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# INVOLUTORY PAIRS OF VERTICES IN TRANSITIVE GRAPHS

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A transitive graph is such a graph $G$ that for any two vertices $u, v$ of $G$ an automorphism $\alpha$ of $G$ exists so that $\alpha(u) = v$. An involutory pair of vertices in a graph $G$ is such a pair $u$, $v$ that $u \neq v$ and there exists an automorphism $\alpha$ of $G$ for which $\alpha(u) = v$, $\alpha(v) = u$.

V. G. Vizing [3] has set the problem of characterizing transitive graphs which any pair of distinct vertices is involutory. Here this problem will not be solved completely, but we shall give a sufficient condition for such a graph.

Before proving the theorem we wish to make some remarks concerning the terminology. We use the term "transitive graph", but various authors use other terms for this concept. V. G. Vizing uses the Russian term "pravilnyi" which means "regular", but the term "regular graph" in English means something else — a graph in which all the vertices have equal degrees. L. Lovász uses the term "symmetric graph", but this can lead to misunderstandings, because this term is used by other authors for other concepts. (For example, by this term P. Erdős denotes a graph with at least one nonidentical automorphism, i. e. a much weaker concept.)

**Theorem.** *Let $G$ be a transitive undirected graph, let $\mathfrak{A}(G)$ be its automorphism group. Let there exist an Abelian subgroup $\mathfrak{A}_0(G)$ of $\mathfrak{A}(G)$ such that for any two vertices $u$, $v$ of $G$ there exists an automorphism $\alpha \in \mathfrak{A}_0(G)$ such that $\alpha(u) = v$. Then any pair of vertices in $G$ is involutory.*

Proof. Let $u$ be a vertex of $G$. Then any vertex of $G$ can be expressed as $\alpha(u)$, where $\alpha \in \mathfrak{A}_0(G)$. Obviously it may happen that $\alpha(u) = \beta(u)$ for $\alpha \in \mathfrak{A}_0(G)$, $\beta \in \mathfrak{A}_0(G)$, $\alpha \neq \beta$. We shall prove that then also $\alpha^{-1}(u) = \beta^{-1}(u)$. As $\alpha(u) = \beta(u)$, we have $\beta^{-1}\alpha(u) = \beta^{-1}\beta(u) = u$. As $\mathfrak{A}_0(G)$ is Abelian, $u = \alpha\beta^{-1}(u)$. Therefore also $\alpha^{-1}(u) = \alpha^{-1}\alpha\beta^{-1}(u) = \beta^{-1}(u)$. As $(\alpha^{-1})^{-1} = \alpha$, $(\beta^{-1})^{-1} = \beta$ we can also prove that if $\alpha(u) \neq \beta(u)$, then $\alpha^{-1}(u) \neq \beta^{-1}(u)$. Now let $u$, $v$ be two arbitrary distinct vertices of $G$. For any vertex $x$ of $G$ let $\beta_x$ be such an automorphism of $G$ that $\beta_x \in \mathfrak{A}_0(G)$, $\beta_x(u) = x$. Now let $\gamma$ be the mapping of the vertex set of $G$ such that $\gamma(x) = \beta_v\beta_x^{-1}(u)$. First we shall prove that $\gamma$ is a bijection. If $\gamma(x) = \gamma(y)$ for some vertices $x$, $y$ of $G$, then $\beta_v\beta_x^{-1}(u) = \beta_v\beta_y^{-1}(u)$, i. e. $\beta_x^{-1}(u) = \beta_y^{-1}(u)$ and $\beta_x(u) = \beta_y(u)$, which means $x = y$; the mapping $\gamma$ is injective. Assume that there exists a vertex $y$ of $G$ for which $y \neq \gamma(x)$ for any $x$. We have $y \neq \beta_v\beta_x^{-1}(u)$, this means $\beta_v^{-1}(y) \neq \beta_x^{-1}(u)$, because $\beta_v^{-1}$ is a bijection. But if we denote $z = \beta_v^{-1}(y)$, then $z = \beta_z(u)$ and we have $\beta_z(u) \neq \neq \beta_x^{-1}(u)$ for any $x$. Let $w = \beta_z^{-1}(u)$; we have $w = \beta_w(u) = \beta_z^{-1}(u)$, therefore $\beta_w^{-1}(u) = \beta_z(u) = z = \beta_v^{-1}(y)$. This means $y = \beta_v\beta_w^{-1}(u) = \gamma(w)$, which is a contradiction with the assumption $y \neq \gamma(x)$ for any $x$. Thus $\gamma$ is a bijection. Now we shall prove that $\gamma \in \mathfrak{A}(G)$. Let two vertices $x, y$ be joined (or not joined) by an edge. We have $x = \beta_x(u)$, $y = \beta_y(u)$. Then $u$, $\beta_x^{-1}\beta_y(u)$ are joined (or not joined respectively) by an edge and, as $\beta_x^{-1}\beta_y = \beta_y\beta_x^{-1}$ also $\beta_x^{-1}(u)$, $\beta_y^{-1}(u)$ are joined (or not joined, respectively) by an edge. Since also $\beta_v$ is an automorphism, $\gamma(x) = \beta_v\beta_x^{-1}(u), \gamma(y) = \beta_v\beta_y^{-1}(u)$ are joined (or not joined, respectively) by an edge and $\gamma$ is an automorphism of $G$. We have $\gamma(u) = \beta_v\beta_u^{-1}(u) = \beta_v(u) = v$, $\gamma(v) = \beta_v\beta_v^{-1}(u) = u$ and $u$, $v$ form an involutory pair of vertices. As $u$, $v$ were chosen arbitrarily, any pair of vertices in $G$ is involutory.

For directed graphs this theorem does not hold. Let $G$ be a cycle (directed circuit) of the odd length $r$, let $u_1, \ldots, u_r$ be its vertices, let $\overrightarrow{u_iu_{i+1}}$ for $i = 1, \ldots, r - 1$, $\overrightarrow{u_ru_1}$ be its edges. The automorphism group of $G$ is a cyclic group of the order $r$, therefore it is Abelian. It is generated by the mapping $\alpha$ such that $\alpha(u_i) = u_{i+1}$ for $i = 1, \ldots, r - 1$, $\alpha(u_r) = u_1$. For any two vertices $u_i, u_j$ there exists exactly one automorphism of $G$ which maps $u_i$ onto $u_j$, namely $\alpha^{j-i}$. But $\alpha^{j-i}(u_j) = u_{2j-i}$, where the subscript $2j - i$ is taken modulo $r$. If $u_{2j-i} = = u_i$, this means that $2j - i \equiv i \pmod r$, i. e. $j \equiv i \pmod r$, which means $j = i$. Thus no pair of distinct vertices in $G$ is involutory.

The result of the Theorem is a generalization of the results in [1] and [2]. In [1] and [2] a proof is given that if the automorphism group of a transitive graph is Abelian, then it is isomorphic to the direct product of $k$ copies of a cyclic group of the order 2, where $k$ is a positive integer different from 2, 3, 4. As each non-unit element of such a group has the order 2, any two vertices of such a graph are involutory.

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