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Matematický časopis, Vol. 21 (1971), No. 2, 82--86

Persistent URL: http://dml.cz/dmlcz/126433

A CONTRIBUTION TO THE THEORY OF PERMUTATIONS

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A permutation of n distinct elements is said to be of the cycle class $(k_1, k_2, \ldots, k_n)$ or for the sake of brevity, of the class (k), if it is a permutation with k_1 unit cycles, k_2 2-cycles, ... and k_n n-cycles. It is convenient to write a cycle with its smallest element in the first position.

If $C(k_1, k_2, \ldots, k_n)$ is the number of permutations of the class (k), so that

$$k_1 + 2k_2 + \ldots + nk_n = n,$$

then

$$C(k_1, k_2, \ldots, k_n) = \frac{n!}{1^{k_1}k_1!2^{k_2}k_2! \ldots n^{k_n}k_n!}$$

The multiple ordinary generating function

$$C_n(t_1, t_2, \ldots, t_n) = C_n = \sum C(k_1, k_2, \ldots, k_n)t_1^{k_1}t_2^{k_2} \ldots t_n^{k_n} =$$
$$= \sum \frac{n!}{k_1!k_2! \ldots k_n!} \left(\frac{t_1}{1}\right)^{k_1} \left(\frac{t_2}{2}\right)^{k_2} \ldots \left(\frac{t_n}{n}\right)^{k_n}$$

for the numbers $C(k_1, k_2, \ldots, k_n)$ is called the cycle indicator of the symmetric group. The sum runs over all non-negative integral values of k_1 to k_n so that $k_1 + 2k_2 + \ldots + nk_n = n$.

Finally, the exponential generating function for the cycle indicators C_n is

$$(1) \quad \exp uC = \sum_{n=0}^{\infty} C_n(t_1, t_2, \ldots, t_n)u^n/n! = \exp\,(ut_1 + u^2t_2/2 + u^3t_3/3 + \ldots)$$

and by differentiation of this equation with respect to the variable u, we obtain

$$(2) \qquad C_{n+1} = \sum_{k=0}^{n} (n)_k t_{k+1} C_{n-k},$$

a basic recurrence formula for the cycle indicators. The arguments of all C's have been omitted.

And now how do we proceed when solving the different emumerative problems of the theory of permutations. Briefly put, we may say that every such problem makes us find a certain special cycle indicator, and thus also the corresponding generating function (1). The above equation supplies us in some cases with the wanted indicators either directly, or by means of its derivations with respect to the variable u, which give us the recurrence relations for the said indicators. (See [1]).

This method of solution can, however, be simplified. As we shall namely show in the following paragraph, the basic recurrence formula (2) can be proved by a purely combinatorial method without a derivation of equation (1). This procedure of proving is essentially the same as the method employed in paper [2].

And from this equation (2), as we can see from several examples in the last paragraph, the recurrence relations for the wanted indicators may be derived.

As to further informations, the reader is advised to consult the quoted literature.

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The indicator C_{n+1} represents all the permutations of $(n + 1)$ different elements, decomposed into cycles, and we shall obtain relation (2) when we realise how the permutations of all elements originate from the permutations of the first n elements.

First we shall add the last element to all these permutations as a unitary cycle. Every 1-cycle, however, is represented by the variable t_1, so that in this way we get the first member

$$t_1 C_n$$

in the indicator C_{n+1}.

Further we put the last element into the single cycles in all the permissible places. The number of all the k-cycles formed of the first n elements is

$$\binom{n}{k} (k - 1)!,$$

for first we form all the k-combinations, so that the smallest elements occupy the first positions, while the elements 2-nd — n-nd are there upon permuted in all possible ways. Into each of these cycles we can afterwards insert the last element in the k places, so that we obtain in this way altogether

$$(n)_k$$

$(k + 1)$-cycles with the last element.

Each of these cycles is contained in several permutations, and in each of them it has for a complement some permutation of the remaining $(n - k)$ elements, and with respect to symmetry all of them are permutations of these elements. Their representative is the indicator C_{n-k}.

Now each $(k + 1)$-cycle is represented by the variable t_{k+1} so that in this way we obtain the remaining members of the sum

$$(n)_k t_{k+1} C_{n-k},$$

where $k = 1, 2, \ldots, n$.

3

1. If the cycle length (the number of elements they contain) is ignored, how many permutations of n distinct elements have exactly k cycles?

By setting each t_i equal to t, we have the indicator

$$C_n(t, t, \ldots, t) = c_n(t)$$

and if we write

$$c_n(t) = \sum_{k=0}^{n} c(n, k) t^k,$$

then $c(n, k)$ are the numbers asked for.

Now relation (2) yields

$$c_{n+1}(t) = t\sum_{k=0}^{n} (n)_k c_{n-k}(t) = tc_n(t) + t\sum_{k=1}^{n} (n)_k c_{n-k}(t) = tc_n(t) + t\sum_{k=0}^{n-1} (n)_{k+1} c_{n-1-k}(t) =$$

$$= tc_n(t) + nt\sum_{k=0}^{n-1} (n - 1)_k c_{n-1-k}(t) = tc_n(t) + nc_n(t)$$

so that for $c_n(t)$ the following recurrence relation holds

$$c_{n+1}(t) = (t + n) c_n(t)$$

Therefore

$$c_n(t) = t(t + 1) \ldots (t + n - 1), \quad n > 0$$

and

$$c(n, k) = (-1)^{k+n} s(n, k),$$

where $s(n, k)$ are the Stirling numbers of the first kind.

2. If the cycle length is ignored, how many permutations of n distinct elements have exactly k cycles, not one of which is a unitary cycle?

The wanted number $d(n, k)$ is clearly the coefficient of t^k in the indicator

$$C_n(0, t, t, \ldots, t) = d_n(t) = \sum_{k=0}^{n} d(n, k) t^k.$$

Again, from the basic formula (2), it follows

$$d_{n+1}(t) = t\sum_{k=1}^{n}(n)_k d_{n-k}(t) = tnd_{n-1}(t) + t\sum_{k=2}^{n}(n)_k d_{n-k}(t) =$$

$$= tnd_{n-1}(t) + nt\sum_{n=1}^{k-1}(n-1)_k d_{n-1-k}(t) = tnd_{n-1}(t) + nd_n(t),$$

so that we have for the indicators $d_n(t)$ the following recurrence relation

$$d_{n+1}(t) = nd_n(t) + ntd_{n-1}(t),$$

and, hence, by equating the coefficients of t^k, the recurrence relation

$$d(n+1, k) = nd(n, k) + nd(n-1, k-1)$$

for the numbers $d(n, k)$.

Note. By the way, we should point out that

$$d(n, k) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}c(n-j, k-j),$$

because permutations with k cycles, $c(n, k)$ in number, may be divided into permutations with j unit cycles, where $j = 0, 1, \ldots, n$.

Now, the number of the permutations in which there are j 1-cycles is evidently

$$\binom{n}{j}c(n-j, k-j)$$

and the stated relation is based on the principle of inclusion and exclusion.

3. How many permutations of n distinct elements have exactly k cycles, not one of which is a r-cycle, $1 \leq r \leq n$. The cycle length is ignored. (See 3]).

We set

$$t_i = t, i = 1, 2, \ldots, n, i \neq r, t_r = 0,$$

and we denote the corresponding indicator by

$$d_n(t, r) = \sum_{k=0}^{n} d(n, k, r)t^k,$$

where $d(n, k, r)$ are the wanted numbers.

Then from equation (2) we have

$$d_{n+1}(t, r) = t\sum_{k=0}^{n}(n)_k d_{n-k}(t, r) - t(n)_{r-1}d_{n-r+1}(t, r) =$$

$$= td_n(t, r) + nt\sum_{k=0}^{n-1}(n-1)_k d_{n-1-k}(t, r) - t(n)_{r-1}d_{n-r+1}(t, r) =$$

$$= td_n(t, r) + n\{d_n(t, r) + t(n-1)_{r-1}d_{n-r}(t)\} - t(n)_{r-1}d_{n+1-r}(t, r),$$

so that

$$d_{n+1}(t, r) = (n + t)d_n(t, r) - t(n)_{r-1} d_{n-r+1}(t, r) + t(n)_r d_{n-r}(t, r).$$

This corresponds to

$$d(n + 1, k, r) = nd(n, k, r) + d(n, k - 1, r) -$$

$$-(n)_{r-1}d(n - r + 1, k - 1, r) + (n)_r d(n - r, k - 1, r),$$

a recurrence formula for the numbers $d(n, k, r)$.

4. How many permutations of n distinct elements have cycles, the lengths of which are the r^{th} powers of the natural numbers: $1^r, 2^r, 3^r, \ldots$, where $r \geq 1$ is also a natural number? (See [4])

If

$$b_n(t, r) = \sum_{k=0}^{n} b(n, k, r)t^k$$

is the corresponding indicator and $b(n, k, r)$ the numbers in demand, then from relation (2) it follows

$$b_{n+1}(t, r) = t \sum_{i=1}^{s} (n)_{i^r-1} b_{n-i^r+1}(t, r)$$

and

$$b(n + 1, k, r) = \sum_{i=1}^{s} (n)_{i^r-1} b(n - i^r + 1, k - 1, r),$$

where

$$s = [(n + 1)^{1/r}]$$

The even and odd permutations as well as different furthter problems may be treated in the same manner.

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Recived May 8, 1969

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