_0^x \sum_{k=0}^{5} (-1)^k \frac{u^{2k}}{[2(k+1)]!}\,du = \frac{\pi}{4}\,. \tag{19}$$

From (18) we see that necessarily $x_1 > x_0$. For $x = 2$ the left side of (19) has the value greater than $\frac{8}{9} > \frac{\pi}{4}$. Clearly therefore $x_0 < 2$.

Further, for $x \in \left\langle 0, \frac{47}{10} \right\rangle$ we have

$$(20)\qquad \frac{x\int_0^x p(u)\,du}{4\int_0^x p(u)\,du-1}\leqslant\frac{x\int_0^x\sum_{k=0}^{6}(-1)^k\frac{u^{2k}}{[2(k+1)]!}\,du}{4\int_0^x\sum_{k=0}^{5}(-1)^k\frac{u^{2k}}{[2(k+1)]!}\,du-\pi}\,.$$

For the selected value of $x=\frac{47}{10}$ we get as an upper estimate of the right-hand side of (20) the value 2.689388.

Analogously for $x\in\left\langle 0,\frac{47}{10}\right\rangle$ we have

$$(21)\qquad \frac{1}{4\int_0^x p(u)\,du-1}\leqslant\frac{\pi}{4\int_0^x\sum_{k=0}^{5}(-1)^k\frac{u^{2k}}{[2(k+1)]!}\,du-\pi}\,.$$

As an upper estimate for the right-hand side of (21) for $x=\frac{47}{10}$ we get the value 0.409999π.

Using these upper estimates for the right-hand sides of (20) and (21), we obtain (9) from (12), (13) and (14). This completes the proof of Lemma 1.

As a consequence of Lemma 1 we get:

Lemma 2. *For every $T>0$ we have*

$$(22)\qquad \Delta\leqslant\frac{A'}{T}+B'\int_{-T}^{T}\left(1-\frac{|u|}{T}\right)\left|\Phi\left(\frac{u}{\sigma}\right)-\exp\left(-\frac{u^2}{2}\right)\right|\frac{du}{|u|}$$

where $A'=2.145822$, $B'=0.205$, Δ is defined by (4) *and $\Phi(t)$ is the characteristic function of the random variable X of* (1).

Proof. In Lemma 1, put $R(x)=F(x\sigma)$, $S(x)=G(x)$, where $F(x)$ is the distribution function of the random variable X of (1), $\sigma>0$ and $G(x)$ is the distribution function of the normal distribution defined in (4). Evidently

$$q=\sup_x|G'(x)|=(2\pi)^{-\frac{1}{2}}\,.$$

The relation (9) yields

$$(23) \qquad \Delta \leqslant \frac{2.145822}{T} + 0.409999 \int_0^1 (1-t) \times$$

$$\times \left| \Phi\left(\frac{tT}{\sigma}\right) - \exp\left(-\frac{(tT)^2}{2}\right) \right| \frac{dt}{t}.$$

Using the substitution $u = tT$ in the integral on the right-hand side of the equation (23), we get the relation (22).

Remark 1. Lemmas 1 and 2 are evidently a refinement of the well-known Esseen's inequality.

Now let $\Phi_k(t)$, $\bar{\beta}_k$, $\bar{\gamma}_k$, $\bar{\bar{\beta}}_k$, σ_k, $k = 1, \ldots, n$, $\Phi(t)$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\bar{\beta}}$, $\sigma > 0$ have the same meaning as before. Let $k = 1, \ldots, n$ be the subscripts of independent random variables X_k. We define a decomposition of the set of all subscripts $\{1, \ldots, n\}$ as follows:

Definition 1. *Let $T > 0$, $\alpha > 0$, $\sigma > 0$ be given reals ($\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$, where σ_i^2, $i = 1, \ldots, n$, are the variances of X_i). We shall say that a subscript k belongs to the set A iff*

$$(24) \qquad \bar{\beta}_k^{\frac{1}{2}} \leqslant \frac{\alpha\sigma}{T}.$$

We shall say that a subscript k belongs to the set A^c iff it does not belong to the set A.

The following lemmas hold:

Lemma 3. *Let $T > 0$, $l > 1$, $0 < \alpha \leqslant \sqrt{2}$ be given reals. Suppose that*

$$(25) \qquad 1 - \frac{\bar{\gamma}T}{\alpha\sigma^3} - \frac{2\bar{\bar{\beta}}}{\sigma^2} - \frac{\alpha^2}{T^2} \geqslant \frac{1}{l}.$$

Then

$$(26) \qquad \int_{-T}^{T} \left| \Phi\left(\frac{u}{\sigma}\right) - \exp\left(-\frac{u^2}{2}\right) \right| |u|^{-1}\, du \leqslant$$

$$\leqslant \int_{-T}^{T} e^{-\frac{u^2}{2l}} \sum_{k=1}^{n} \left| \Phi_k\left(\frac{u}{\sigma}\right) - \exp\left(-\frac{\sigma_k^2 u^2}{2\sigma^2}\right) \right| |u|^{-1}\, du.$$

Proof. $\left(\text{For } \alpha = \frac{4}{3} \text{ see [2]}\right)$. Define $\bar{\beta}_A$, $\bar{\gamma}_A$, $\bar{\bar{\beta}}_A$ and $\bar{\beta}_{A^c}$, $\bar{\gamma}_{A^c}$, $\bar{\bar{\beta}}_{A^c}$ in the same way as $\bar{\beta}$, $\bar{\gamma}$, $\bar{\bar{\beta}}$ with the exception that the sums are over all $k \in A$ or $k \in A^c$ respectively.

For every real y the following well-known inequalities hold:

$$\left|e^{iy} - 1 - iy + \frac{y^2}{2}\right| \leqslant \frac{|y|^3}{6}, \quad |e^{iy} - 1 - iy| \leqslant \frac{y^2}{2}. \tag{27}$$

Since $\alpha_k = 0$ for $k = 1, \ldots, n$ by assumption, using this and (27) we have for $k = 1, \ldots, n$

$$\Phi_k\left(\frac{u}{\sigma}\right) = 1 - \frac{u^2\bar{\beta}_k}{2\sigma^2} + \frac{\Theta_1|u|^3\bar{\gamma}_k}{6\sigma^3} + \frac{\Theta_2 u^2\bar{\bar{\beta}}_k}{2\sigma^2}, \tag{28}$$

where $|\Theta_1| \leqslant 1$, $|\Theta_2| \leqslant 1$. Further for $k \in A$, $|u| < T$ from (28) we conclude that

$$\left|\Phi_k\left(\frac{u}{\sigma}\right)\right| \leqslant \exp\left\{-\frac{u^2}{2\sigma^2}\left[\bar{\beta}_k - \frac{T\bar{\gamma}_k}{3\sigma} - \bar{\bar{\beta}}_k\right]\right\}. \tag{29}$$

Taking the product and the sum over all $k \in A$ we get for $|u| < T$ the estimate

$$\prod_{k \in A}\left|\Phi_k\left(\frac{u}{\sigma}\right)\right| \leqslant \exp\left\{-\frac{u^2}{2\sigma^2}\left[\bar{\beta}_A - \frac{T\bar{\gamma}_A}{3\sigma} - \bar{\bar{\beta}}_A\right]\right\} = \tag{30}$$

$$= \exp\left\{-\frac{u^2}{2\sigma^2}\left[\bar{\beta} - \bar{\beta}_{A^c} - \frac{T}{3\sigma}(\bar{\gamma} - \bar{\gamma}_{A^c}) - \bar{\bar{\beta}}_A\right]\right\}.$$

Evidently $0 \leqslant \bar{\beta}_{A^c} \leqslant \bar{\beta}$, $0 \leqslant \bar{\gamma}_{A^c} \leqslant \bar{\gamma}$, $0 \leqslant \bar{\bar{\beta}}_A \leqslant \bar{\bar{\beta}}$.
For $k \in A^c$ we have from the moment inequality

$$\bar{\gamma}_k \geqslant \bar{\beta}_k^{\frac{3}{2}} > \frac{\alpha\sigma}{T}\bar{\beta}_k.$$

Summing over all $k \in A^c$ we get

$$\bar{\gamma}_{A^C} > \frac{\alpha\sigma}{T}\bar{\beta}_{A^c}. \tag{31}$$

Using (31), (30) yields

$$\prod_{k \in A}\left|\Phi_k\left(\frac{u}{\sigma}\right)\right| \leqslant \exp\left\{-\frac{u^2}{2\sigma^2}\left[\bar{\beta} - \frac{T\bar{\gamma}}{3\sigma} - \frac{T}{\sigma}\left(\frac{1}{\alpha} - \frac{1}{3}\right)\bar{\gamma}_{A^C} - \bar{\bar{\beta}}_A\right]\right\}. \tag{32}$$

If $0 < \alpha \leqslant \sqrt{2}$, then $\frac{1}{\alpha} - \frac{1}{3} > 0$ and we get an upper estimate for the right-hand side of (32) for $\bar{\gamma}_{A^c} = \bar{\gamma}$. Using this and the equality $\sigma^2 = \bar{\beta} + \beta$ (32) gives the estimate

$$(33) \qquad \prod_{k \in A} \left| \Phi_k \left(\frac{u}{\sigma} \right) \right| \leqslant \exp \left\{ -\frac{u^2}{2} \left[1 - \frac{T\bar{\gamma}}{\alpha\sigma^3} - \frac{2\bar{\beta}}{\sigma^2} \right] \right\}.$$

By induction we easily prove that for arbitrary complex u_k, v_k, $k = 1, \ldots, n$

$$(34) \qquad u_1 \ldots u_n - v_1 \ldots v_n = \sum_{k=1}^{n} u_1 \ldots u_{k-1}(u_k - v_k) v_{k+1} \ldots v_n \, .$$

Now for $k = 1, \ldots, n$ put

$$(35) \qquad u_k = \Phi_k \left(\frac{u}{\sigma} \right), \quad v_k = \exp \left(-\frac{\sigma_k^2 u^2}{2\sigma^2} \right), \quad |u| < T \, .$$

Foj $k \in A$ we may use (29) to prove that an upper estimate for $\left| \Phi_k \left(\frac{u}{\sigma} \right) \right|$ is not less than $\exp \left(-\frac{\sigma_k^2 u^2}{2\sigma^2} \right)$. Therefore it is possible to use (29) as an upper estimate for u_k as well as for v_k. For $k \in A^c$ we use the estimate $|u_k| \leqslant 1$, $|v_k| \leqslant 1$. If $j \in A^c$, then the absolute value of the multiplier of $u_j - v_j$ in (34) with u_k and v_k defined by (35), is not greater than the right-hand side of (33). If $j \in A$, then this multiplier is not greater than the right-hand side of (33) multiplied by

$$(36) \qquad \exp \left(\frac{u^2 \bar{\beta}_k}{2\sigma^2} \right) \leqslant \exp \left(\frac{u^2 \alpha^2}{2T^2} \right).$$

Thus for u_k and v_k defined in (35) the absolute value of the right-hand side of (34) is smaller than

$$(37) \qquad \sum_{k=1}^{n} |u_k - v_k| \exp \left\{ -\frac{u^2}{2} \left[1 - \frac{T\bar{\gamma}}{\alpha\sigma^3} - \frac{2\bar{\bar{\beta}}}{\sigma^2} - \frac{\alpha^2}{T^2} \right] \right\}.$$

Now (26) is a direct consequence of (37) if the condition (25) is satisfied.

Lemma 4. *For all* $T > 0$, $l > 0$, $0 \leqslant \varkappa \leqslant \frac{1}{2}$ *we have*

$$(38) \qquad \int_{-T}^{T} e^{-\frac{u^2}{2l}} \sum_{k=1}^{n} \left| \Phi_k \left(\frac{u}{\sigma} \right) - \exp \left(-\frac{u^2 \sigma_k^2}{2\sigma^2} \right) \right| |u|^{-1} \, du \leqslant$$

$$\leqslant \left[2\varkappa^2 l^2 + \frac{\sqrt{2\pi} l^{\frac{3}{2}}}{6} + (1 - 2\varkappa l\right] \frac{\bar{\gamma}}{\sigma^3} + (3 - 2\varkappa) l \frac{\bar{\bar{\beta}}}{\sigma^2}.$$

Proof. Using (28) we get for $k = 1, \ldots, n$

$$(39) \qquad \left|\Phi_k\left(\frac{u}{\sigma}\right) - \exp\left(-\frac{u^2\sigma_k^2}{2\sigma^2}\right)\right| \leqslant \left|\Phi_k\left(\frac{u}{\sigma}\right) - 1 + \frac{u^2\bar{\beta}_k}{2\sigma^2}\right| +$$

$$+ \left|\exp\left(-\frac{u^2\sigma_k^2}{2\sigma^2}\right) - 1 + \frac{u^2\bar{\beta}_k}{2\sigma^2}\right| \leqslant \frac{|u|^3\bar{\gamma}_k}{6\sigma^3} +$$

$$+ \frac{u^2\bar{\bar{\beta}}_k}{2\sigma^2} + \left|\exp\left(-\frac{u^2\sigma_k^2}{2\sigma^2}\right) - 1 + \frac{u^2\bar{\beta}_k}{2\sigma^2}\right|.$$

For $x \geqslant 0$, $0 \leqslant e^{-x} - 1 + x \leqslant \frac{x^2}{2}$. Using this we have

$$(40) \qquad -\frac{u^2\bar{\bar{\beta}}_k}{2\sigma^2} \leqslant \exp\left(-\frac{u^2\sigma_k^2}{2\sigma^2}\right) - 1 + \frac{u^2\bar{\beta}_k}{2\sigma^2} \leqslant$$

$$\leqslant \exp\left(-\frac{\varkappa u^2\bar{\beta}_k}{\sigma^2}\right) - 1 + \frac{\varkappa u^2\bar{\beta}_k}{\sigma^2} + \frac{(1 - 2\varkappa)u^2\bar{\beta}_k}{2\sigma^2} \leqslant$$

$$\leqslant \frac{\varkappa^2 u^4 \bar{\beta}_k^2}{2\sigma^4} + \frac{(1 - 2\varkappa)u^2\bar{\beta}_k}{2\sigma^2}.$$

From (40) we conclude that

$$(41) \qquad \left|\exp\left(-\frac{u^2\sigma_k^2}{2\sigma^2}\right) - 1 + \frac{u^2\bar{\beta}_k}{2\sigma^2}\right| \leqslant \max\left\{\frac{u^2\bar{\bar{\beta}}_k}{2\sigma^2},\right.$$

$$\left.\frac{\varkappa^2 u^4 \bar{\beta}_k^2}{2\sigma^4} + \frac{(1 - 2\varkappa)u^2\bar{\beta}_k}{2\sigma^2}\right\} \leqslant \frac{\varkappa^2 u^4 \bar{\beta}_k^2}{2\sigma^4} +$$

$$+ \frac{(1 - 2\varkappa)u^2\bar{\beta}\bar{\beta}_k}{2\sigma^4} + \frac{(1 - 2\varkappa)u^2\bar{\bar{\beta}}\bar{\beta}_k}{2\sigma^4} + \frac{u^2\bar{\bar{\beta}}_k}{2\sigma^2}.$$

Summing over $k = 1, \ldots, n$ and using the moment inequality (41) and (39) gives

$$(42) \qquad \sum_{k=1}^{n} \left|\Phi_k\left(\frac{u}{\sigma}\right) - \exp\left(-\frac{u^2\sigma_k^2}{2\sigma^2}\right)\right| \leqslant \frac{|u|^3\bar{\gamma}}{6\sigma^3} +$$

$$+\frac{u^2\bar{\bar{\beta}}}{2\sigma^2}+\frac{\varkappa^2u^4\bar{\beta}^2}{2\sigma^4}+\frac{(1-2\varkappa)u^2\bar{\beta}^2}{2\sigma^4}+\frac{(1-2\varkappa)u^2\bar{\beta}\bar{\bar{\beta}}}{2\sigma_4}+$$

$$+\frac{u^2\bar{\bar{\beta}}}{2\sigma^2}\leqslant\frac{|u|^3\bar{\gamma}}{6\sigma^3}+\frac{u^2\bar{\bar{\beta}}}{2\sigma^2}+\frac{\varkappa^2u^4\bar{\gamma}}{2\sigma^3}+$$

$$+\frac{(1-2\varkappa)u^2\bar{\gamma}}{2\sigma^3}+\frac{(1-2\varkappa)u^2\bar{\bar{\beta}}}{2\sigma^2}+\frac{u^2\bar{\bar{\beta}}}{2\sigma^2}=$$

$$=\left(\frac{\varkappa^2u^4}{2}+\frac{|u|^3}{6}+\frac{(1-2\varkappa)u^2}{2}\right)\frac{\bar{\gamma}}{\sigma^3}+\frac{(3-2\varkappa)u^2}{2}\frac{\bar{\bar{\beta}}}{\sigma^2}.$$

Furthermore,

$$(43)\qquad \int_{-\infty}^{\infty} e^{-\frac{u^2}{2l}}u^2\,du=\sqrt{2\pi}l^{\frac{3}{2}},\qquad \int_{-\infty}^{\infty} e^{-\frac{u^2}{2l}}|u|\,du=2l,$$

$$\int_{-\infty}^{\infty} e^{-\frac{u^2}{2l}}|u|^3\,du=4l^2.$$

Now (38) is a consequence of (43) and (42) and the proof is complete.

The main result of this paper is given by

Theorem 1. *Let* X_k, $k=1,\dots,n$ *be independent random variables. For* $k=1,\dots,n$ *let* $E(X_k)=0$, $E(X_k^2)=\sigma_k^2<\infty$ *and* $\sigma^2=\sum_{k=1}^{n}\sigma_k^2>0$. *Then*

$$(44)\qquad \Delta\leqslant 4{,}35\left(\frac{\bar{\gamma}}{\sigma^3}+\frac{\bar{\bar{\beta}}}{\sigma^2}\right)$$

with Δ *defined by* (4) *and* $\bar{\gamma}$, $\bar{\bar{\beta}}$ *by* (3).

Proof. For arbitrary $T>0$ we have

$$(45)\qquad \Delta\leqslant\frac{A'}{T}+B'\int_{-T}^{T}\left|\Phi\left(\frac{u}{\sigma}\right)-\exp\left(-\frac{u^2}{2}\right)\right||u|^{-1}\,du$$

where $A'=2.145822$, $B'=0.205$, Δ is defined by (4) and $\Phi(t)$ is the characteristic function of the random variable X defined by (1). Using Lemma 3 for $\alpha=\sqrt{2}$ and Lemma 4 we get

$$\Delta \leqslant \frac{A'}{T} + B'\left[A(l, \varkappa)\frac{\bar{\gamma}}{\sigma^3} + B(l, \varkappa)\frac{\bar{\bar{\beta}}}{\sigma^2}\right], \tag{46}$$

where

$$A(l, \varkappa) = 2\varkappa^2 l^2 + \frac{\sqrt{2\pi} l^{\frac{3}{2}}}{6} + (1 - 2\varkappa)l, \tag{47}$$

$$B(l, \varkappa) = (3 - 2\varkappa)l, \quad l > 1, \; 0 \leqslant \varkappa \leqslant \frac{1}{2}$$

and $T > 0$ is chosen so that

$$1 - \frac{1}{l} \geqslant \frac{\bar{\gamma}T}{\sqrt{2}\sigma^3} + \frac{2\bar{\bar{\beta}}}{\sigma^2} + \frac{2}{T^2}. \tag{48}$$

From (46) and (47) we have

$$\Delta \leqslant \frac{A'}{T} + B'[\max\{A(l,\varkappa), B(l, \varkappa)\}]\left(\frac{\bar{\gamma}}{\sigma^3} + \frac{\bar{\bar{\beta}}}{\sigma^2}\right). \tag{49}$$

Suppose that for some $C > 0$, $T > 0$, $l > 1$, $0 \leqslant \varkappa \leqslant \frac{1}{2}$ the inequality

$$\frac{A'}{T} + B'[\max\{A(l, \varkappa), B(l, \varkappa)\}]\left(\frac{\bar{\gamma}}{\sigma^3} + \frac{\bar{\bar{\beta}}}{\sigma^2}\right) \leqslant C\left(\frac{\bar{\gamma}}{\sigma^3} + \frac{\bar{\bar{\beta}}}{\sigma^2}\right) \tag{50}$$

is satisfied together with the condition (48).
Then

$$\Delta \leqslant C\left(\frac{\bar{\gamma}}{\sigma^3} + \frac{\bar{\bar{\beta}}}{\sigma^2}\right). \tag{51}$$

Without loss of generality we may assume that

$$\frac{\bar{\gamma}}{\sigma^3} + \frac{\bar{\bar{\beta}}}{\sigma^2} \leqslant \frac{1}{C}, \qquad C > 0. \tag{52}$$

In the opposite case the inequality (51) is satisfied trivially, since $\Delta \leqslant 1$.

Choose $T > 0$ in such a way that for selected $C = C_0$, $l = l_0$, $\varkappa = \varkappa_0$ the inequality in (51) is attained. In this case

$$\frac{1}{T} = \frac{1}{A'}[C_0 - \max\{A(l_0, \varkappa_0), B(l_0, \varkappa_0)\}]\left(\frac{\bar{\gamma}}{\sigma^3} + \frac{\bar{\bar{\beta}}}{\sigma^2}\right). \tag{53}$$

Since $T > 0$ and also $\frac{\bar{\gamma}}{\sigma^3} + \frac{\bar{\bar{\beta}}}{\sigma^2} > 0$, evidently

$$C_0 - \max\{A(l_0, \varkappa_0), B(l_0, \varkappa_0)\} > 0 . \tag{54}$$

From (53) and (52) we derive for $T > 0$ the estimate

$$\frac{1}{T} \leqslant \frac{1}{A' C_0} [C_0 - \max\{A(l_0, \varkappa_0), B(l_0, \varkappa_0)\}] . \tag{55}$$

By computing $\frac{\bar{\gamma}}{\sigma^3}$ from (53) and substituting into (48) we get the following:

$$1 - \frac{1}{l_0} \geqslant \frac{A'}{\sqrt{2}} [C_0 - \max\{A(l_0, \varkappa_0), B(l_0, \varkappa_0)\}]^{-1} + \left(2 - \frac{T}{\sqrt{2}}\right) \frac{\bar{\bar{\beta}}}{\sigma^2} + \frac{2}{T^2} . \tag{56}$$

To prove (44) it is sufficient to prove that for selected l_0, $\varkappa_0$, C_0 a solution T of (53) is also a solution T of (56). It is easily proved by direct computation that for $l_0 = 4{,}1$, $\varkappa_0 = 0.375$, $\max\{A(l_0, \varkappa_0), B(l_0, \varkappa_0)\} = B(l_0, \varkappa_0) = 9.225$. Moreover, for $C_0 = 4.35$ from (55) we see that in this case $T > 3.796177 > 2\sqrt{2}$. For such l_0, $\varkappa_0$ and C_0 the inequality (56) is satisfied; this completes the proof.

Remark 2. In [2] Theorem 1 Feller proved (44) with the constant 6 instead of 4.35 obtained here.

Using our Theorem 1, other theorems in [2], which give analogous results for arbitrary random variables, may be similarly improved. Before we formulate these theorems, we introduce the following notation:
For $k = 1, \ldots, n$ let

$$\pi_k = P(\bar{\bar{X}}_k \neq 0), \quad p = \sum_{k=1}^{n} \pi_k, \quad \lambda_k = \frac{\bar{\alpha}_k^2}{\pi_k} \quad \text{for} \quad \pi_k \neq 0 . \tag{57}$$

For $\pi_k = 0$ we define $\lambda_k = 0$ if $\bar{\alpha}_k = 0$ and $\lambda_k = \infty$ if $\bar{\alpha}_k \neq 0$.

Theorem 2. *If*

$$\sigma^2 \geqslant \bar{\beta} + \sum_{k=1}^{n} \lambda_k \tag{58}$$

then

$$(59) \qquad \Delta \leqslant 4{,}35\left(\frac{\bar{\gamma}}{\sigma^3} + \frac{\sigma^2 - \bar{\beta}}{\sigma^2}\right) + p\,.$$

Theorem 3. *Suppose that*

$$(60) \qquad \int_{-{}^*t_k}^{t_k'} x \,\mathrm{d}F_k(x) \leqslant 0 \quad \text{and} \quad \int_{-t_k}^{{}^*t_k'} x \,\mathrm{d}F_k(x) \geqslant 0$$

for some $-\infty \leqslant -{}^*t_k \leqslant -t_k$ *and* $t_k' \leqslant {}^*t_k' \leqslant \infty$.
If

$$(61) \qquad \sigma^2 \geqslant \sum_{k=1}^{n} \int_{-{}^*t_k}^{{}^*t_k'} x^2 \,\mathrm{d}F_k(x), \text{ then (59) holds.}$$

These theorems may be proved in the same way as the original Theorems 2 and 3 in [2] except that our Theorem 1 is used instead of that given in [2].

REFERENCES

[1] BERRY, A. C.: The accuracy of the Gaussian approximation to the sum of independent variates. Trans. Amer. Math. Soc., 49, 1941, 122—136.

[2] FELLER, W.: On the Berry-Esseen Theorem. Z. Wahrscheinlichkeitstheor. verw. Geb., 10, 1968, 261—268.

[3] LOÈVE, M.: Probability Theory. 2nd ed. Princeton, Van Nostrand 1960.

[4] ОСИПОВ, Л. В.: Уточнение теоремы Линдеберга. Теория вероят. и ее примен. XI, 2, 1966, 339—342.

[5] ЗОЛОТАРЕВ, В. М.: О близости распределений двух сумм независимых случайных величин. Теория вероят. и ее примен. X, 3, 1965, 519—525.

Received August 14, 1972

I. katedra matematiky
Prírodovedeckej fakulty Univerzity P. J. Šafárika
Komenského 14
041 54 Košice