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ON THE PROJECTIVE TENSOR PRODUCT OF VECTOR-VALUED MEASURES II

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1. Let X and Y be locally convex spaces. Let $(S, \mathscr{S})$ and $(T, \mathscr{T})$ be measurable spaces ($\mathscr{S}$ and $\mathscr{T}$ being sigma rings [1]), and let $\mu : \mathscr{S} \to X$ and $\nu : \mathscr{T} \to Y$ be sigma additive vector-valued measures. As shown in [10], there need not exist, in general, a projective tensor product of the vector measures μ and ν, i. e. a sigma additive vector measure $\lambda : \mathscr{S} \otimes_\sigma \mathscr{T} \to X \hat{\otimes} Y$ such that $\lambda(E \times F) = \mu(E) \otimes \nu(F)$, $E \in \mathscr{S}$, $F \in \mathscr{T}$.

Those locally convex spaces X, for which for any locally convex space Y and any vector measure $\mu : \mathscr{S} \to X$ and $\nu : \mathscr{T} \to Y$ such a measure λ exists, we called in [4] admissible factors and we have found some such locally convex spaces. For example, every nuclear space is an admissible factor [3] and every Banach space with an absolute basis is an admissible factor [4].

In this paper we give some conditions imposed on either a vector measure μ or ν under which there exists a projective tensor product of these vector measures. For example, the finiteness of the variation of either μ or ν is such a condition.

2. Let a locally convex topology on X be generated by a family of semi-norms $\{| \ |_\alpha\}_{\alpha \in A}$. Similarly $\{| \ |_\beta\}_{\beta \in B}$ for Y.

The projective tensor product of X and Y is a locally convex space $X \otimes Y$, the topology of which is generated by a family of semi-norms $\gamma = \alpha \otimes \beta$, $\alpha \in A$, $\beta \in B$:

$$|z|_\gamma = \inf \{ \sum_{i=1}^{n} |x_i|_\alpha \, |y_i|_\beta : z = \sum_{i=1}^{n} x_i \otimes y_i \}$$

with the property that $|x \otimes y|_\gamma = |x|_\alpha \, |y|_\beta$ for all $x \in X$, $y \in Y$ [cf. 2, 8, 11, 13]. The locally convex space $X \otimes Y$ can be imbedded in a complete locally convex space which is unique (to within isomorphism) and is denoted by $X \hat{\otimes} Y$ (cf. [13], p. 94).

In the sequel we shall make use of the following

Theorem 1. *Let T be a set, $\mathscr{T}$ a sigma ring of subsets of T, and $\nu : \mathscr{T} \to Y$*

a vector measure. Then for every $\beta \in B$ there exists a finite nonnegative measure v_β such that

$$\lim_{v_\beta(F)\to 0} |\nu(F)|_\beta = 0$$

and

$$v_\beta(F) \leqq \sup_{H\subset F} |\nu(H)|_\beta \textit{ for } F \textit{ in } \mathscr{T}.$$

This theorem is proved in [5] and [6] in the case where $T \in \mathscr{T}$ and Y is a Banach space. An elementary proof was given in [7]. The condition $T \in \mathscr{T}$ can be dropped due to the fact (proved in [9]) that there exists a set T_0 in $\mathscr{T}$ such that $|\nu(F)|_\beta = 0$ for every set F in $\mathscr{T}$ disjoint from T_0 (cf. also [15]).

In some cases vector measures have the finite variation (cf. [16]). Recall that if $\mathscr{R}$ is a ring of sets and $\mu : \mathscr{R} \to X$ is a vector measure, the variation of μ is the set function $|\mu|_\alpha$ defined by the relation

$$|\mu|_\alpha(E) = \sup \sum_{i=1}^{n} |\mu(E_i)|_\alpha, \ E \in \mathscr{R}, \ \alpha \in A,$$

where the supremum is taken for all finite disjoint families $\{E_i\} \subset \mathscr{R}$ such that $\bigcup_i E_i = E$.

The semivariation of μ with respect to Y is the set function $\|\mu\|_\gamma^Y$ defined by the equality

$$\|\mu\|_\gamma^Y(E) = \sup \Big|\sum_{i=1}^{n} \mu(E_i) \otimes y_i\Big|_\gamma, \ E \in \mathscr{R}, \ \gamma = \alpha \otimes \beta,$$

where the supremum is taken over all finite families $\{E_i\}$ of disjoint sets of $\mathscr{R}$, $\bigcup_i E_i = E$ and all families $\{y_i\}$ of Y such that $|y_i|_\beta \leqq 1$ (cf. [16], I. IV. 1).

Definition. *Let $\mu : \mathscr{S} \to X$ be a vector measure. We say that μ is dominated with respect to Y by a nonnegative finite measure m_γ on $\mathscr{S}$ if and only if*

$$\|\mu\|_\gamma^Y(E) \to 0 \textit{ if } m_\gamma(E) \to 0, \ E \in \mathscr{S}, \ \gamma = \alpha \otimes \beta. \ \text{(cf. [18])}.$$

Theorem 2. *Let $\mu : \mathscr{S} \to X$ and $\nu : \mathscr{T} \to Y$ be vector measures. Let μ be dominated with respect to Y by m_γ, $\gamma = \alpha \otimes \beta$, $\alpha \in A$, $\beta \in B$.*

Then there exists the projective tensor product $\lambda = \mu \hat{\otimes} \nu : \mathscr{S} \otimes_\sigma \mathscr{T} \to X \hat{\otimes} Y$ of the vector measures μ and ν.

Proof. If a set G is of the form

$$G = \bigcup_{i=1}^{k} E_i \times F_i, \tag{1}$$

where the union is disjoint and $E_i \in \mathscr{S}$, $F_i \in \mathscr{T}$, $i = 1, \ldots, k$, then in view of the additivity condition we define the function λ by the equality

(2) $$\lambda(G) = \sum_{i=1}^{k} \mu(E_i) \otimes \nu(F_i).$$

It is easy to see that the function λ is unambigously defined by the last equality on the ring $\mathscr{S} \otimes \mathscr{T}$ of sets of the form (1) and that it is additive on $\mathscr{S} \otimes \mathscr{T}$.

We must prove that λ can be extended to a sigma additive function on the sigma ring $\mathscr{S} \otimes_\sigma \mathscr{T}$ generated by the ring $\mathscr{S} \otimes \mathscr{T}$ with values in $X \hat{\otimes} Y$. It is known (see e. g. [9], § 4) that such an extension (if it exists) is only one. To prove an existence if suffices to show that there exists a nonnegative bounded measure λ_γ, $\gamma = \alpha \otimes \beta$, on $\mathscr{S} \otimes \mathscr{T}$ such that

$$\lim_{\lambda_\gamma(G) \to 0} |\lambda(G)|_\gamma = 0, \; G \in \mathscr{S} \otimes \mathscr{T},$$

because then λ is evidently sigma additive and can be extended to a sigma additive function on $\mathscr{S} \otimes_\sigma \mathscr{T}$ ([9], Theorem 4.2).

By the Theorem on exhaustion of a measure ([17], 17 (3)) there exists a set S_0 in $\mathscr{S}$ such that $m_\gamma(E) = 0$, hence $\|\mu\|_\gamma^Y(E) = 0$, for every set E in $\mathscr{S}$ disjoint from S_0. Using Saks' lemma ([5], IV. 9. 7) it can be proved that there exists such a $K_\gamma^1 < \infty$ that $\|\mu\|_\gamma^Y(S_0) \leqq K_\gamma^1 < \infty$ and from the monotony of $\|\mu\|_\gamma^Y$ it follows that $\|\mu\|_\gamma^Y(H) \leqq \|\mu\|_\gamma^Y(S_0) \leqq K_\gamma^1 < \infty$ for every $H \subset S_0$, $H \in \mathscr{S}$. Further there exists a K_β, $0 < K_\beta < \infty$ such that $K_\beta = \sup_{F \in \mathscr{T}} |\nu(F)|_\beta < \infty$ ([5], IV. 10.2).

Let ε and δ be such two positive numbers that $m_\gamma(E) < \delta$, $E \in \mathscr{S}$ imply $\|\mu\|_\gamma^Y(E) < \varepsilon$ and $\nu_\beta(F) < \delta$, $F \in \mathscr{T}$ imply $|\nu(F)|_\beta < \varepsilon$ (Theorem 1). We wish to prove that there exists a K_γ, $0 < K_\gamma < \infty$, such that for every set of the form (1) with E_i mutually disjoint the inequality

$$m_\gamma \times \nu_\beta(\bigcup_{i=1}^{k} E_i \times F_i) < \delta^2$$

implies

$$|\lambda(\bigcup_{i=1}^{k} E_i \times F_i)|_\gamma < \varepsilon K_\gamma,$$

where

$$\lambda(\bigcup_{i=1}^{k} E_i \times F_i) = \sum_{i=1}^{k} \mu(E_i) \otimes \nu(F_i).$$

In fact, put

$$D = \{s \in S : \nu_\beta((\bigcup_{i=1}^{k} E_i \times F_i)_s) < \delta\}.$$

Then

$$\delta^2 > (m_\gamma \times \nu_\beta)\,(\bigcup_{=1}^{k} E_i \times F_i) = \int_{\cup E_i} \nu_\beta((\bigcup_{i=1}^{k} E_i \times F_i)_s)\,\mathrm{d}m_\gamma(s) \geqq$$

$$\geqq \int_{\cup E_i - D} \nu_\beta((\bigcup_{i=1}^{k} E_i \times F_i)_s)\, \mathrm{d}m_\gamma(s) \geqq \delta m_\gamma(\bigcup_{i=1}^{k} E_i - D),$$

hence

$$m_\gamma(\bigcup_{i=1}^{k} E_i - D) < \delta$$

and therefore

$$\|\mu\|_\gamma^Y(\bigcup_{i=1}^{k} E_i - D) < \varepsilon.$$

We may suppose that

$$\nu_\beta(F_i) < \delta,\ i = 1, \ldots, p,$$

hence

$$|\nu(F_i)|_\beta < \varepsilon, \text{ i. e. } \frac{|\nu(F_i)|_\beta}{\varepsilon} < 1,$$

$$\nu_\beta(F_i) \geqq \delta,\ i = p + 1, \ldots, k,$$

and

$$D = E_1 \cup \ldots \cup E_p.$$

Now

$$|\lambda(\bigcup_{i=1}^{k} E_i \times F_i)|_\gamma = |\sum_{i=1}^{k} \lambda(E_i \times F_i)|_\gamma = |\sum_{i=1}^{k} \mu(E_i) \otimes \nu(F_i)|_\gamma \leqq$$

$$\leqq |\sum_{i=1}^{p} \mu(E_i) \otimes \nu(F_i)|_\gamma + |\sum_{i=p+1}^{k} \mu(E_i) \otimes \nu(F_i)|_\gamma =$$

$$= \left|\sum_{i=1}^{p} \mu(E_i) \otimes \frac{\nu(F_i)}{\varepsilon}\right|_\gamma . \varepsilon + \left|\sum_{i=p+1}^{k} \mu(E_i) \otimes \frac{\nu(F_i)}{K_\beta}\right|_\gamma K_\beta \leqq$$

$$\leqq \|\mu\|_\gamma^Y(\bigcup_{i=1}^{p} E_i)\varepsilon + \|\mu\|_\gamma^Y(\bigcup_{i=p+1}^{k} E_i)K_\beta =$$

$$= \|\mu\|_\gamma^Y(D)\varepsilon + \|\mu\|_\gamma^Y(\bigcup_{i=1}^{k} E_i - D)K_\beta < K_\gamma^1\varepsilon + \varepsilon K_\beta = \varepsilon K_\gamma,$$

where $K_\gamma = K_\gamma^1 + K_\beta < \infty$.

It is easy to see that λ is sigma additive on the ring $\mathscr{S} \otimes \mathscr{T}$ and can be extended to a sigma additive function (again denoted by) $\lambda : \mathscr{S} \otimes_\sigma \mathscr{T} \to X \hat{\otimes} Y$ ([9], Theorem 4.2, cf. also [15]).

Corollary. *If μ has the finite variation then there exists a projective tensor product of the vector measures $\mu : \mathscr{S} \to X$ and $\nu : \mathscr{T} \to Y$.*

In this case the finite variation of μ acts the part of m_y, since

$$\|\mu\|_\gamma^Y(E) = \sup |\sum_{i=1}^{k} \mu(E_i) \otimes y_i|_\gamma \leqq \sup \sum_{i=1}^{k} |\mu(E_i)|_\alpha = |\mu|_\alpha(E).$$

Theorem 3. *If $\mu : \mathscr{S} \to X$ and $\nu : \mathscr{T} \to Y$ are vector measures with a finite variation $|\mu|_\alpha$ and $|\nu|_\beta$, respectively, then there exists the projective tensor product $\lambda = \mu \hat{\otimes} \nu : \mathscr{S} \otimes_\sigma \mathscr{T} \to X \hat{\otimes} Y$ of the vector measures μ and ν, and*

$$(3) \qquad |\mu \hat{\otimes} \nu|_{\alpha\otimes\beta} = |\mu|_\alpha \times |\nu|_\beta, \ \alpha \in A, \ \beta \in B.$$

Proof. The existence of a vector measure $\lambda = \mu \hat{\otimes} \nu$ follows from the Corollary. To prove the equality (3), take disjoint sets $G_n = \bigcup_{i=1}^{k_n} E_i^n \times F_i^n$ in $\mathscr{S} \otimes \mathscr{T}$, $n = 1, \ldots, l$. Then we have

$$\sum_{n=1}^{l} |\lambda(G_n)|_\gamma = \sum_{n=1}^{l} | \sum_{i=1}^{k_n} \mu(E_i^n) \otimes \nu(F_i^n)|_\gamma \leqq$$

$$\leqq \sum_{n=1}^{l} \sum_{i=1}^{k_n} |\mu(E_i^n)|_\alpha \, |\nu(F_i^n)|_\beta \leqq \sum_{n=1}^{l} \sum_{i=1}^{k_n} |\mu|_\alpha(E_i^n) \, |\nu|_\beta(F_i^n) = \sum_{n=1}^{l} |\mu|_\alpha \times |\nu|_\beta(G_n) =$$

$$= |\mu|_\alpha \times |\nu|_\beta(\bigcup_{n=1}^{l} G_n).$$

It follows that for any $G \in \mathscr{S} \otimes \mathscr{T}$ we have $|\lambda|_\alpha(G) \leqq |\mu|_\alpha \times |\nu|_\beta(G)$, hence for G in $\mathscr{S} \otimes_\sigma \mathscr{T}$.

On the other hand, for any $E \times F$ in $\mathscr{S} \otimes \mathscr{T}$ and for any $\varepsilon > 0$ there exist disjoint sets $\{E_i\}$, $\bigcup_i E_i = E$, $\{F_j\}$, $\bigcup_j F_j = F$ such that we have

$$|\mu|_\alpha \times |\nu|_\beta(E \times F) = |\mu|_\alpha(E) \, |\nu|_\beta(F) \leqq$$

$$\leqq (\sum_i |\mu(E_i)|_\alpha + \varepsilon) \, (\sum_j |\nu(F_j)|_\beta + \varepsilon) =$$

$$= \sum_i \sum_j |\mu(E_i) \otimes \nu(F_j)|_\gamma + \varepsilon(\sum_i |\mu(E_i)|_\alpha + \sum_j |\nu(F_j)|_\beta) + \varepsilon^2;$$

$\varepsilon > 0$ being arbitrary we have

$$|\mu|_\alpha \times |\nu|_\beta(E \times F) \leqq |\mu \otimes \nu|_\gamma(E \times F).$$

Therefore

$$|\mu|_\alpha \times |\nu|_\beta(E \times F) = |\mu \otimes \nu|_\gamma(E \times F)$$

It follows that

$$|\mu|_\alpha \times |\nu|_\beta(G) = |\mu \otimes \nu|_\gamma(G)$$

for any G in $\mathscr{S} \otimes_\sigma \mathscr{T}$. The proof is completed.

Remark. We have seen that if μ has a finite variation so μ is dominated by $|\mu|_\alpha$ with respect to any Y. If we take $X = l^1(I)$, the Banach space of all unconditionally (in this case also absolutely) summable numerical functions $[\xi_i, i \in I]$ defined on an indexed set I, where the norm is $\|[\xi_i, i \in I]\| = \sum_{i \in I} |\xi_i|$ ([2], II. 2. (1) or [12], 1.1), we can find a vector measure $\mu : \mathscr{S} \to l^1(I)$ which does not have the finite variation, nevertheless μ is dominated by a nonnegative finite measure. In this case for every $E \in \mathscr{S}$ we have $\mu(E) = [\xi_i(E), i \in I]$, hence ξ_i, $i \in I$ form a bounded family of uniformly sigma additive scalar measures. Let $\{E_r\}_{r=1}^k \subset \mathscr{S}$ be disjoint sets and $E = \bigcup_{r=1}^k E_r$. It follows from ([12], 7. 2. 2, cf. also [2], IV. 2. 5) that for y_r in Y, $r = 1, \ldots, k$, $|y_r|_\beta \leqq 1$,

$$|\sum_{r=1}^k [\xi_i(E_r), I] \otimes y_r|_\beta = \sum_{i \in I} |\sum_{r=1}^k \xi_i(E_r) y_r|_\beta \leqq \sum_{i \in I} \sum_{r=1}^k |\xi_i(E_r)| \leqq \sum_{i \in I} \sum_{r=1}^k |\xi_i|(E_r) =$$

$$= \sum_{i \in I} |\xi_i|(E).$$

For every $i \in I$ there exists a finite nonnegative measure m_i on $\mathscr{S}$ such that $m_i(E) \leqq |\xi_i(E)| \leqq |\xi_i|(E)$, and $|\xi_i|(E) \to 0$ for $m_i(E) \to 0$.

Let $\sigma \subset I$ be an arbitrary finite subset. Take the finite sum for E in $\mathscr{S}$:

$$\sum_{i \in \sigma} m_i(E) \leqq \sum_{i \in \sigma} |\xi_i(E)| \leqq \sum_{i \in I} |\xi_i(E)| < K < \infty.$$

Define the set function m_β on $\mathscr{S}$ by the relation:

$$m_\beta(E) = \sum_{i \in I} m_i(E) = \sup \{\sum_{i \in \sigma} m_i(E) : \sigma \subset I\} \leqq K.$$

The function m_β is a finite nonnegative measure ([1], I. 10) with this property: If $m_\beta(E) \to 0$, then $m_i(E) \to 0$ uniformly in i, i. e. $|\xi_i|(E) \to 0$ also uniformly in i, henceforth also $\sum_{i \in \sigma} |\xi_i|(E) \to 0$ for an arbitrary $\sigma \subset I$, and thus also $\sum_{i \in I} |\xi_i|(E) \to 0$. Since for every β in B $\|\mu\|_\beta^Y(E) \leqq \sum_{i \in I} |\xi_i|(E)$, it follows that $\|\mu\|_\beta^Y(E) \to 0$, if $m_\beta(E) \to 0$. Thus every μ is dominated by an m_β.

Let now $\mathscr{S}$ be a sigma algebra of all subsets of the set of natural numbers. Let $X = l^1(I)$ be infinite-dimensional and $\{c_n\}_{n=1}^\infty$ be any sequence of positive numbers such that $\sum_{n=1}^\infty c_n^2 < \infty$; then there exists in $X = l^1(I)$ a sequence $\{x_n\}_{n=1}^\infty$ such that $\|x_n\| = c_n$ and $\sum_{n=1}^\infty x_n$ is unordered convergent ([2], IV. 1. 2).

Let us define $\mu(\{n\}) = x_n$. Then $\|\mu(\{n\})\| = c_n \leqq |\mu|(\{n\})$. If we choose $\{c_n\}_{n=1}^\infty$ in such a way that $\sum_{n=1}^\infty c_n = \infty$, the variation of μ cannot be finite.

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