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Matematický časopis, Vol. 24 (1974), No. 4, 371--380

Persistent URL: http://dml.cz/dmlcz/126557

OSCILLATION OF SOLUTIONS OF NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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We consider the nonlinear delay differential equation of the n-th order of the following form

(1) $$y^{(n)}(t) + p(t)f(y[h_1(t)], \ldots, y^{(n-2)}[h_{n-1}(t)]) = 0, \quad n \geq 2,$$

where

(2) $$p \in C[R_+ = [0, \infty), (0, \infty)]$$

(3) $f \in C[R^{n-1}, R]$, $x_1 f(x_1, \ldots, x_{n-1}) > 0$ for $x_1 \neq 0$, $(x_1, \ldots, x_{n-1}) \in R^{n-1}$,

(4) $h_i \in C[R_+, R]$, $h_i(t) \leq t$ for $t \in R_+$, $\lim_{t \to \infty} h_i(t) = \infty$, $(i = 1, \ldots, n-1)$.

The following papers [2, 3, 5, 6, 7, 8, 9] deal with the oscillatory properties of solutions of some nonlinear delay differential equations of the n-th order.

In this paper we shall prove a necessary and sufficient condition for all solutions of the equation (1) to be oscillatory in the case n is even and to be either oscillatory or tending monotonically to zero as $t \to \infty$ when n is odd.

Let

$$H(t) = \min\{H_1(t), \ldots, H_{n-1}(t)\},$$

where

$$H_i(t) = \inf\{h_i(x);\ x \geq t \in R_+\}, \quad (i = 1, \ldots, n-1)$$

and

$$D_K = \{(x_1, \ldots, x_n) : K \leq |x_1| \leq 2K, \quad |x_i| \leq K, \quad (i = 1, \ldots, n)\}$$

A function $y(t)$ is said to be a solution of (1) on the interval $[H(t_0), \infty)$, $t_0 \geq 0$ with an initial function $\varphi \in C^{n-2}[[H(t_0), t_0], R]$, if

(i) $y(t) \in C^n[[t_0, \infty), R]$, $y(t) \in C^{n-2}[[H(t_0), \infty), R]$

(ii) $y^{(k)}(t) = \varphi^{(k)}(t)$ for $t \in [H(t_0), t_0]$, $(k = 0, 1, \ldots, n-2)$,

$$y^{(n-1)}(t_{0+}) = y_0^{(n-1)}$$

(*iii*) $y(t)$ satisfies (1) for every $t \geq t_0$.

In the following we shall always suppose that all the functions in (1) and the initial conditions φ, $y^{(n-1)}$ guarantee the existence and uniqueness of a solution of the equation (1) for every $t \geq t_0$.

A solution $y(t)$ of (1) is called *oscillatory* on the interval $J = [t_0, \infty)$, if the set of zeros of $y(t)$ is not bounded from the right. A solution $y(t)$ of (1) is called *nonoscillatory* on J, if there exists a number $a \in J$ such that $y(t) \neq 0$ for $t > a$.

Theorem 1. *Let the functions in* (1) *satisfy* (2), (3), (4) *and in addition*
(*a*) $f(x_1, \ldots, x_{n-1})$ *is nondecreasing* [*nonincreasing*] *in* $x_1, x_2, x_4, \ldots, x_{n-2} \in R$ [*in* $x_3, x_5, \ldots, x_{n-1} \in R$] *for n even.*
If n is odd then $f(x_1, \ldots, x_{n-1})$ *is nondecreasing* [*nonincreasing*] *in* $x_1, x_3, \ldots, x_{n-2} \in R$ [*in* $x_2, x_4, \ldots, x_{n-1} \in R$].
Then

$$\int^{\infty} t^{n-1} p(t)\, dt < \infty \tag{5}$$

is a necessary and sufficient condition for the existence of a solution $y(t)$ of (1) *with the property*

$$(V) \quad |y(t)| \to C, \quad |y^{(k)}(t)| \to 0 \textit{ for } t \to \infty,\ 0 < C \textit{ constant},$$
$$(k = 1, \ldots, n-1).$$

Proof. I. The necessary condition. Let $y(t)$ be a nonoscillatory solution of (1), having the property (V). Without the loss of generality we can suppose that $y(t) > 0$ for $t \geq H(t_1) \geq t_0 \in R$. Then $y[h_1(t)] > 0$ for $t \geq t_1$. Integrating (1) j- times ($j = 1, \ldots, n-1$) from $t (t \geq t_1)$ to ∞, we obtain

$$(-1)^{j-1} y^{(n-j)}(t) = \int_t^{\infty} \frac{(s-t)^{j-1}}{(j-1)!}\, p(s) \tag{6}$$
$$\times f(y[h_1(s)], \ldots, y^{(n-2)}[h_{(n-1)}(s)])\, ds, \quad (j = 1, \ldots, n-1), \quad t \geq t_1.$$

From (6), with regard to (2), (3), for $j = n-1$, we get $(-1)^n y'(t) > 0$.

If n is even, then the nonoscillatory solution $y(t)$ of (1), having the property (V), is increasing and therefore $0 < C - y(t) < \infty$.

When n is odd, then $y(t)$ is a decreasing solution of (1), having the property (V) and therefore $0 < y(t) - C < \infty$.

Integrating (6), for $j = n-1$, from t to ∞, we obtain

$$(7)\qquad (-1)^n(C - y(t)) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s) f(y[h_1(s)],\dots,y^{(n-2)}[h_{n-1}(s)])\,\mathrm{d}s$$

As the function f is continuous, there exists $t_2 \geq t_1$, such that for $t \geq t_2$ we have

$$f(y[h_1(t)],\dots,y^{(n-2)}[h_{n-1}(t)]) \geq \frac{1}{2} f(C, 0, \dots, 0) > 0\,.$$

From (7), using the last inequality, we get

$$\infty > (-1)^n(C - y(t)) > \frac{1}{2} f(C, 0, \dots, 0) \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\, p(s)\,\mathrm{d}s, \quad t \geq t_2$$

The last inequality implies

$$\int_t^\infty s^{n-1} p(s)\,\mathrm{d}s < \infty$$

and thus the necessary condition is proved.

II. The sufficient condition. The existence of a solution $y(t)$ of (1), having the property (P), will be proved by the method of successive approximations.

a) Let n be an even number.

Let us define the sequence of functions $\{y_m(t), \dots, y_m^{(n-1)}(t)\}_{m=0}^{\infty}$, $t > H(T)$ as follows:

$$(8)\qquad y_0(t) = \frac{C}{2}, \quad y_0^{(n-i)}(t) = 0, \quad (i = 1, \dots, n-1), \quad t \geq H(T), \quad C > 0$$

$$(9)\qquad y_{m+1}(t) = \begin{cases} \dfrac{C}{2} + \displaystyle\int_T^t \frac{(s-T)^{n-1}}{(n-1)!}\, p(s) f(y_m[h_1(s)],\dots,y_m^{(n-2)}[h_{n-1}(s)])\,\mathrm{d}s \,+ \\[2ex] + \displaystyle\int_t^\infty \frac{(s-T)^{n-1} - (s-t)^{n-1}}{(n-1)!}\, p(s) f(y_m[h_1(s)],\dots,y_m^{(n-2)} \times \\[2ex] \times [h_{n-1}(s)])\,\mathrm{d}s, \quad t \geq T \\[2ex] \dfrac{C}{2}, \quad H(T) \leq t \leq T \end{cases}$$

$$(10)\qquad y_{m+1}^{(n-i)}(t) = \begin{cases} (-1)^{i-1}\displaystyle\int_t^{\infty} \frac{(s-t)^{i-1}}{(i-1)!}\, p(s) f(y_m[h_1(s)], \dots, y_m^{(n-2)} \\ \quad \times [h_{n-1}(s)])\,\mathrm{d}s, \quad t \geq T, \quad (i = 1, \dots, n-1) \\ 0, \quad H(T) \leq t < T, \quad (i = 1, \dots, n-1), \end{cases}$$

where T is chosen so that

$$(11)\qquad M \int_T^{\infty} (s-T)^{i-1}\, p(s)\,\mathrm{d}s < \frac{C}{2}, \quad i = 1, \dots, n$$

and

$$M = \max\{1, M_{C/2}, \sup_{D_{C/2}} |f(x_1, \dots, x_{n-1})|\}.$$

By mathematical induction, with regard to (8)–(11) and the assumption (a), it is easy to prove that

$$(12)\qquad \frac{C}{2} < y_m(t) \leq C, \quad 0 < (-1)^{i-1} y_m^{(n-i)}(t) \leq \frac{C}{2}, \quad t \geq T,$$
$$(i = 1, \dots, n-1;\ m = 0, 1, \dots)$$

$$(13)\qquad y_{m+1}(t) \geq y_m(t), \quad (-1)^{i-1} y_{m+1}^{(n-i)}(t) \geq (-1)^{i-1} y_m^{(n-i)}(t), \quad t \geq T,$$
$$(i = 2, \dots, n-1;\ m = 0, 1, \dots)$$

From (8), (9), (10) it is obvious that the functions $y_m^{(n-i)}(t)$, $(i = 1, \dots, n)$ are continuous for $t \geq T$.

In view of (12), (13) the sequences $\{y_m^{(n-i)}(t)\}_{m=0}^{\infty}$, $(i = 2, \dots, n, t \geq T)$ of the continuous functions are uniformly convergent on $[T, A] \subset [T, \infty)$ $(A \in R,\ A > T)$ and convergent on $[T, \infty)$, i.e. $\lim y_m^{(n-i)}(t) = y^{(n-i)}(t)$, $(i = 2, \dots, n)$ exist on $[T, \infty)$.

The function $y(t)$ is a solution of the integral equation

$$y(t) = \begin{cases} \dfrac{C}{2} + \displaystyle\int_T^{t} \frac{(s-T)^{n-1}}{(n-1)!}\, p(s) f(y[h_1(s)], \dots, y^{(n-2)}[h_{n-1}(s)])\,\mathrm{d}s \\ + \displaystyle\int_t^{\infty} \frac{(s-T)^{n-1} - (s-t)^{n-1}}{(n-1)!}\, p(s) f(y[h_1(s)], \dots, y^{(n-2)}[h_{n-1}(s)])\,\mathrm{d}s, \quad t > T \\ \dfrac{C}{2}, \quad H(T) < t < T \end{cases}$$

and it has the property (P).

If we carry the initial point T to the point $t_0 = \sup\{t;\ H(t) < T,\ t \geq T\}$, then the solution $y(t)$ with the property (Γ) is the solution of (1) on the interval $[H(t_0) \geq T, \infty)$.

b) Let n be an odd number.

Let us define the sequence of functions $\{y_m(t), \ldots, y_m^{(n-1)}(t)\}_{m=0}^{\infty}$, $t \geq H(T)$, as follows:

$$(8') \qquad y_0(t) = C > 0, \quad y_0^{(n-i)}(t) = 0, \quad (i = 1, 2, \ldots, n-1), \quad t \geq H(T),$$

$$(9') \qquad y_{m+1}(t) = \begin{cases} C + \int\limits_t^{\infty} \dfrac{(s-t)^{n-1}}{(n-1)!}\, p(s) f(y_m[h_1(s)], \ldots, y_m^{(n-2)}[h_{n-1}(s)])\,\mathrm{d}s, & t \geq T \\ y_{m+1}(T), \quad H(T) \leq t < T \end{cases}$$

and

$$y_{m+1}^{(n-i)}(t), \quad (i = 1, 2, \ldots, n-1)$$

is defined by (10).

The point T is chosen so that

$$(11') \qquad M \int\limits_T^{\infty} (s-T)^{i-1} p(s)\,\mathrm{d}s < C, \quad i = 1, \ldots, n,$$

where $M = \max\{1, M_C = \sup\limits_{D_C} f(x_1, \ldots, x_{n-1})\}$.

By mathematical induction, in view of (8'), (9'), (10), (11') and the assumption (a), it easy to prove that

$$(12') \qquad C < y_m(t) \leq 2C, \quad 0 < (-1)^{i-1} y_m^{(n-i)}(t) < C, \quad t \geq T,$$
$$(i = 1, \ldots, n-1, \quad m = 0, 1, \ldots)$$

$$(13') \qquad (-1)^{i-1} y_{m+1}^{(n-i)}(t) > (-1)^{i-1} y_m^{(n-i)}(t), \quad t > T,$$
$$(i = 2, \ldots, n;\quad m = 0, 1, \ldots)$$

From (8'), (9'), (10) it is obvious that the functions $y_m^{(n-i)}(t)$, $(i = 1, \ldots, n)$ are continuous for $t > T$.

(12'), (13') imply that the sequences $\{y_m^{(n-i)}(t)\}_{m=0}^{\infty}$, $(i = 2, \ldots, n,\ t \geq T)$ of the continuous functions are uniformly convergent on $[T, B] \subset [T, \infty)$, $(B \in R,\ B > T)$ and convergent on $[T, \infty)$, i.e. $\lim\limits_{m\to\infty} y_m^{(n-i)}(t) = y^{(n-i)}(t)$, $(i = 2, \ldots, n)$ exist for $t \geq T$.

The function $y(t)$ is a solution of the integral equation

$$y(t) = \begin{cases} C - \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} p(s) f(y[h_1(s)], \ldots, y^{(n-2)}[h_{n-1}(s)])\, ds, & t \geq T \\ y(T_+), \ H(T) < t < T \end{cases}$$

and it has the property (V).

Similarly as above it follows that $y(t)$, having the property (V), is the solution of (1) on the interval $[H(t_0) - T, \infty)$.

Thus Theorem 1 is proved.

Lemma 1. *Let $y(t)$ be a function such that its derivatives up to order $n-1$ inclusive by are absolutely continuous and of constant sign in the interval (t_0, ∞) and $y(t)y^{(n)}(t) \leq 0$. Then there exists a number $k \in \{0, 1, \ldots, n-1\}$, $n-k$ is odd and such that*

$$(14) \qquad y(t)y^{(i)}(t) \geq 0, \quad (i = 0, 1, \ldots, k),$$

$$(-1)^{n+i+1} y(t)y^{(i)}(t) > 0, \quad (i = k+1, \ldots, n), \quad t \geq t_0,$$

$$(15) \qquad |y^{(k)}(t)| > t^{n-k-1}|y^{(n-1)}(2^{n-k-1}t)|, \quad t > t_0,$$

$$(16) \qquad |y^{(k-i)}(t)| \geq B_i t^{n-k+i-1}|y^{(n-1)}(t)|, \quad (i = 1, 2, \ldots, k), \quad t \geq 2^{n-k}t_0,$$

where

$$B_i = \frac{2^{-(n-k+i)^3}}{(n-k)\ldots(n-k+i-1)}.$$

The proof of Lemma 1 can be found in [6, Lemma 2].

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled and, in addition, suppose that*

(b) $h_i(t) = t + g_i(t)$, $-r \leq g_i(t) \leq 0$, $(i = 1, \ldots, n-1)$, $t \in R_+$, $r > 0$

(c) *there exists a number $\alpha > 1$ such that*

$$\liminf_{x_1 \to \infty} \frac{|f(x_1, x_2, \ldots, x_{n-1})|}{|x_1|^{\alpha}} \neq 0.$$

1. *If n is an even number, then a necessary and sufficient condition for all solutions of the equation (1) to be oscillatory is*

$$(17) \qquad \int^{\infty} t^{n-1} p(t)\, dt = \infty.$$

2. *When n is an odd number, then (17) is the necessary and sufficient condition for all solutions of (1) to be either oscillatory or to tend monotonically to zero as $t \to \infty$ together with their first $n-1$ derivatives.*

Proof. I. The necessary condition follows from Theorem 1.

II. Condition (17) is sufficient. Let (17) hold and let $y(t)$ be a solution of the equation (1) such that $y(t) > 0$ for $t \geq t_0 - r$. Then $y_t[g_1(t)] > 0$ for $t \geq t_0$ and with regard to (2), (3), we have

$$y^{(n)}(t) = -p(t)f(y_t[g_1(t)], y'_t[g_2(t)], \ldots, y_t^{(n-2)}[g_{n-1}(t)]) < 0, \; t > t_0.$$

From $y^{(n)}(t) < 0$ it follows that there exists $t_1 \geq t_0$ such that $y^{(j)}(t)$ ($j = 0, 1, \ldots, n-1$) have constant sign for $t > t_1$. Then, by Lemma 1, for $y(t)$ and its derivatives (14)—(16) hold.

1. Let n be an even number. Then, by Lemma 1, there exists an odd number $k \in \{1, \ldots, n-1\}$ such that $y(t)y^{(i)}(t) > 0$, $i = 0, \ldots, k$, $t > t_1$. Therefore, in view of Theorem 1, we have $y(\infty) = \infty$.

2. If n is an odd number, then by Lemma 1, there exists a number $k \in \{0, 2, \ldots, n-1\}$ such that $n + k$ is odd. Let $k = 0$. Then (14) implies $y'(t) < 0$ for $t > t_1$. By Theorem 1 and (14) we have $y^{(i)}(\infty) = 0$, $i = 0, 1, \ldots, n-1$. Thus we proved that $y(t)$ tends monotonically to zero as $t \to \infty$ together with its first $n-1$ derivatives.

If $k \in \{2, 4, \ldots, n-1\}$, then $y'(t) > 0$, $y''(t) > 0$ and therefore $y(\infty) = \infty$.

In general, the case $y(\infty) = \infty$, $y'(t) > 0$ for $t \geq t_1$ can occur only for $k \in \{1, 2, \ldots, n-1\}$.

Integrating (1) from $t (t \geq t_1)$ to ∞ and then using $y^{(n-1)}(\infty) = C \geq 0$, we obtain

$$\text{(18)} \qquad y^{(n-1)}(t) \geq y^{(n-1)}(t) - y^{(n-1)}(\infty) =$$

$$= \int_t^\infty p(s)f(y_s[g_1(s)], y'_s[g_2(s)], \ldots, y_s^{(n-2)}[g_{n-1}(s)])\, ds, \; t > t_1$$

A) Let $k = 1$. Then (15) implies

$$y'(t) \geq t^{n-2}y^{(n-1)}(2^{n-2}t), \quad t \geq t_1$$

From (18), with regard to the last inequality and the assumption (a), we have

$$y'(t) > t^{n-2} \int_{2^{n-2}t}^\infty p(s)f(y_s[-r], y'(g_2(s)], \ldots, y_s^{(n-2)}(g_{n-1}(s)])\, ds, \; t \geq t_1.$$

Multiplying the last inequality by $\{y_t[-r]\}^{-\alpha}$ and using $y'(t) > 0$, $y''(t) < 0$ and assumption (a), we get

$$\text{(19)} \qquad \frac{y'_t[-r]}{\{y_t[-r]\}^\alpha} \geq \frac{y'(t)}{\{y_t[-r]\}^\alpha} \geq$$

$$t^{n-2} \int_{2^{n-2}t}^{\infty} p(s) \frac{f(y_s[-r], y'_s[g_2(s)], \ldots, y_s^{(n-2)}[g_{n-1}(s)])}{\{y_s[-r]\}^{\alpha}} \, \mathrm{d}s$$

According to the assumption (c) and the fact $\lim_{t\to\infty} y_t[-r] = \infty$, we can choose $T_1 > t_1$ such that for $s > 2^{n-2}T_1$ we have

$$\frac{f(y_s[-r], y'_s[g_2(s)], \ldots, y_s^{(n-2)}[g_{n-1}(s)])}{\{y_s[-r]\}^{\alpha}} \geq d > 0$$

Then (19) implies

$$\frac{y'_t[-r]}{\{y_t[-r]\}^{\alpha}} > d\, t^{n-2} \int_{2^{n-2}t}^{\infty} p(s)\, \mathrm{d}s, \quad t \geq T_1$$

Integrating the last inequality from T_1 to $t (t > T_1)$, we obtain

(20)
$$\frac{1}{\alpha - 1} \left(\frac{1}{\{y_{T_1}[-r]\}^{\alpha-1}} - \frac{1}{\{y_t[-r]\}^{\alpha-1}} \right)$$

$$\geq \frac{d}{n-1} \int_{2^{n-2}T_1}^{2^{n-2}t} [(2^{2-n}s)^{n-1} - T_1^{n-1}]\, p(s)\, \mathrm{d}s + d\, \frac{t^{n-1} - T_1^{n-1}}{n-1}$$

$$\cdot \int_{2^{n-2}t}^{\infty} p(s)\, \mathrm{d}s \geq \frac{d}{n-1} \int_{2^{n-2}T_1}^{2^{n-2}t} [(2^{2-n}s)^{n-1} - T_1^{n-1}]\, p(s)\, \mathrm{d}s .$$

The first expression in (20) is bounded because $\lim_{t\to\infty} y_t[-r] = \infty$, $\alpha > 1$ and therefore

(21)
$$\int_{2^{n-2}T_1}^{\infty} [(2^{2-n}s)^{n-1} - T_1^{n-1}]\, p(s)\, \mathrm{d}s < \infty .$$

If we choose $2^{2-n}s \geq 2^{2-n}t_2 \geq 2^{1/(n-1)}T_1$, then $(2^{2-n}s)^{n-1} - T_1^{n-1} \geq (2^{-n}s)^{n-1}/2$ and so (21) implies

$$\int_{t_2}^{\infty} s^{n-1} p(s)\, \mathrm{d}s < \infty.$$

which contradicts (17).

B, Let $k \in \{2, 3, \ldots, n-1\}$. Then from (16) we get

(22)
$$y'(t) \geq B_{k-1} t^{n-2} y^{(n-1)}(t), \quad t > t_3 = 2^{n-k} t_1,$$

where

$$B_{k-1} = \frac{1}{(n-1)^2 (n-k) \dots (n-2)}.$$

From (18), in view of the monotonicity of the function $y^{(n-1)}(t)$, the assumption (a) and (22), we get

$$y'[t-r] > B_{k-1}[t-r]^{n-2} \int_t^{\sigma} p(s) f(y_s[-r], y'_s[g_2(s)], \dots, y_s^{(n-2)}[g_{n-1}(s)])\, ds,$$

$$t > t_1 > t_3 + r.$$

Further, exactly as in the case A, (i.e. we multiply the last inequality by $\{y_t[-r]\}^{-\alpha}$, use (c) and finally we integrate from $T_2 = 2^{n-2}T_1 + t_1$ to t) we get

$$\int_{T_2}^{\sigma} \frac{(s-r)^{n-2} - (t_1 - r)^{n-1}}{\dots} p(s)\, ds < \infty.$$

That contradicts (17).

The proof of Theorem 2 is complete.

Theorem 2 generalizes Theorem 2 [4].

Theorem 1 and Theorem 2 can be generalized to the equation

(23) $$y^{(n)}(t) + \sum_{i=1}^{m} p_i(t) f_i(y[h_{1,i}(t)], \dots, y^{(n-2)}[h_{n-1,i}(t)]) = 0, \quad n \geq 2.$$

The proof of Theorem 1′ [Theorem 2′] is very similar to the proof of Theorem 1 [Theorem 2] and we omit it here.

Theorem 1′. *Let the functions p_i, f_i, $h_{j,i}$ ($i = 1, \dots, m$; $j = 1, \dots, n-1$) satisfy (2), (3), (4) and, in addition let f_i ($i = 1, \dots, n-1$) satisfy the assumption (a) in Theorem 1.*

$$\sum_{i=1}^{m} \int^{\infty} t^{n-1} p_i(t)\, dt < \infty$$

is a necessary and sufficient condition for the existence of a solution $y(t)$ of (23), having the property (Γ).

Theorem 2′. *Let the assumptions of Theorem 1′ be satisfied and in addition let*

(b) $h_{j,i}(t) = t + g_{j,i}(t)$, $-r < g_{j,i}(t) < 0$ ($j = 1, \dots, n-1$; $i = 1, \dots, m$),

$$t \in R_+, \quad r > 0;$$

(c) *let there exist a number $\alpha > 1$ such that*

$$\liminf_{x_1 \to \infty} \frac{|f_i(x_1, \ldots, x_{n-1})|}{|x_1|^\alpha} \neq 0, \quad (i = 1, \ldots, m)$$

1) *If n is an even number, then*

$$\sum_{i=1}^{m} \int^{\infty} t^{n-1} p_i(t)\, dt = \infty \tag{25}$$

is a necessary and sufficient condition for all solutions of (23) *to be oscillatory.*

2) *Let n be an odd number. Then* (25) *is a necessary and sufficient condition for all solutions of* (23) *to be either oscillatory or to tend monotonically to zero as $t \to \infty$ together with their first $n - 1$ derivatives.*

Acknowledgment. The author wishes to thank the referee for his helpful suggestions.

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Received January 5, 1973

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