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ON A QUESTION OF J. HASHIMOTO

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Dedicated to Professor Štefan SCHWARZ on the occasion of his sixtieth birthday

1. Introduction

According to [1], a distributive lattice with universal bounds O and I can be characterized as an algebraic system with a ternary operation (xyz) such that

(1) $(OaI) = a$,

(2) $(aba) = a$,

(3) $(abc) = (bac) = (bca)$,

(4) $((abc)de) = ((ade)b(cde))$,

identically, provided

(5) $a \wedge b = (aOb)$, $a \vee b = (aIb)$.

The ternary operation (xyz) has the meaning of

(6) $(abc) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.

J. Hashimoto [3] proved that in any modular lattice the ternary operation (xyz) defined by

(7) $(abc) = ((b \vee c) \wedge a) \vee (b \wedge c) = (b \vee c) \wedge (a \vee (b \wedge c))$ satisfies the identities

(8) $(abb) = b$,

(9) $((ade)b(cde)) = (a(bde)(cde))$,

(10) $(abc) = (acb)$,

and that if (xyz) is a ternary operation on a set A containing elements O and I for which (1), (8), (9), (10) hold identically, then under $\wedge$ and $\vee$ from (5) A is a modular lattice satisfying (7).

To show that $(A, \wedge, \vee)$ is a lattice Hashimoto used only (1), (8), (10), and

(11) $((adc)bc) = (a(bcd)c) = (a(bdc)c)$;

(11) is a consequence of (8)—(10) (see [3, Lemma, Theorem 2 and its proof]). He put the question what the system was in which (9) is replaced by (11) in his theorem.

In this paper we give a characterization of modular lattices with a least element by means of the ternary operation (7) (Theorem 1) which yields that Hashimoto's theorem remains valid also when replacing (9) by (11) (see Corollary 1 of Theorem 1). We constructed Theorem 1 as such a modification of [2, Satz 6] that Satz 6 and Satz 5 of [2] are obtainable as consequences of Theorem 1 and the results in [3], [1]. Moreover, we prove that the systems of identities {(2), (3), (4)}, {(2), (3), (9)}, and {(2), (3), (11)} for a ternary operation are equivalent (Theorem 2; {(2), (3), (11)} is evidently equivalent to {(2), (3), (12)}).

2. The results

Lemma. *The identities and implications* (2), (10), *and* (13)—(19) *below hold for any ternary operation* (xyz) *satisfying* (8) *and*

(12) $((adc)bc) = (ac(bcd))$

identically.

(13) $(aab) = a$.

(14) $((abc)bc) = (acb)$.

(15) $((abc)ac) = (ac(abc)) = (abc)$.

(16) $(ab(cab)) = (abc)$.

(17) $(abc) = c$ *implies* $(bac) = c = (cab)$.

(18) $(bac) = (cab)$ *implies* $(abc) = (bac)$.

(19) $(a(abc)(dbc)) = (abc)$.

Remark. It is obvious from the Lemma that the system {(8), (12)}, consisting of only two identities, is equivalent to {(8), (10), (11)}. We could not use (11) instead of (12); the ternary operation (xyz) on a set with more than one element defined by $(abc) = c$ for any a, b, c satisfies both (8) and (11), while (10) does not hold.

Theorem 1. *Let* M *be a set with a ternary operation* (xyz) *satisfying the identities* (8), (12) *and*

(20) *There is an element* O *and for any* a, b *an element* u *exists such that* $(Oau) = a$, $(Obu) = b$.

Then $(M, \wedge, \vee)$ *with the operations* $\wedge$ *and* $\vee$ *defined by*

(21) $a \wedge b = (aOb)$, $a \vee b = (aub)$, *where* u *is an element of* M *for which* $(Oau) = a$, $(Obu) = b$

is a modular lattice with a least element O *in which* (7) *holds for any* a, b, c.

Conversely, the operation (7) *in a modular lattice with* O *satisfies* (20) *and the identities* (8), (12).

Corollary 1. *Let* M *be a set containing elements* O, I *(not necessarily different) and* (xyz) *a ternary operation on* M *satisfying* (1), (8), (10), (11) *identically. Then* $(M, \wedge, \vee)$, *where the operations* $\wedge$ *and* $\vee$ *are given by* (5), *is a modular lattice with a least element* O *and a greatest element* I *in which* (7) *holds for any* a, b, c.

Conversely, the operation (7) *in a modular lattice with* O *and* I *satisfies the identities* (1), (8), (10), *and* (11).

Corollary 2. *Distributive lattices with a least element* O *can be characterized as sets with a ternary operation* (xyz) *satisfying* (20) *and the identities* (3), (8), (12).

Distributive lattices with universal bounds O, I *can be characterized as sets containing the elements* O *and* I *with a ternary operation* (xyz) *satisfying* (1), (3), (8) *and* (12) *identically.*

In both assertions the ternary operation (xyz) *has the meaning of* (6).

Actually, for the distributive case we can prove a stronger result. Compare the first section of our introduction, Hashimoto's Theorem 3 cited in Remark 2 below and the second assertion of Corollary 2; the following Theorem 2 shows that certain systems of identities for the ternary operation (xyz) are equivalent also without requiring anything concerning special elements O and I or even O alone.

Theorem 2. *The systems of identities*

(a) (2), (3), (4);

(b) (2), (3), (9);

(c) (2), (3), (12);

(c′) (13) *and*

(22) $((adc)bc) = ((bcd)ac)$;

for a ternary operation (xyz) *are equivalent.*

Remark 1. (13) in (c′) can be replaced neither by (2) nor by (8): the ternary operation (xyz) in the Remark following the Lemma satisfies (2), (8), and (22), but (3) does not hold.

Remark 2. Since {(3), (8), (12)} and {(2), (3), (12)} are clearly equivalent, Theorem 2 enables us to extend Corollary 2 by other postulate systems in an obvious manner. A characterization of bounded distributive lattices is also given in the following Hashimoto's Theorem 3 [3]:

"Let A be any algebraic system with a ternary operation (abc) and elements O, I, such that (1), (2),

(23) $((dae)b(dce)) = ((ebd)a(ecd))$,

identically. Then A is a distributive lattice under (5) in which (6) holds."

However, the system of identities (2), (23) is not equivalent to (b). For example, it can happen that the value of (abc) does not depend on b — take a lattice with more than one element and set $(abc) = a \wedge c$ for each a, b, c; then both (2) and (23) hold identically, but for $a < b$ we have $(abb) = a \wedge b = = a \neq b$.

Remark 3. Unlike the identities (4) and (9), those of (c) all contain at most 4 parameters. We show by means of an example that even if assuming the existence of the elements O and I satisfying (1) for each a, there exists no system of identities (which do not contain constants!), with at most 3 parameters, equivalent to (c).

Let $M = \{O, I, u, v\}$ and set $(OuI) = (Ouv) = (Iuv) = u$, $(OvI) = v$, (xyz) being invariant under all permutations of the elements x, y, z and $(xxy) = x$ for all x, y, z from M. It can be easily seen that any nonvoid proper subset N of M is closed under our ternary operation. Moreover, $(N, (---))$ satisfies (c) identically, whence any identity with at most three parameters which is a consequence of (c) does hold in M while (12) for $a = u$, $b = O$, $c = v$, $d = I$ is not satisfied.

3. The proofs

Proof of the Lemma. By (12), $((aaa)ba) = (aa(baa))$; this together with (8) gives (2). Similarly, (13) follows from $((aba)ba) = (aa(bab))$ and (2). (12) and (2) imply also (14): $((abc)bc) = (ac(bcb)) = (acb)$. Now we prove (10) applying (13), (14), (12), (14), and (2): $(acb) = ((acb)(acb)c) = (((abc)bc)(acb)c) = ((abc)c((acb)cb)) = ((abc)c(abc)) = (abc)$.

The identities (2), (8), and (10) are used freely in the work to follow.

(12), (14), and (12) give (15): $((abc)ac) = (ac(abc)) = (ac((abc)cb)) = = ((abc)(abc)c) = (abc)$. (16) is obtained applying (12) to $((aab)cb)$. Next let $(abc) = c$. Then, by (16), $(bac) = (bc(abc)) = (bcc) = c$; similarly, $(cab) = = (cb(abc)) = (cbc) = c$. (18) follows from (15), (17), and (16): we have $(ba(bac)) = (bac)$ by (15), whence (17) gives $(ab(bac)) = (bac)$. Thus, by (16), $(abc) = (ab(cab)) = (ab(bac)) = (bac)$. To prove (19) we first use (15), three times (12) and (15) again: $((dbc)a(abc)) = ((dbc)a((abc)ac)) = (((dbc)ac)(abc)a) = = ((dc(abc))a(abc)) = (d(abc)(a(abc)c)) = (d(abc)(abc)) = (abc)$. This together with (17) implies (19).

Proof of Theorem 1. Suppose $(M, (---))$ satisfies (20) and the identities (8), (12). From now on we shall use (2), (8), (10) and (17) freely (see the Lemma).

First we show

(24) $(aOb) = (bOa) = (Oab)$ for any a, b from M.

By (20), there exists an element u in M with $(Oau) = a$, $(O(bOa)u) = (bOa)$. Hence (12) gives $(aOb) = ((uaO)bO) = (uO(bOa)) = (bOa)$. This together with (18) implies (24).

It can be easily proved now that $a \leqq b$ if and only if $(Oab) = a$ is a partial ordering of M under which O is the least element; $(M, \leqq)$ is directed. Only the transitivity of $\leqq$ requires some computation. Let $a \leqq b$, $b \leqq c$; then $(Oac) = (O(Oab)c) = (cO(aOb)) = ((cbO)aO) = (baO) = a$ by (24) and (12), whence $a \leqq c$.

Further we have

(25) $a \leqq b \leqq c$ implies $(abc) = (bac) = (cab) = b$.

Indeed, $(abc) = ((aOb)cb) = (ab(cbO)) = (abb) = b$ and (25) follows from (17).

(26) $a, b \leqq u$ implies $(aub) = (bua) = (uab)$.

We apply successively (12), (15), (12), (15): $(aub) = ((Oau)bu) = (Ou(bua)) = (Ou((bua)ub)) = ((Obu)(bua)u) = (b(bua)u) = (bua)$; (26) holds by (18).

(27) $a, b \leqq u, v$ implies $(aub) = (avb)$.

If $a, b \leqq u \leqq w$, we have $(awb) = ((uwb)a(awb)) = (ua(awb)) = (ua(baw)) = ((uwa)ba) = (uba) = (aub)$ by (19) and (17), (25), (26), (12), (25), and (26). By (20), there exists an w with $u, v \leqq w$, whence $(aub) = (awb) = (avb)$.

Having shown that $a \vee b$ does not depend on the choice of a suitable element u in (21), we can prove that $(M, \wedge, \vee)$ is a lattice. The commutativity of $\wedge$ and $\vee$ follows from (24) and (26). Let u be an element of M with a, b, c, $a \vee b$, $b \vee c < u$; we have by (12) $(a \wedge b) \wedge c = ((aOb)Oc) = (aO(cOb)) = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = ((aub)uc) = (au(cub)) = a \vee (b \vee c)$. Finally, $a \wedge (b \vee a) = (aO(bua)) = ((bua)Oa) = (ba(Oau)) = (baa) = a$ and $a \vee (b \wedge a) = (au(bOa)) = ((bOa)ua) = (ba(uaO)) = (baa) = a$, where u was an element of M with a, b, $a \wedge b \leqq u$. Hence $(M, \wedge, \vee)$ is a lattice, its partial ordering being $<$; the latter is obvious from the definition of $\wedge$.

To prove that this lattice is modular, we show

(28) $b \leq c$ implies $b \vee (a \wedge c) = (abc) = (b \vee a) \wedge c$.

Since b, $a \wedge c \leqq c$, we have $b \vee (a \wedge c) = ((a \wedge c)cb) = ((aOc)bc) = (a(bOc)c) = (abc)$. Obviously $b \leqq (b \vee a) \wedge c \leqq c$, whence by (25) and (12) $(b \vee a) \wedge c = (((b \vee a) \wedge c)bc) = (((b \vee a)Oc)bc) = ((b \vee a)c(bcO)) = ((b \vee a)cb) = ((aub)cb) = (ab(cbu)) = (abc)$; u was an element of M with $a, b, c \leqq u$.

It remains to prove (7); for this purpose two auxiliary assertions proved useful:

(29) $b \wedge c \leqq (abc) \leqq b \vee c$,

(30) $a \wedge (b \vee c) \leqq (abc) \leqq a \vee (b \wedge c)$.

By (19), $b \wedge c = (Obc) = (O(Obc)(abc))$; similarly, $b \vee c = (ubc) = (u(ubc)(abc)) = (b \vee c) \vee (abc)$, where u was an element of M with $b, c, (abc) \leqq u$. Hence (29) holds. Now we make use of (19), (28), and (29): $(abc) = (a(abc)(Obc)) = (a(abc)(b \wedge c)) = (a \vee (b \wedge c)) \wedge (abc)$; similarly, $(abc) = (a(abc)(ubc)) = (a(abc)(b \vee c)) = (abc) \vee (a \wedge (b \vee c))$, where u was an element of M with $b, c \leqq u$.

The identitity (7) is a consequence of (29), (30) and the modularity.

For the second part of Theorem 1, it can be easily computed that (20) holds for $u = a \vee b$, while the validities of (8) and (12) are a result of Hashimoto (see our introduction and the Lemma).

Proof of Corollary 1. Since the systems of identities $\{(8), (12)\}$ and $\{(8), (10), (11)\}$ are equivalent (see the Lemma), Corollary 1 is a direct consequence of Theorem 1. Computation is needed only to prove $(OaI) = a$ in the second assertion.

Proof of Corollary 2. It is sufficient to show that if the ternary operation (7) in a modular lattice satisfies the identities (3), then this lattice is distributive. Now we apply a usual procedure, which was used also by J. Hashimoto in order to prove a similar assertion. There exist distinct elements a, b, c in the modular nondistributive five-element lattice such that $(abc) = a$, $(bac) = b$, whence (3) implies the distributivity.

Proof of Theorem 2. Since by (4), (3), (4), and (3) again $((ade)b(cde)) = ((abc)de) = ((bac)de) = ((bde)a(cde)) = (a(bde)(cde))$, (b) is a consequence of (a); setting $e = c$ in (9) makes clear that (b) implies (c).

Next let M be a set on which a ternary operation (xyz) satisfying (c) is defined; we prove (a). The fact that (abc) is invariant under all permutations of the elements a, b, c will be used freely now.

Define a ternary relation R in M by aub if and only if $(aub) = u$ (we shall write abc instead of $(a, b, c) \in R$). The relation R has the following three properties:

(B_1) aba implies $a = b$.

(F) abc together with acd imply dba.

(D) To a, b, and c there corresponds a unique element w such that awb bwc and cwa hold.

(B_1) is obvious from (2); $(abc) = b$ and $(acd) = c$ give by (12) $(dba) = (d(abc)a) = ((dac)ba) = (cba) = b$, which proves (F). By (15), (abc) has the required properties of w in (D); we show that (abc) is the only such element. Suppose $(aub) = (buc) = (cub) = u$; then by (19) and (12) $(abc) = (a(abc)(ubc)) = (a(abc)u) = (ac(uab)) = (acu) = u$. Hence the fact that (c) implies (a) is a consequence of the following theorem of M. Sholander [4]:

Let abc be a ternary relation in a set S, satisfying the conditions (D), (B_1), and (F). Then setting $(abc) = w$, where w is the uniquely determined element from (D), we obtain a median semilattice (in [4] the term median semilattice is used for a set with a ternary operation (xyz) satisfying the identities (a)).

The proof of Theorem 2 will be complete if we show that (c′) is equivalent to (c). Obviously (c) implies (c′). Conversely, let (c′) hold for a ternary operation (xyz) identically. We first prove

(31) $((abc)cb) = (cab)$

and (2).

Indeed, by (22) and (13) $((abc)cb) = ((ccb)ab) = (cab)$. Next we apply successively (13), (22) with $a = (aba)$, $b = a$, $c = b$, $d = a$, (31) to $((aba)ab)$, and finally (13) twice: $(aba) = ((aba)(aba)b) = (((aba)ab)ab) = ((aab)ab) = (aab) = a$.

Now we have $(acb) = ((aba)cb) = ((cab)ab) = (((abc)cb)ab) = ((abc)(abc)b) = (abc)$ by (2), (22), (31), (22) with $a = (abc)$, $b = a$, $c = b$, $d = c$, and (13). We proved $(abc) = (acb)$, which together with (31) gives $(bac) = ((acb)bc) = ((abc)cb) = (cab)$, whence (abc) is invariant under all permutations of the elements a, b, and c. Thus (12) follows from (c′), too.

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