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SOME MORE REMARKS ON CERTAIN ALGEBRAIC IDENTITIES

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1.

Let

$$a_i \quad (i = 0, 1, \ldots, n)$$

and

$$x_j \quad (j = 1, 2, \ldots, m)$$

be given complex numbers, the a_i are distinct while x_j are arbitrary.

If we put

$$(1) \qquad S(m, n) = \sum_{i=0}^{n} \frac{(a_i - x_1)(a_i - x_2) \ldots (a_i - x_m)}{(a_i - a_0) \ldots (a_i - a_{i-1})(a_i - a_{i+1}) \ldots (a_i - a_n)},$$

then

$$(2) \qquad S(n + 1, n) = \sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j,$$

$$(3) \qquad S(n, n) = 1,$$

$$(4) \qquad S(m, n) = 0, \quad m < n.$$

Three proofs of these formulas are known: one by induction (Bartoš [1]), one using the calculus of residues (Kaucký [1]) and one by means of the Lagrange interpolation formula (Carlitz [2])

By the method used in the last two proofs we can evaluate also the sums $S(n + 2, n)$, $S(n + 3, n), \ldots$

In this article I am going to show that the formulas (2), (3) and (4) are simple consequences of certain well-known relations.

2.

For this purpose we denote

$$f(x) = (x - x_1)(x - x_2) \ldots (x - x_m) = \sum_{k=0}^{m} (-1)^k \sigma_k x^{m-k}.$$

If

$$(5) \qquad \Delta(a_0, a_1, \ldots, a_n) = \begin{vmatrix} 1 & a_0 & \ldots & a_0^n \\ 1 & a_1 & \ldots & a_1^n \\ \cdot\cdot & \cdot\cdot & \cdot\cdot\cdot\cdot & \cdot\cdot \\ 1 & a_n & \ldots & a_n^n \end{vmatrix} = \prod_{\substack{k,l=0 \\ k>l}}^{n} (a_k - a_l)$$

is the Vandermonde's determinant, then according (to [3], p. 9)

$$(6) \qquad S(m, n) = \begin{vmatrix} 1 & a_0 & \ldots & a_0^{n-1} & f(a_0) \\ 1 & a_1 & \ldots & a_1^{n-1} & f(a_1) \\ \cdot\cdot & \cdot\cdot & \cdot\cdot\cdot & \cdot\cdot & \cdot\cdot \\ 1 & a_n & \ldots & a_n^{n-1} & f(a_n) \end{vmatrix} : \Delta(a_0, a_1, \ldots, a_n) =$$

$$= \sum_{k=0}^{m} (-1)^k \sigma_k \frac{\begin{vmatrix} 1 & a_0 & \ldots & a_0^{n-1} & a_0^{m-k} \\ 1 & a_1 & \ldots & a_1^{n-1} & a_1^{m-k} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot\cdot & \cdot\cdot & \cdot\cdot \\ 1 & a_n & \ldots & a_n^{n-1} & a_n^{m-k} \end{vmatrix}}{\Delta(a_0, a_1, \ldots, a_n)} .$$

From this equation we derive immediately the formulas (4) and (3). In fact if $m < n$, all the determinants in numerators vanish because they have in the last columns the numbers

$$a_0^l, a_1^l, \ldots a_n^l$$

where $0 \leqq l \leqq n - 1$. So we get the formula (4).

If $m = n$, then obviously

$$S(n, n) = \sigma_0 = 1$$

and this is formula (3).

Now we still have to prove equation (2). However with the use of a further well-known formula ([3], p. 9)

$$(7) \qquad \begin{vmatrix} 1 & a_0 & \ldots & a_0^{n-1} & a_0^{n+1} \\ 1 & a_1 & \ldots & a_1^{n-1} & a_1^{n+1} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot\cdot & \cdot\cdot & \cdot\cdot \\ 1 & a_n & \ldots & a_n^{n-1} & a_n^{n+1} \end{vmatrix} : \Delta(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} a_i,$$

the equation (6) gives

$$S(n + 1, n) = \sum_{i=0}^{n} a_i - \sigma_1 = \sum_{i=0}^{n} a_i - \sum_{j=1}^{n+1} x_j$$

which is the formula (2).

3.

As we have already pointed out it is possible to calculate easily for example with the help of the calculus of residues the sums $S(m, n)$ also for $m > n + 1$. In reference to the method desccribed in the above paragraph this does not hold good. To demonstrate this let us calculate the value of the sum $S(n + 2, n)$.

As we can see from equation (6) we must know the value of the quotient

$$(8) \qquad \begin{vmatrix} 1 & a_0 & \dots & a_0^{n-1} & a_0^{n+2} \\ 1 & a_1 & \dots & a_1^{n-1} & a_1^{n+2} \\ .. & .. & \dots & .. & .. \\ 1 & a_n & \dots & a_n^{n-1} & a_n^{n+2} \end{vmatrix} : \Delta(a_0, a_1, \dots, a_n)$$

For $n = 1$ the value of the quotient is

$$\begin{vmatrix} 1 & a_0^3 \\ 1 & a_1^3 \end{vmatrix} : (a_1 - a_0) = a_0^2 + a_0 a_1 + a_1^2.$$

For $n = 2$ it is also easy to show that

$$\begin{vmatrix} 1 & a_0 & a_0^4 \\ 1 & a_1 & a_1^4 \\ 1 & a_2 & a_2^4 \end{vmatrix} : (a_2 - a_1)(a_2 - a_0)(a_1 - a_0) =$$

$$= a_0^2 + a_1^2 + a_2^2 + a_0 a_1 + a_0 a_2 + a_1 a_2$$

We may therefore assume that the quotient value (8) will be

$$(9) \qquad \sum_{i=0}^{n} a_i^2 + \sum_{\substack{i,j=0 \\ i<j}}^{n} a_i a_j$$

which can be proved by induction.

We have just shown that for $n = 1,2$ this statement is correct. Let us therefore assume that statement holds also if $(n - 1)$ is inserted in place of n.

We subtract now in the determinant in the numerator of (8) the first column times a_n from the second, from the third column the second times a_n, etc. until from the nth column the preceeding column also multiplied by a_n. Finally we subtract from the last column the last but one multiplied by a_n^3.

Thus we obtain a determinant with the numbers

$$1 \quad 0 \quad \dots \quad 0 \quad 0$$

in the last row. The remaining rows are as follows

$$\begin{matrix} 1 & a_0 - a_n & a_0(a_0 - a_n) & \dots a_0^{n-2}(a_0 - a_n) & a_0^{n-1}(a_0^3 - a_n^3) \\ 1 & a_1 - a_n & a_1(a_1 - a_n) & \dots a_1^{n-2}(a_1 - a_n) & a_1^{n-1}(a_1^3 - a_n^3) \\ .. & \dots & \dots & \dots \quad \dots & \dots \quad \dots \\ 1 & a_{n-1} - a_n & a_{n-1}(a_{n-1} - a_n) & \dots a_{n-1}^{n-2}(a_{n-1} - a_n) & a_{n-1}^{n-1}(a_{n-1}^3 - a_n^3) \end{matrix}$$

Expanding this determinant according to the elements of the last row and reducing the quotient by the product

$$(a_n - a_0)(a_n - a_1) \dots (a_n - a_{n-1})$$

we see that the quotient (8) has been reduced to

$$\begin{vmatrix} 1 & a_0 & \dots a_0^{n-2} & a_0^{n-1}(a_0^2 + a_0 a_n + a_n^2) \\ 1 & a_1 & \dots a_1^{n-2} & a_1^{n-1}(a_1^2 + a_1 a_n + a_n^2) \\ \dots & \dots & \dots \dots & \dots \dots \dots \\ 1 & a_{n-1} & \dots a_{n-1}^{n-2} & a_{n-1}^{n-1}(a_{n-1}^2 + a_{n-1}a_n + a_n^2) \end{vmatrix} : \Delta(a_0, a_1, \dots, a_{n-1})$$

The above quotient may be decomposed into three parts. The first

$$\begin{vmatrix} 1 & a_0 & \dots a_0^{n-2} & a_0^{n+1} \\ 1 & a_1 & \dots a_1^{n-2} & a_1^{n+1} \\ \dots & \dots & \dots \dots & \dots \\ 1 & a_{n-1} & \dots a_{n=1}^{n-2} & a_{n-1}^{n+1} \end{vmatrix} : \Delta(a_0, a_1, \dots, a_{n-1})$$

has by assumption the value

$$\sum_{i=0}^{n-1} a_i^2 + \sum_{\substack{i,\, j=0 \\ i<j}}^{n-1} a_i a_j$$

The second part

$$a_n \begin{vmatrix} 1 & a_0 & \dots a_0^{n-2} & a_0^n \\ 1 & a_1 & \dots a_1^{n-2} & a_1^n \\ \dots & \dots & \dots \dots & \dots \\ 1 & a_{n-1} & \dots a_{n-1}^{n-2} & a_n^n \end{vmatrix} : \Delta(a_0, a_1, \dots, a_{n-1})$$

has in accord with formula (7) — if n is replaced by $(n - 1)$ — the value

$$a_n \sum_{i=0}^{n-1} a_i .$$

And finally the third part has evidently the value a_n^2.

Summing up all these results we see that the quotient (8) really has the value (9).

Having put down, further, for the sake of simplification,

$$g(x) = (x - a_0)(x - a_1) \dots (x - a_n) = \sum_{k=0}^{n+1} (-1)^k \tau_k x^{n+1-k},$$

we see that expression (9) is equal to

$$\tau_1^2 - \tau_2 .$$

Thus we have now everything to enable us to find the value of the sum

$S(n+2, n)$. According to formula (6), in which we replace m with $(n+2)$, the following holds

$$S(n+2, n) = \tau_1^2 - \tau_2 - \sigma_1\tau_1 + \sigma_2 = \sigma_2 + \tau_1(\tau_1 - \sigma_1) - \tau_2.$$

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