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CHAIN CONDITIONS IN THE DISTRIBUTIVE PRODUCT OF LATTICES

ZUZANA LADZIANSKA

This paper is concerned with a generalization of the distributive free product and the ordinal sum of distributive lattices, so-called the $\mathscr{D}$ — poproduct of distributive lattices. The notion of the $\mathscr{D}$ — poproduct was first introduced by Balbes and Horn [1] under the name of the order sum. Generally, the notion of the $\mathscr{K}$ — poproduct for an arbitrary equational class of lattices was introduced in [7].

We begin with some preliminary notions.

Let P be a poset and let L_p, $p \in P$ be pairwise disjoint lattices.

Let $Q = \bigcup_{p \in P} L_p$ be partially ordered in the following way: $a, b \in Q$, $a \leqq b$ if and only if one of the conditions (1) and (2) holds:

(1) there is a $p \in P$ such that $a, b \in L_p$ and the relation $a \leqq b$ in L_p holds;

(2) there are $p, r \in P$ such that $a \in L_p$, $b \in L_r$ and the relation $p < r$ in the poset P holds.

If f is a mapping from Q into M then f_p denotes its restriction on L_p.

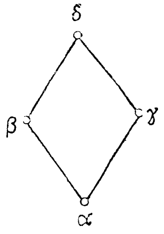
Definition (see [7]). *Let $\mathscr{K}$ be an equational class of lattices. Let L_p, $L \in \mathscr{K}$ let P be a poset. A lattice L is the $\mathscr{K}$ — poproduct of the lattices $L_p, p \in P$, if:*

1. *there is an isotone injection $i : Q \to L$ such that for each $p \in P$, i_p is a lattice homomorphism;*
2. *if $M \in \mathscr{K}$, then for every isotone mapping $f : Q \to M$, such that for each $p \in P$, f_p is a lattice homomorphism, there exists uniquely a lattice homomorphism $\Psi : L \to M$ such that $\Psi . i = f$*

We denote by $\mathscr{D}$ the class of all distributive lattices. The $\mathscr{D}$ — poproduct will be called also the distributive poproduct.

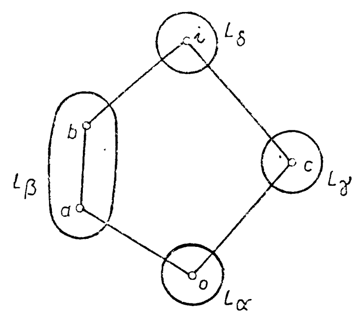
Theorem 1 from [7] says that the $\mathscr{K}$ — poproduct is a generalization of the $\mathscr{K}$ free product and the ordinal sum of lattices: the $\mathscr{K}$ — poproduct forms the $\mathscr{K}$ free product iff P is an anti-chain and the ordinal sum iff P is a chain.

Remark. If $\mathscr{K}_1 \subseteq \mathscr{K}_2$ are two equational classes of lattices and L is



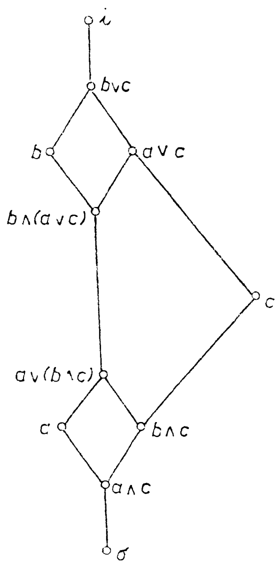
P

Fig. 1



Q

Fig. 2



L

Fig. 3

the $\mathscr{K}_2$ — poproduct of lattices $L_p \in \mathscr{K}_1$ $(p \in P)$, then L need not be in as the following example shows.

Example. Let $P = \{\alpha, \beta, \gamma, \delta\}$, $\beta \wedge \gamma = \alpha$, $\beta \vee \gamma = \delta$ (see fig. 1.). Let $L_\alpha = \{o\}$, $L_\beta = \{a, b\}$, $a < b$, $L_\gamma = \{c\}$, $L_\delta = \{i\}$ (see fig. 2). Then L (see fig. 3) is the $\mathscr{L}$ — poproduct of L_p, $p \in P$, but not the $\mathscr{D}$ — poproduct.

In this paper the word problem for the $\mathscr{D}$ — poproduct is solved and the following theorem about the chain condition is proved: if $\mathfrak{m}$ is a regular cardinal greater than $\aleph_0$, then the $\mathscr{D}$ — poproduct of L_p, $p \in P$ does not contain a chain of the cardinality $\geqq \mathfrak{m}$ iff P and every L_p, $p \in P$ does not contain a chain of the cardinality $\geqq \mathfrak{m}$. The existence of the $\mathscr{D}$ — poproduct follows from [1].

We shall consider distributive lattices with 0,1. We shall use the methods of [6]. Similarly to [6], all results are applicable to the category of distributive lattices.

1. The word problem

Lemma 1. *Let L be a distributive lattice with* 0,1 *and let $x, y \in L$. Let M be a two element chain* $\{0,1\}$. *If $x \not\leq y$, then there exists a lattice homomorphism $\Phi : L \to M$ such that $\Phi(x) = 1$, $\Phi(y) = 0$.* The proof follows from the Stone theorem ([3], Theorem 7.15).

Let L be the $\mathcal{D}$-poproduct of the family $(L_p, p \in P)$. The lattice operations in L will be denoted by $\wedge, \vee$. Let $Q = \bigcup_{p\in P} L_p$. A finite nonempty subset $X \subseteq Q$ is said to be *reduced* if for every two distinct elements $x, y \in X$ holds: if $x \in L_p$, $y \in L_r$, $p, r \in P$, then $p \parallel r$. For every finite nonempty set X there are unique reduced sets $X^\wedge$, $X^\vee$ such that $\wedge X = \wedge(X^\wedge)$, $\vee X = \vee(X^\vee)$. If X is given, let $X' = \{\wedge(X \cap L_i) | i \in P_X\}$, where $P_X = \{p \in P | X \cap L_p \neq \emptyset\}$. Then $X^\wedge$ is the set of $x \in X'$ such that if $x \in L_p$, there is no $y \in L_r \cap X'$, $r < p$. The set $X^\vee$ is constructed dually.

Since L is a distributive lattice generated by Q, each element a of L can be written (in a nonunique manner) as $a = \wedge(\vee X | X \in J)$, where J is a finite family of finite reduced subsets of Q. Conversely any such family yields an element $\wedge(\vee X | X \in J)$ of L.

Theorem 1. *Let L be a distributive lattice generated by the poset $Q = \bigcup_{p\in P} L_p$. Then L is the $\mathcal{D}$ — poproduct of the L_p, $p \in P$ if and only if in L there holds: Let P_1, P_2 be finite subsets of P. Let $x_i \in L_i$ for $i \in P_1$ and $y_j \in L_j$ for $j \in P_2$. Then $\bigwedge_{i\in P_1} x_i \leq \bigvee_{j\in P_2} y_j$ implies that there is at least one pair i, j $(i \leq j)$, $i \in P_1$, $j \in P_2$ such that $x_i \leqq y_j$.*

Proof. The part „only if" of the theorem has been proved in [1], Lemma 1.9. We shall prove the sufficiency of the condition. Denote by L^* the poproduct of L_p, $p \in P$. We shall show $L^* \cong L$. Let f be the identity mapping $Q \to L$, then there exists a homomorphism $\Phi : L^* \to L$ extending f, hence for $q \in Q$ there holds $\Phi(q) = f(q) = q$. We shall show that Φ is an isomorphism. Φ maps L^* onto L, because L is generated by Q. To prove that Φ is one-to-one it is enough to prove that $a, b \in L^*$, $\Phi(a) \leqq \Phi(b)$ implies $a \leqq b$. Let $a, b \in L^*$, $\Phi(a) \leq \Phi(b)$. The elements a, b could be written in the form $a = \vee(\wedge X | X \in J)$, $b = \wedge(\vee Z | Z \in K)$, where X, Z are reduced subsets of Q, hence $\Phi(\wedge X) = \wedge X$, $\Phi(\vee Z) = \vee Z$. Because Φ is a homomorphism, for every pair X, Z we have $\wedge X \leq \Phi(a) \leqq \Phi(b) \leqq \vee Z$ in L, therefore according to the assumption there are $x \in X$, $z \in Z$ such that $x \leqq z$. Then in L^* there holds $\wedge X \leqq x \leqq z \leqq \vee Z$ for every pair X, Z. Therefore $a \leqq b$ in L^*. The theorem is proved.

Definition 1. *A finite family J of finite reduced subsets of Q is said to be a $\vee$-representation of $a \in L$ if $a = \vee(\wedge X | X \in J)$. The family J is said to be a $\wedge$-representation of $a \in L$ if $a = \wedge(\vee X | X \in J)$.*

Given a $\wedge$-representation J of an element $a \in L$ we can write, using the distributivity, $a = \vee(\wedge(F(J)) | F \in C(J))$, where $C(J)$ denotes the set of choice functions on J, that is, the set of functions $F : J \to \cup J$ such that $F(X) \in X$ for each $X \in J$. Hence $a = \vee(\wedge(F(J)^\wedge) | F \in C(J))$ holds. Since the set $C(J)$ is finite we can consider a subset $C_{\text{red}}(J) \subseteq C(J)$, the set of *re-*

duced choice functions such that the set $\{\wedge(F(J)^\wedge) | F \in C_{\mathrm{red}}(J)\}$ is the set of all maximal elements of the set $\{\wedge(F(J)^\wedge) | F \in C(J)\}$. Thus $a = \vee(\wedge(F(J)^\wedge) | F \in C_{\mathrm{red}}(J))$. The family $\{F(J)^\wedge | F \in C_{\mathrm{red}}(J)\}$ is said to be a *normal* $\wedge$ *— representation* of a. A *normal* $\vee$ *— representation* is defined dually.

Each element $a \in L$ has a normal $\vee$ — representation and a normal $\wedge$ — representation. From the definition it follows that if J_1 is a normal $\wedge$ — representation of a, $a = \vee(\wedge X | X \in J_1)$, then $X, X' \in J_1$ implies $X \not\subset X'$.

Lemma 2. *Let L be the distributive poproduct of the distributive lattices* $(L_p, p \in P)$. *If* X, Y *are finite reduced subsets of* Q, *then* $\wedge X \leqq \wedge Y$ *in L if and only if for each* $y \in Y$ *there is an* $x \in X$ *such that* $x \leqq y$.

Proof. The sufficiency is clear and the necessity follows from Theorem 1. Let $y \in Y$, then $\wedge X \leqq y$, X is reduced, so there exists $x \in X$ such that $x \leqq y$.

Theorem 2. *Let L be the distributive poproduct of the distributive lattices* $(L_p, p \in P)$. *Let* $a, b \in L$ *and let* J_1 *be a* $\vee$ *— representation of a and* J_2 *a normal* $\vee$ *— representation of b. Then* $a \leqq b$ *if and only if the following condition holds:*

For each $X \in J_1$ *there is a* $Y \in J_2$ *such that* $\wedge X \leqq \wedge Y$, *that is, for each* $y \in Y$ *there is an* $x \in X$ *such that* $x \leqq y$.

Corollary. *The normal* $\vee$ *— representation of any element of L is uniquely defined.*

Proof of Theorem 2. The sufficiency is clear. Now let $a, b \in L$, $a \leqq b$, $a = \vee(\wedge X | X \in J_1)$, $b = \vee(\wedge Y | Y \in J_2)$. Because J_2 is a normal $\vee$ — representation, it has arised from some $\wedge$ — representation K: $b = \wedge(\vee Z | Z \in K)$, where K is such that $J_2 = \{F(K)^\wedge | F \in C_{\mathrm{red}}(K)\}$ holds. Thus $\vee(\wedge X | X \in J_1) \leqq \wedge(\vee Z | Z \in K)$. It follows that for every pair $X \in J_1$, $Z \in K$ holds $\wedge X \leqq \vee(\wedge X | X \in J_1) \leqq \wedge(\vee Z | Z \in K) \leqq \vee Z$. Let $X \in J_1$. By Theorem 1 there are $x \in X$ and $G(Z) \in Z$ such that $x \leqq G(Z)$. Then $\wedge X \leqq x \leqq G(Z)$. Therefore for each $Z \in K$ there is $G(Z) \in Z$ such that $\wedge X \leqq G(Z)$. It follows $\wedge X \leqq \wedge(G(Z) | Z \in K) = \wedge(G(K)^\wedge)$. By the definition of $C_{\mathrm{red}}(K)$ there is $F \in C_{\mathrm{red}}(K)$ such that $\wedge(G(K)^\wedge) \leqq \wedge(F(K)^\wedge)$. Therefore to each $X \in J_1$ there exists $Y = F(K)^\wedge \in J_2$ so that $\wedge X \leqq \wedge Y$. The rest of the condition follows by Lemma 2. Thus the theorem is proved.

Proof of corollary. Let $a = \vee(\wedge X | X \in J_1) = \vee(\wedge Y | Y \in J_2)$ and let J_1, J_2 be normal $\vee$ — representations. Let $X \in J_1$. Then there exists $Y \in J_2$ such that $\wedge X \leqq \wedge Y$. Similarly there is $X' \in J_1$ such that $\wedge Y \leqq \wedge X'$. Then $\wedge X \leqq \wedge Y \leqq \wedge X'$, but because of the normality of J_1 we have $X = X' = Y$. Similar arguments prove that to every $Y \in J_2$ there is $X \in J_1$ such that $X = Y$. Thus $J_1 = J_2$.

2. The chain conditions for regular cardinals

Let $\mathfrak{m}$ be an infinite cardinal. A poset P is said to satisfy the strong (weak) *chain condition* for $\mathfrak{m}$, if every chain in P has cardinality $<\mathfrak{m}$ ($\leqq \mathfrak{m}$). It will be denoted $R(\mathfrak{m})$ ($R'(\mathfrak{m})$).

Theorem 3. *Let L be the distributive poproduct of the distributive lattices L_p, p P. Let $\mathfrak{m}$ be a regular cardinal, $\mathfrak{m} > \aleph_0$. Then there holds: L obeys $R(\mathfrak{m})$ if and only if P and each $L_p (p \in P)$ obey $R(\mathfrak{m})$. L obeys $R(\aleph_0)$ if and only if P is finite and each L_p $(p \in P)$ obey $R(\aleph_0)$, i.e. P and each L_p $(p \in P)$ are finite.*

Corollary 1. *Let $\mathfrak{m}$ be an infinite cardinal. Then there holds: L obeys $R'(\mathfrak{m})$ if and only if P and each L_p $p \in P$) obey $R'(\mathfrak{m})$.*

Corollary 1 immediately follows from Theorem 3, because $\mathfrak{m}' > \aleph_0$ is regular for $\mathfrak{m}'$ the succesor of $\mathfrak{n}$.

Corollary 2. *Let $\mathfrak{m}$ be a regular cardinal, $\mathfrak{m} > \aleph_0$. Then the following holds: The distributive free product of the distributive lattices L_i, $i \in I$ obeys $R(\mathfrak{m})$ if and only if each L_i $(i \in I)$ obeys $R(\mathfrak{m})$.*

Corollary 2 implies Theorem 4 from [5].

Proof of the Theorem 3.

1) the necessity is clear: if we take the ordinal sum of $\mathfrak{n}$ lattices, $\mathfrak{n} \geqq \mathfrak{m}$ and if P is a chain with P $\mathfrak{n}$, or the free product of lattices at least one of which does not obey $R(\mathfrak{m})$, then in L, $R(\mathfrak{m})$ fails to hold.
2) the sufficiency: Throughout the proof, the following lemma proved in [3] and [4] will be useful:

Lemma 3. *Let Λ be a chain and let $\mathscr{H} = (H_\lambda | \lambda \in \Lambda)$ be a family of finite sets. For each pair λ, μ such that $\lambda \leqq \mu$ let there be a relation $\Phi_{\lambda\mu} \leqq H_\lambda \times H_\mu$ with the domain (codomain) H_λ satisfying the two conditions:*
(i) *$\Phi_{\lambda\lambda}$ is equality for all $\lambda \in \Lambda$;*
(ii) *if $\lambda \leqq \mu \leqq \nu$, then $\Phi_{\mu\nu} . \Phi_{\lambda\mu} \leqq \Phi_{\lambda\nu}$.*
Then there is a family $(x_\lambda \in H | \lambda \in \Lambda)$ such that $\langle x_\lambda, x_\mu \rangle \in \Phi_{\lambda\mu}$ if $\lambda \leqq \mu$.

Now let L be the distributive poproduct of the distributive lattices L_p $(p \in P)$ with 0, 1. Let P obey $R(\mathfrak{m})$ and let each L_p $(p \in P)$ obey $R(\mathfrak{m})$ for $\mathfrak{m} > \aleph_0$ and regular.

If J is a — representation of $a \in L$, we call $\bar{\bar{J}}$ the *rank* of the representation and $\sum_{X \in J} X$ the *length* of the representation ($a = \bigvee(\bigwedge X | X \in J)$).

If $H \subseteq L$, then a — representation of H, $J(H)$, is a family $(J_a | a \in H)$, where J_a is a — representation of a. If n is an integer and the rank of J_a is n for each $a \in H$, then $J(H)$ is said to have the rank n. A — representation $J(H)$ of H is said to be *special* if for each a. $b \in H$, the following conditions hold ($J_a \in J(H)$ and $J_b \in J(H)$ are — representations of a and b, respectively):

(1) if J_a 1, then $x, y \in X_a$, $\{X_a\} = J_a$ and $x \leqq y$ imply that x y; if $J_a > 1$, then $X, Y \in J_a$ and $\wedge X \leqq \wedge Y$ imply that X Y;
(2) if $J_a = 1$, $J_b = 1$, then $a \leqq b$ imply that for each $y \in Y$, $\{Y\}$ J_b there is an $x \in X$, $\{X\}$ J_a such that $x \leqq y$;
if $J_a > 1$ or $J_b > 1$, then $a \leqq b$ imply that for each $X \in J_a$ there is Y J_b such that $\wedge X \leqq$ Y.

Each $H \subseteq L$ has a special — representation: by Theorem 2 a normal representation is special. A special representation need not be normal as the example in [6] shows.

We shall show that if C is a chain in L, then $C < \mathfrak{m}$. Let $J(C)$ be a special representation of C. For each $n < \aleph_0$ let C_n $\{a$ C rank J_a $n\}$. Then $J(C_n) = (J_a\ a \in C_n)$ is a special — representation of C_n of rank n. We shall show by induction according to a rank of the representation that $C_n < \mathfrak{m}$.

Lemma 4. *Let C be a chain in L that has a special — representation of rank 1. Then $C < \mathfrak{m}$.*

Proof. Let $J(C)$ be a special $\vee$ representation of C of rank 1. For each integer n let $C^{(n)} = \{a \in C \mid \text{length } J_a \; n\}$. Then $J(C^{(n)})$ $(J_a\ a \in C^{(n)})$ is a special representation of $C^{(n)}$ of length n. We shall show by induction according to the length of the representation that $\overline{\overline{C^{(n)}}} < \mathfrak{m}$.

If n 1, then $C^{(1)}$ is a chain in Q, so $\overline{\overline{C^{(1)}}} < \mathfrak{m} \cdot \mathfrak{m}$ $\mathfrak{m}$.

Now suppose that for all $k < n$ there is $\overline{\overline{C^{(k)}}} < \mathfrak{m}$ and $C^{(n)} \geq \mathfrak{m}$.

For $a \in C^{(n)}$, $J_a = \{X_a\}$, a $\wedge X_a$. We use Lemma 3 for A $C^{(n)}$, H_λ $= X_a$. $a \leqq b : \Phi_{ab} = \{\langle x, y\rangle \mid x \in X_a,\ y \in X_b,\ x \leqq y\}$. Then there is a family $\Psi = (x_a\ x_a \in X_a,\ a \in C)$ such that $x_a, x_b\rangle \in \Phi_{ab}$ if $a \leqq b$. Since Ψ is a special — representation of rank 1 and length 1 of a chain in L, $\Psi < \mathfrak{m}$. Because $\mathfrak{m}$ is regular, there is a subset $C^{(n)\prime} \subseteq C^{(n)}$ such that $C^{(n)\prime} \geqq \mathfrak{m}$ and if $a, b \in C^{(n)\prime}$ and $x_a, x_b \in \Psi$, then $x_a = x_b$. The family $\mathscr{G}$ $(\{X_a$ $\{x_a\}\}\ a \in C^{(n)\prime})$ has cardinality $\geqq \mathfrak{m}$. $\mathscr{G}$ is a representation of rank 1, length n 1 of some subset $G \subseteq L$. It is a special representation — condition (1) follows from the speciality of $J(C^{(n)})$ and condition (2) as well: let $a \leqq b$, $a, b \in C^{(n)}$ and $y \in X_b - \{x_b\}$. Then there is $x \in X_a$ such that $x \leqq y$. If x x_a, then $x_b = x_a = x$, hence $x_b \leqq y$ and the speciality of $J(C^{(n)})$, $y \in C^{(n)}$ implies $x_b = y$. Thus $x \neq x_a$ and $\mathscr{G}$ is a special representation of the chain G. Thus $G < \mathfrak{m}$, which is a contradiction. Therefore $\overline{\overline{C^{(n)}}} < \mathfrak{m}$. Since $\mathfrak{m} > \aleph_0$ and regular, there holds $C = \sum_{n \; \aleph_0} \overline{\overline{C^{(n)}}} < \mathfrak{m}$. Lemma 4 is proved.

Lemma 5. *Let C be a chain in L that has a special — representation of rank n. Then $\bar{\bar{C}} < \mathfrak{m}$.*

Proof. Let n be the smallest integer such that there is a chain $C \subsetneqq L$ where $\bar{\bar{C}} \geqq \mathfrak{m}$ and C has a special representation $J(C)$ of rank n. Note that by Lemma 4 $n > 1$. We use Lemma 3 for $A = C$, $H_\lambda = J_a$, $a < b \cdot \Phi_{ab} = \{X, Y \mid X \in J_a, Y \in J_b \wedge X \leqq \wedge Y\}$. Then there is a family $\chi = (X_a \mid X_a \in J_a, J_a \in J(C), a \in C)$ such that $\wedge X_a \leqq \wedge X_b$, whenever $a \leqq b$. Since χ is a special — representation of rank 1 of a chain in L, by lemma 4 $\bar{\bar{\chi}} < \mathfrak{m}$. Since $\mathfrak{m}$ is regular, there is a subset $C' \subseteq C$ such that $\bar{\bar{C'}} \geqq \mathfrak{m}$ and if $a, b \in C'$ and $X_a, X_b \in \chi$, then $X_a = X_b$. The family $\mathscr{H} = (J_a - \{X_a\} \mid a \in C')$ has a cardinality $\geqq \mathfrak{m}$. $\mathscr{H}$ is a representation of rank $n - 1$ of some subset $H \subseteq L$. It is a special representation, condition (1) follows from the speciality of $J(C)$ and condition (2) in the first case from Lemma 2 and in the second one as follows: let $a \leqq b$, $a, b \in C'$ and $X \in J_a - \{X_a\}$. Then there is $Y \in J_b$ such that $\wedge X \leqq \wedge Y$. If $Y = X_b$, then $X_a = X_b = Y$, hence $X_a = Y$, $\wedge X \leqq \wedge Y = \wedge X_a$ and the speciality of $J(C)$, $a \in C$ implies $X = X_a$. Thus $Y \in J_b - \{X_b\}$ and so H is a chain with a special — representation $\mathscr{H}$. However, rank $\mathscr{H} = n - 1$ and $\bar{\bar{H}} \geqq \mathfrak{m}$, contradicting the minimality of n. Lemma 5 is proved.

Now let C be a chain in L that has a special — representation C. Then $C = \bigcup_{\aleph_0} C_n$, where $C_n = \{a \in C \mid \text{rank } J_a = n\}$. It was shown that $\bar{\bar{C_n}} < \mathfrak{m}$. Since $\mathfrak{m} > \aleph_0$ and regular, $\bar{\bar{C}} = \sum_{\aleph_0} \bar{\bar{C_n}} < \mathfrak{m}$ holds. The first part of theorem 3 is proved.

To prove the second part of the theorem, we note that an infinite distributive lattice contains an infinite chain. Let P be finite and each L_p ($p \in P$) contain only finite chains, then each L_p is finite, $Q = \bigcup_{p \in P} L_p$ is a finite set and $\bar{\bar{L}} \leq 2^{2^Q}$. Conversely, if some L_p contain an infinite chain or P is infinite, then Q is infinite and $L \subsetneqq Q$ is infinite. Theorem 3 is proved.

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