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THE RECONSTRUCTION OF A CONNECTED GRAPH FROM ITS SPANNING TREES

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Throughout this paper a graph is to be understood as an undirected graph without loops and multiple edges. Following the well-known Kelly-Ulam conjecture (see [4] and [8]) every graph $\mathscr{G}$ having at least three vertices is uniquely reconstructable from its subgraphs $\mathscr{G}-x$ where x passes through all the vertices of the graph $\mathscr{G}$. There exist a number of papers proving the Kelly-Ulam conjecture for special classes of finite graphs. J. Fisher [1] has proved that the conjecture is not valid for infinite graphs and has so given a counterexample to the question raised by F. Harary [3]. In this paper we shall deal with a connected graph and all its spanning trees. By a spanning tree $\mathscr{K}$ of a connected graph $\mathscr{G}$ we understand the maximum tree contained in $\mathscr{G}$ as a subgraph. In bibliography there already exist references dealing with the structures of the spanning trees of a given graph. So e.g. B. Zelinka [9] has paid attention to a finite connected graph whose all spanning trees are mutually isomorphic and has described all the graphs having this property. The investigation of a connected finite graph whose no two spanning trees are mutually isomorphic would represent a similar problem. It can be easily proved that such a graph exists even if in addition we require for it to have a given number m of the spanning trees ($m \neq 2$). However in this paper we will be concerned with another question resembling the Kelly-Ulam conjecture – namely with the problem whether a graph is uniquely determined by the structure of all its spanning trees. In order to make the explanation concise we shall say that a (finite) tree is of the snake type if it has two vertices of degree 1. Moreover, we will call a tree a Y-graph if it has three vertices of degree 1 (see [7], page 25). If we denote u the vertex of degree 3 and v_1, v_2, and v_3 the vertices of degree 1 in a given Y-graph we shall say that this graph is of the type (d_1, d_2, d_3), provided d_i is the distance of u from v_i in the usual metric ($i = 1, 2, 3$).

We shall say that a connected graph $\mathscr{G}$ is uniquely reconstructable from all its spanning trees if the following property is true:

P. Let $\mathscr{G}^*$ be a connected graph with the same number of vertices as $\mathscr{G}$. Let

$M = \{\mathscr{K}_i\}$ be an ordered set of all spanning trees of the graph $\mathscr{G}$ and $M^* = \{\mathscr{K}_i^*\}$ an ordered set of all spanning trees of the graph $\mathscr{G}^*$. Moreover let $|M| = |M^*|$ and $\mathscr{K}_i$ be isomorphic with $\mathscr{K}_i^*$ for every i. Then $\mathscr{G}$ and $\mathscr{G}^*$ are isomorphic.

It is trivial to show that each tree and each circuit have the property P. Unlike the Kelly-Ulam conjecture a finite graph can be found that is not uniquely reconstructable from all its spanning trees. An example is given in Fig. 1. Graph $\mathscr{G}_0$ plotted on the left-hand side has eight spanning trees, of which two are of the snake type and the remaining ones are Y-graphs: two spanning trees of the type (1, 1, 6), two of the type (1, 2, 5) and two of the type (1, 3, 4). The graph $\mathscr{G}_0^*$ plotted on the right-hand side also has eight spanning trees isomorphic in turn with those described above. And yet $\mathscr{G}_0$ and $\mathscr{G}_0^*$ are not isomorphic. It can also be easily seen that $\mathscr{G}_0$ and $\mathscr{G}_0^*$ are (with respect to the isomorphism) the sole two graphs determined by the structure of the particular eight trees.

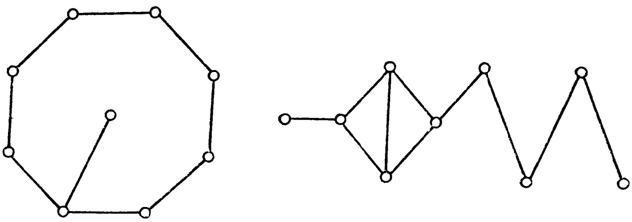

Fig. 1

An example not having the property P can be found also among infinite connected graphs. Let us choose the complete bipartite graph $\langle 2, \aleph_0 \rangle$ and prove that it is not uniquely reconstructable in the sense explained above. Let us consider thus the graph $\mathscr{G} = \langle 2, \aleph_0 \rangle$, whose vertices belong to two classes A, B where $|A| = 2$, $|B| = \aleph_0$. Let us put $A = \{u, v\}$ and let w be another vertex not belonging to $A \cup B$. Let us choose as $\mathscr{G}^*$ the graph arising from $\mathscr{G}$ by completing it with the vertex w and the edge vw. Obviously $\mathscr{G}$ and $\mathscr{G}^*$ are not isomorphic.

Each of the graphs $\mathscr{G}$, $\mathscr{G}^*$ has an uncountable number of the spanning trees and $|M| = |M^*|$. Let us order the spanning trees of the graph $\mathscr{G}$ and also the spanning trees of the graph $\mathscr{G}^*$ and prove that two corresponding spanning trees are isomorphic. The spanning trees of the graph $\mathscr{G}$ can be split into three types. The first type is such that either u or v is an end-vertex of a spanning tree. The number of these spanning trees is countable. The second type is such that either u or v is of a finite degree > 1 in the given spanning tree. Also here is the number of the spaning trees countable. At last the third type is a spanning tree, in which both u and v have an infinite degree. The number of the cases is uncountable here. The same classification can be implemented for the spanning trees of the graph $\mathscr{G}^*$ and hence there follows what was to be proved.

The two presented examples concern the graphs lacking the property P but from a few attempts we can see that finite connected graphs with a small number of vertices are uniquely reconstructable. Another illustration of the graphs with the property P is provided by the results found in [6]: If a graph (with at least eight vertices) lacks less than five edges in order to be a complete graph, then its structure is already determined by the number of its spanning trees. Or, in other words the „nearly" complete graphs have obviously the property P.

In the remaining part of this paper we shall describe in two theorems classes of finite connected graphs that are uniquely reconstructable from their spanning trees.

Theorem 1. *A complete bipartite graph* $\langle 2, n\rangle$ *has the property* P.

Proof. The statement is obvious for $n = 1, 2$. Let therefore $n \geq 3$. A graph $\langle 2, n\rangle$ has altogether $n2^{n-1}$ spanning trees separable into two groups. The first group contains the spanning trees with diameters 3 and the spanning trees of the second one have diameters 4. Let a connected graph $\mathscr{G}^*$ on $n + 2$ vertices have in turn isomorphic spanning trees with these trees. Let us question if $\mathscr{G}^*$ can contain a circuit of a length at least 6. If yes, let $u_1, u_2, \ldots, u_6, \ldots$ denote in turn the vertices of this circuit. We can certainly construct a spanning tree $\mathscr{K}$ containing five edges u_1u_2, u_2u_3, u_3u_4, u_4u_5, u_5u_6. However $\mathscr{K}$ is then of diameter at least 5, which is impossible. Let us discuss therefore the case that $\mathscr{G}^*$ contains a pentagon $\mathscr{C}_5$ with the vertices u_1, u_2, u_3, u_4, u_5. From the existence of another vertex w not belonging to $\mathscr{C}_5$ and connected with some u_i $(1 \leq i \leq 5)$ the existence of a spanning tree of a diameter at least 5 can be derived. Graph $\mathscr{G}^*$ has therefore only 5 vertices and the calculation shows possible numbers of 5, 11, 21, 24, 40, 45, 75 or 125 spanning trees. Since none of these numbers is of the form $n\,2^{n-1}$ we have to reject the assumption that $\mathscr{G}^*$ contains $\mathscr{C}_5$. Let $\mathscr{G}^*$ contain a quadrangle $\mathscr{C}_4$ with the vertices u_1, u_2, u_3, u_4 in turn. Since $n \geq 3$, $\mathscr{G}^*$ has to have at least one more vertex v. Let it be next to the vertex u_1. Then u_4 is not incident, except u_1u_4, u_3u_4 (and possibly u_2u_4), with any other edge xu_4 where x is outside $\mathscr{C}_4$. If this case occurred, x could neither coincide with v (a pentagon!) nor be x v (a walk of length 5). The same is valid for u_2. The vertices outside $\mathscr{C}_4$ are next either to u_1 or u_3 and each of them is either an end-vertex (let k be their number) or the next one to u_1 coincides with the next one to u_2 (their number would be $n - k - 2$). If namely v were next to u_1 and w next to u_3 (v and w being outside $\mathscr{C}_4$), then neither the edge vw would be possible (a pentagon) nor the edges yv or yw could exist (y is a new vertex). If C did not contain any of the diagonals u_1u_3, u_2u_4, then $\mathcal{G}^*$ would have so many spanning trees as the complete bipartite graph $2, n - k$ has i.e. $(n - k) \cdot 2^{n-k-1}$. Hence $k = 0$ and $\mathcal{G}^*$ is isomorphic with $2, n$. Let therefore the edge u_2u_4 exist. Then $n - k - 2$ must be zero (a pentagon) and $\mathcal{G}^*$ has either 8 or 16 spanning trees. However, none of these numbers fits the form $n\, 2^{n-1}$. Let u_2u_4 do not exist and let the edge u_1u_3 be here. That leads us to the consideration: We are calculating the number $(n - k - 2)\, 2^{n-k-1}$ of the spanning trees of the graph $\mathcal{G}^*$. But this number does not equal $n\, 2^{n-1}$ and so the case with the circuit C_4 is exhausted. It remains to discuss the case when the maximum circuit in $\mathcal{G}^*$ is a triangle C_3 with the vertices u_1, u_2, u_3. Then each block of the graph $\mathcal{G}^*$ is either composed of one edge or of three edges and the number of its spanning trees is therefore 3^α, where α is the number of the triangles in $\mathcal{G}^*$. Nor has the equation $n\, 2^{n-1} = 3^\alpha$ in this case a solution and so we reject also the last assumption.

We have thus reached the conclusion that $\mathcal{G}^*$ is isomorphic with the graph $2, n$ and have proved that $2, n$ is uniquely reconstructable from its spanning trees.

In the second theorem we will deal with „a wheel" (see e.g. [7]). We are denoting this graph as $\mathcal{W}_n$ ($n \geq 3$) and we are defining it as follows: We are choosing a circuit C_n of a length n and a vertex u outside C_n and we are connecting u with each vertex of the circuit C_n by an edge.

The construction of the graph $\mathcal{W}_n$ is so described. We have proved in [5] the validity of

$$(1) \qquad k(\mathcal{W}_n) = \left(\frac{3 + \sqrt{5}}{2}\right)^n + \left(\frac{3 - \sqrt{5}}{2}\right)^n - 2$$

for $k(\mathcal{W}_n)$ denoting the number of the spanning trees of the graph $\mathcal{W}_n$. We shall need here also one auxiliary result of [5], namely the graph $\mathcal{H}_n$ and the formula

$$(2) \qquad k(\mathcal{H}_n) = \frac{(3 + \sqrt{5})^n - (3 - \sqrt{5})^n}{2^n \sqrt{5}}.$$

The graph $\mathcal{H}_n$ is obtained from $\mathcal{W}_n$ by removing one edge of the circuit C_n. It is convenient to extend the definition of $\mathcal{H}_n$ to $n = 1$ and $n = 2$. The graph $\mathcal{H}_1$ is the complete graph 2 and $\mathcal{H}_2$ is the complete graph 3.

Theorem 2. *A graph $\mathcal{W}_n$ has the property P.*

Proof. We shall distinguish two cases. The first of them comprises $n \leq 6$, the second one is for $n > 6$.

a) The most composite of the subcases $n = 3, 4, 5$, and 6 is the last one. Since the three preceding ones are analogous, we will deal here only with $n = 6$.

According to (1) the graph $\mathscr{W}_6$ has 320 spanning trees, each of which is isomorphic with one of 11 possible trees on seven vertices (see Fig. 2). Let

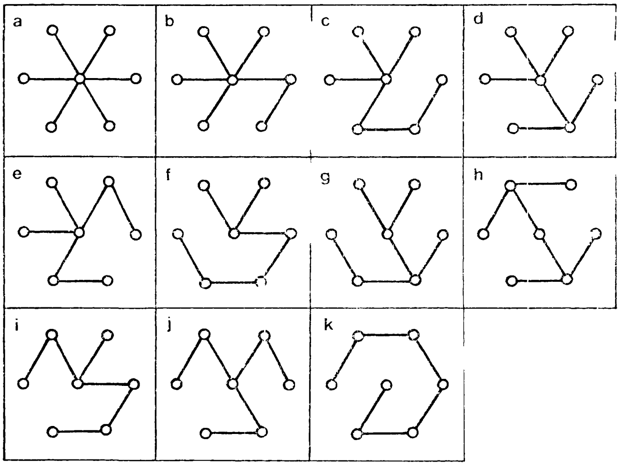

Fig. 2

the spanning trees of the graph $\mathscr{W}_6$ be ordered in an arbitrary manner and let $\mathscr{G}$ be a connected graph on seven vertices having all its spanning trees isomorphic in turn with the spanning trees of the graph $\mathscr{W}_6$. Let us consider the spanning tree $\mathscr{K}_1$ of the graph $\mathscr{G}$, which is isomorphic with the graph in Fig. 2a, and let u denote the vertex of degree 6 in $\mathscr{K}_1$. Let v_i $(i = 1, 2, \ldots, 6)$ be the remaining vertices in $\mathscr{K}_1$. We are asking the question whether u can be a cut vertex of the graph $\mathscr{G}$. If yes, how many blocks are incident with this cut vertex? We are rejecting immediately the cases of 6 and 5 blocks. Let 4 blocks be the case. Then either of the patterns $2 + 2 + 1 + 1$ and $3 + 1 + 1 + 1$ is taken into account. The first pattern indicates that two of the blocks are triangles and the remaining two have one edge each. However, then $k(\mathscr{G}) = 9$ (a contradiction). The second pattern indicates that three edges being incident with u belong to one block and each of the other blocks has one edge. However, then $k(\mathscr{G}) \leq 16$ (a contradiction). The case of three blocks yields the patterns $2 + 2 + 2$ or $3 + 2 + 1$ or $4 + 1 + 1$. We have to reject the first pattern, for $k(\mathscr{G}) = 27$, the second one results in contradiction $k(\mathscr{G}) < 16 \cdot 3$ and the third one in $k(\mathscr{G}) \leq 5^3$. The case of two blocks can be described by the patterns $3 + 3$, resp. $4 + 2$, resp. $5 + 1$. The first one is rejected for $k(\mathscr{G}) < 16 \cdot 16$ the second one yields a triangle as a block and therefore ends in a contradiction $3\ k(\mathscr{G})$ and the third one will be described in more details. The block containing 5 edges uv_i cannot be a complete graph for then $k(\mathscr{G})$ 6^4. If it lacked one edge in order to be complete,*) we would get $k(\mathscr{G})$ 864. If it lacked two edges, then $k(\mathscr{G})$ 576 or 540 and if it lacked three edges, then $k(\mathscr{G})$ would be 360, 336, 324, 300 respectively; if it lacked more edges than 3, $k(\mathscr{G})$ would be less than 300. On the whole we see that u cannot be a cut vertex of the graph $\mathscr{G}$.

We shall thus start to construct the graph $\mathscr{G}$ without cut vertices in such a way that we shall successively complete the above mentioned tree $\mathscr{K}_1$ by edges. Without loss of generality we can take the edges v_1v_2, v_2v_3. Let us ask the question whether $\mathscr{G}$ can then also have both edges v_2v_4, v_2v_5. If yes, then v_2v_6 does not exist, for $\mathscr{G}$ would have at least two isomorphic spanning trees with the graph in Fig. 2a. The so far constructed graph has 16 spanning trees that are isomorphic with the graph in Fig. 2c, whereas $\mathscr{W}_6$ has only 12 of them. There exists therefore at most one of the edges v_2v_4, v_2v_5, v_2v_6; let it be v_2v_4. Let us discuss the case that v_5 and v_6 (apart from u) are then connected with one additional common vertex (e.g. with v_1). However, the so far constructed graph has 10 spanning trees of the type d) in Fig. 2, whereas $\mathscr{W}_6$ has only 6 of them. So we are proceeding to the case that v_5 and v_6 (apart from u) are connected each with a different vertex, e.g. with the edges v_5v_4, v_6v_1. The so far constructed graph is not yet $\mathscr{G}$, for it has only 10 spanning trees of the type b) in Fig. 2, whereas $\mathscr{W}_6$ has 12 of them. We cannot add the edge v_5v_6 to it, for the resulting graph would already have 18 spanning trees of the type c) in Fig. 2, whereas $\mathscr{W}_6$ has only 12 of them. The attachment of the edge v_3v_5 results in 14 spanning trees of the type c) instead of 12 spanning trees in the graph $\mathscr{W}_6$ and it follows from the symmetry that it is impossible to attach v_3v_6 as well. Can we add v_1v_5? This attachment also results in 16 spanning trees of the type c) instead of the correct number 12. We are rejecting v_4v_6 for the same reasons. Let us add v_3v_1 and we see that the originated graph has 10 spanning trees of the type d), whereas $\mathscr{W}_6$ has 6 of them. We cannot add v_3v_4 as well (symmetry). The last matter in this discussion is the attachment of v_1v_4. It would yield 11 spanning trees of the type d) instead of those 6 in $\mathscr{W}_6$. So we are returning in our discussion to the graph that arose by completing the spanning tree $\mathscr{K}_1$ with the edges v_1v_2, v_2v_3 and we assume that none of the edges v_2v_4, v_2v_5, v_2v_6 exists. Without loss of generality we can suppose the edge v_3v_4. The vertex v_5 cannot be connected with v_3, for we would obtain the above rejected case by renumbering the vertices v_i

*) We are using here the formulas derived in [6] and listed in a table on the page 222. The assumption in the quoted reference is $n \geq 8$ but it is obvious that those entries of the table that are needed here are valid also for n 6.

Let us assume the first edge of the two symmetric cases v_4v_5, v_1v_5. The vertex v_6 can also be connected only with v_5 or v_1. We see immediately that we have to introduce both considered connecting edges and we thus obtain a graph isomorphic with $\mathscr{W}_6$. And that concludes the case $n = 6$.

b) Let $n \geq 7$ and let $\mathscr{G}$ be a graph on $(n + 1)$ vertices, the spanning trees of which are isomorphic in turn with the spanning trees of the graph $\mathscr{W}_n$. There exists a vertex u of degree n in the graph $\mathscr{G}$, for $\mathscr{W}_n$ has a spanning tree with a vertex of degree n. Let us construct $\mathscr{G} - u$ and ask the question whether there exists a vertex of degree at least 3 in this graph. If yes, let it be denoted v_1 and let the three edges incident with it be v_1v_2, v_1v_3, v_1v_4. With regard to the assumption $n \geq 7$ there still exist the vertices v_5, v_6, v_7 in the graph $\mathscr{G} - u$, each of which is connected in $\mathscr{G}$ by an edge with the vertex u. In the graph $\mathscr{G}$ we can construct the spanning tree $\mathscr{K}_2$ containing all the edges uv_1, v_1v_2, v_1v_3, v_1v_4, uv_5, uv_6, uv_7. However, $\mathscr{K}_2$ has two neighbouring vertices u, v_1, each of which is of a degree at least 4; therefore $\mathscr{K}_2$ is not isomorphic with any spanning tree of the graph $\mathscr{W}_n$. That is a contradiction and so each vertex of the graph $\mathscr{G} - u$ is of degree 0,1 or 2. If $\mathscr{G} - u$ were a disconnected graph, its components would have either one point each or they would be graphs of the snake type (with the lengths d_1, d_2, $\ldots$, d_r in turn) or circuits (with the lengths D_1, D_2, $\ldots$, D_s in turn). Let us discuss here briefly only the „general" case when the two last mentioned component types of the graph $\mathscr{G} - u$ exist (then $r \geq 1$, $s \geq 1$). Let $a(n)$, resp. $b(n)$ denote the right-hand side**) of the equation (1), resp. (2).
Then we have

$$k(\mathscr{G}) = \prod_{i=1}^{s} a(D_i) \prod_{j=1}^{r} b(d_j + 1).$$

We can readily find that

$$\prod_{i=1}^{s} a(D_i) \leqq a(\sum_{i=1}^{s} D_i),$$

$$\prod_{j=1}^{r} b(d_j + 1) \leqq b(r + \sum_{j=1}^{r} d_j) < a(r + \sum_{j=1}^{r} d_j).$$

Since

$$r + \sum_{j=1}^{r} d_j + \sum_{i=1}^{s} D_i \leqq n,$$

we obtain for the number $k(\mathscr{G})$ the following inequalities:

**) The number $a(n)$ was defined only for $n \geq 3$, but let us here extend (1) also to $n = 2$).

$$k(\mathscr{G}) = a(\sum_{i=1}^{s} D_i)\,a(r + \sum_{j=1}^{r} d_j) < a(r + \sum_{j=1}^{r} d_j + \sum_{i=1}^{s} D_i) \leq a(n)$$

That is a contradiction, for $k(\mathscr{G}) = a(n)$. The graph $\mathscr{G} - u$ is therefore connected. Since $k(\mathscr{H}_n) < k(\mathscr{H}'_n)$, $\mathscr{G} - u$ is a circuit of length n and $\mathscr{G}$ is isomorphic with $\mathscr{H}'_n$. This completes the proof.

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