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OSCILLATION CRITERIA FOR THIRD-ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

This paper is concerned with the oscillatory properties of the third-order linear differential equation

$$(A) \qquad y''' + B(x)y' + C(x)y = 0 \, ,$$

where $B(x)$, $C(x)$, and $B'(x)$ are continuous functions on the interval $(0, \infty)$. In this paper there are proved the comparison theorems, the oscilation criteria and some asymtotic praperties of (A).

Definition 1. *Equation (A) is said to be of class V_1 [class V_2] iff every solution $y(x)$ for which $y(a) = y'(a) = 0 \, y''(a) > 0$ $(0 < a < \infty)$ has the property that $y(x) > 0$ in $(0, a)$ [in (a, ∞)].*

Definition 2. *A solution of (A) will be called oscillatory iff it has an infinity of zeros in $(0, \infty)$ and nonoscillatory iff it has but a finite number of zeros in this interval. Equation (A) is said to be oscillatory iff it has at least one (nontrivial) oscillatory solution, and nonoscillatory iff all of its (nontrivial) solutions are nonoscillatory.*

In papers [1], [2] and [5] some properties of equation (A) of class V_1 [class V_2] have been investigated. If $2C(x) - B'(x) \geqq 0$ $[2C(x) - B'(x) \leqq 0]$, and not identically zero in any interval, then the equation (A) is of class V_1 [of class V_2] (see, e. g. [3], p. 153, [5], p. 266 ...). Since in this paper we shall often suppose that $2C(x) - B'(x) \geqq 0$ $[2C(x) - B'(x) \leqq 0]$ and the sign = can hold in any interval, we first prove some assertions under the above introduced assumption. These assertions slightly generalize the results of papers [1] and [5].

2. Properties of the equation (A) under the assumption $2C(x) - B'(x) \geqq 0$.

Theorem 1. *If equation (A) is oscillatory, and if the inequality $2C(x) -$*

— $B'(x) \geqq 0$ *is satisfied, then any solution of this equation which vanishes at least once is oscillatory.*

Proof. Let $y(x)$ be an oscillatory solution of (A) which vanishes at the point a. First of all we shall prove that any solution of (A) which is linearly independent with $y(x)$ and vanishes at the point a is oscillatory. Let $y'(a) > 0$. Now we prove that the solution $z(x)$ of (A) which satisfies the conditions $z(a) = z'(a) = 0$ is oscillatory. Assume that it is not true, that is, there exists a number b such that $z(x) > 0$ in (b, ∞). The solution $y(x)$ is oscillatory. therefore there exist two consecutive zeros $\alpha, \beta \in (b, \infty)$ of $y(x)$. Hence, by Lemma 1.2 in [4], there exists a number $c > 0$ such that the solution $w(x) = z(x) - cy(x)$ has a double zero at some point $\tau \in (\alpha, \beta)$. The solution $w(x)$ is such that $w(a) = 0$, $w'(a) \neq 0$, $w(\tau) = w'(\tau) = 0$ for $\tau > a$. Multiplying (A) by $w(x)$ we obtain the identity

$$(1) \qquad \left[yy'' - \frac{1}{2}y'^2 + \frac{1}{2}B(x)y^2\right]' = -\frac{1}{2}[2C(x) - B'(x)]y^2 .$$

Putting $w(x)$ instead of $y(x)$ in (1) and integrating over (a, τ) yields

$$(2) \qquad \left[ww'' - \frac{1}{2}w'^2 + \frac{1}{2}B(x)w^2\right]_a^\tau = -\frac{1}{2}\int_a^\tau [2C(x) - B'(x)]w^2 \,\mathrm{d}x ,$$

and after putting the limits we obtain

$$\frac{1}{2}w'^2(a) = -\frac{1}{2}\int_a^\tau 2C(x) - B'(x)]w^2 \,\mathrm{d}x .$$

Clearly, the left-hand side of this last equation is positive while the right-hand side is nonpositive. This contradiction proves that the solution $z(x)$ of (A) is oscillatory too. Similarly, we can prove that if $y(x)$ and $z(x)$ are such that $y(a) = y'(a) = 0$, $y''(a) \neq 0$, $z(a) = 0$, $z'(a) \neq 0$, then $z(x)$ is oscillatory if $y(x)$ is oscillatory. If $y(x)$ and $z(x)$ are such that $y'(a) \neq 0$, $z'(a) \neq 0$, then the solution $u(x)$ of (A) such that $u(a) = u'(a) = 0$, $u''(a) \neq 0$ is oscillatory and so $z(x)$ is oscillatory too.

Let $y(x)$ be an oscillatory solution of (A). Let $z(x)$ be an arbitrary solution of (A) which vanishes at the point x_0. If x_0 is equal to some zero of $y(x)$, then by the above part of the proof $z(x)$ is oscillatory as well. Let $a \neq x_0$ be a zero of $y(x)$. Then there exists a solution $w(x)$ of (A) such that $w(a) = w(x_0) = 0$. This solution is by the above part oscillatory and so $z(x)$ is oscillatory too.

Remark 1. From the proof of Theorem 1 it follows that if the coefficients of (A) are such that $2C(x) - B'(x) \geqq 0$ in $(0, \infty)$, then every solution of (A) with a double zero at the point a has not a simple zero in $(0, a)$.

Lemma 1. *Let the coefficients of* (A) *be such that* $2C(x) - B'(x) \geqq 0$ *in* $(0, \infty)$. *Then there exists a solution of* (A) *which is nonnegative in* $(0, \infty)$.

Proof. We use the method of the proof of Theorem 14 in [5], p. 270. Let $\{x_n\}$ be a sequence of numbers in $(0, \infty)$ and let $\lim x_n = \infty$. Let x_0 be an arbitrary number in $(0, \infty)$ and let $y_n(x)$ be a solution of (A) which satisfies the conditions

$$y_n(x_n) = y_n'(x_n) = 0\,,$$
$$y_n^2(x_0) + y_n'^2(x_0) + y_n''^2(x_0) = 1\,.$$

From identity (1) it follows that $y_n(x) \geqq 0$ in $(0, x_n)$, or $y_n(x) \leqq 0$ in $(0, x_n)$. Without loss of generality we can suppose that $y_n(x) \geqq 0$ for $x \in (0, x_n)$. Let $u_0(x)$, $u_1(x)$, $u_2(x)$ be a fundamental system of (A) satisfying the initial conditions

$$u_i^{(k)}(x_0) = \delta_{i,k}\,, \qquad i, k = 0, 1, 2\,,$$

where $\delta_{i,k}$ is the Kronecker δ. Then $y_n(x) = a_n u_0(x) + b_n u_1(x) + c_n u_2(x)$, where $a_n^2 + b_n^2 + c_n^2 = 1$. Further we can show that there exists a subsequence $\{y_{n_i}(x)\}$ which converges uniformly on any compact subinterval of $(0, \infty)$ to the solution $y(x)$ of (A). Since $y_{n_i}(x) \geqq 0$ in $(0, x_{n_i})$, it implies that $y(x) = \lim y_{n_i}(x) \geqq 0$ in $(0, \infty)$.

Theorem 2. *Let* $2C(x) - B'(x) \geqq 0$ *in* $(0, \infty)$. *Then there exists a solution of* (A) *without zeros.*

Proof. If $2C(x) - B'(x) \equiv 0$ in $(0, \infty)$, then the existence of such solution follows from the property of the self-adjoint equation.

Suppose that equation (A) is oscillatory. If $2C(x) - B'(x) \equiv 0$ in some interval of infinite length, then there exists a solution without zeros in this interval, and by Theorem 1 this solution is without zeros in $(0, \infty)$, since equation (A) is oscillatory. Let $2C(x) - B'(x) \equiv 0$ do not hold in any interval of infinite length. Then there exists a sequence $\{a_n\}$ such that $\lim a_n = \infty$ and $2C(a_n) - B'(a_n) > 0$. Let $y(x)$ be the solution of (A) which was constructed in Lemma 1. By Theorem 1, the solution $y(x)$ of (A) is without zeros, or it has an infinite number of double zeros, i. e. there exists a sequence $\{x_n\}$ such that $\lim x_n = \infty$ and $y(x_n) = y'(x_n) = 0$. Hence, there exist numbers $x_r \neq x_s$ and a number a_p such that $x_r \leqq a_p \leqq x_s$. Then from identity (1) we find

$$\left[yy'' - \frac{1}{2}y'^2 + \frac{1}{2}\,B(x)y^2\right]_{x_r}^{x_s} = -\frac{1}{2}\int\limits_{x_r}^{x_s} [2C(x) - B'(x)]y^2\,\mathrm{d}x\,,$$

i. e.

$$0 = -\frac{1}{2}\int_{x_r}^{x_s} [2C(x) - B'(x)]y^2 \, dx ,$$

which is impossible, since the right-hand side is positive ($2C(x) - B'(x)$ is a positive function in some neighbourhood of the point a_p). Therefore, $y(x)$ must be positive in $(0, \infty)$.

Now we suppose that equation (A) is nonoscillatory. Let $y(x)$ be the solution of (A) constructed in Lemma 1. Let x_0 be the last double zero of $y(x)$. Let $u(x)$ be a solution of (A) which satisfies the following initial conditions at a point $x_1 > x_0$

$$\begin{aligned} u(x_1) &= y(x_1) \\ u'(x_1) &= y'(x_1) \\ u''(x_1) &> y''(x_1) \end{aligned} \tag{3}$$

Then the function $z(x) = u(x) - y(x)$ satisfies the initial conditions

$$z(x_1) = z'(x_2) = 0, \quad z''(x_1) > 0 .$$

From the identity (1) it follows that $z(x) \geqq 0$ in the interval $(0, x_1\rangle$ and hence, $u(x) \geqq y(x) \geqq 0$ in $(0, x_1\rangle$. From the conditions (3) it follows that $u(x)$ and $y(x)$ are linearly independent, therefore they cannot have common double zeros. So $u(x) > 0$ in $(0, x_1\rangle$. We will now show that $z(x)$ has not a zero in (x_1, ∞).

Suppose on the contrary that $z(x)$ has a zero in (x_1, ∞). Then there exists a constant $c \neq 0$ such that the solution $cz(x) - y(x)$ of (A) has a double zero at a point $\tau > x_1$ and a single zero at a point $\delta < x_1$, which contradicts Remark 1. This contradiction shows that $z(x) > 0$ in (x_1, ∞), i. e. $u(x) > y(x) > 0$ in (x_1, ∞). Consequently $u(x) > 0$ in all the interval $(0, \infty)$, and Theorem 2 is thus proved completely.

Remark 2. Theorem 2 generalizes Theorem 14 in paper [5].

Theorem 3. *Suppose that the coefficients of equation (A) satisfy the assumption $2C(x) - B'(x) \geqq 0$, or $2C(x) - B'(x) \leqq 0$ in $(0, \infty)$. Then the equation (A) is oscillatory if and only if its adjoint equation is oscillatory.*

Proof. The adjoint equation of (A) is the equation

$$y''' + B(x)y' + [B'(x) - C(x)]y = 0 . \tag{A'}$$

It can be seen that if the coefficients of (A) satisfy the condition $2C(x) - B'(x) \geqq 0$, then the coefficients of (A') satisfy the condition $2C(x) - B'(x) \leqq$

$\leqq 0$. Hence to prove Theorem 3 it is sufficient to show that if equation (A) is oscillatory and $2C(x) - B'(x) \geqq 0$, or $2C(x) - B'(x) \leqq 0$, then equation (A') is oscillatory too.

Let $2C(x) - B'(x) \geqq 0$. By Theorem 2, there exists a solution $u(x) > 0$ in $(0, \infty)$. Since equation (A) is oscillatory, there exists an oscillatory solution $v(x)$ of (A). Then the function $w(x) = v'(x)u(x) - v(x)u'(x)$ is the solution of the adjoint equation. If x_1, x_2 are two consecutive zeros of $v(x)$, then

$$\int_{x_1}^{x_2} \frac{v'(x)u(x) - v(x)u'(x)}{u^2(x)}\, dx = \left[\frac{v(x)}{u(x)}\right]_{x_1}^{x_2} = 0\,.$$

The last equality implies that $w(x)$ has a zero in any interval (x_1, x_2) and hence, equation (A') is oscillatory.

Let $2C(x) - B'(x) \leqq 0$ in $(0, \infty)$. Let $y(x)$ be a solution of (A) such that $y(x_0) = y'(x_0) = 0$, $y''(x_0) = 1$. Then from the identity (2) it follows that $y(x)$ has not a simple zero in (x_0, ∞), and so $y(x) \geqq 0$ for $x > x_0$. There are now two possibilities; either $y(x)$ has an infinite number of double zeros or there exists a number b such that $y(x) > 0$ in (b, ∞). Equation (A) is oscillatory then there exists its solution $z(x)$, linearly independent with $y(x)$. Then the function $w(x) = z'(x)y(x) - z(x)y'(x)$ is the solution of (A') and it is oscillatory in both cases. In the first case, (A') is oscillatory since $w(x)$ vanishes at the zeros of $y(x)$. In the second case we can use the same way as under the assumption $2C(x) - B'(x) \geqq 0$, but in the interval (b, ∞).

3. Comparison theorems

We begin with the following lemma which generalizes Lemma 1.2 in [4].

Lemma 2. *Let the functions $u(x)$ and $v(x)$ have a derivative of the second order in $(a, b) \subset (0, \infty)$. Let the function $u(x)$ have the properties*

$$u(\alpha) = u'(\alpha) = 0, \quad u''(\alpha) > 0,$$

$$u(\beta) = 0, \quad u(x) > 0 \quad \text{in } (\alpha, \beta), \text{ where } \quad a < \alpha < \beta < b\,.$$

Let the function $v(x)$ be such that

$$v(\alpha) \geqq 0,\ v'(\alpha) > 0,\ v(x) > 0 \quad \text{in} \quad (\alpha, \beta\rangle\,.$$

Then there exists a number $\tau \in (\alpha, \beta)$ and a constant $c > 0$ such that the function $f(x) = v(x) - cu(x)$ has a double zero at the point $\tau, f''(\tau) \geqq 0$ and $f(x) \geqq 0$ in $\langle\alpha, \tau\rangle$.

Proof. Consider the function

$$w(x) = \begin{cases} \dfrac{u(x)}{v(x)} & \text{for} \quad x \in (\alpha, \beta\rangle \\ 0 & \text{for} \quad x = \alpha \end{cases} .$$

From the properties of the functions $u(x)$ and $v(x)$ it follows that the function $w(x)$ is continuous in $\langle\alpha, \beta\rangle$, $w(\alpha) = w(\beta) = 0$ and $w(x) > 0$ in (α, β). Therefore $w(x)$ has a maximum in $\langle\alpha, \beta\rangle$ at a point $\tau \in (\alpha, \beta)$. This maximum we denote as $1/c$, where c is a positive number. Thus

$$w(x) \leqq \frac{1}{c} \quad \text{and} \quad w(\tau) = \frac{1}{c}$$

for all $x \in \langle\alpha, \beta\rangle$, i. e.

$$\frac{u(x)}{v(x)} \leqq \frac{1}{c} \quad \text{and} \quad \frac{u(\tau)}{v(\tau)} = \frac{1}{c}$$

for all $x \in (\alpha, \beta\rangle$.

From the last inequalities and the facts that $v(\alpha) \geqq 0$, $u(\alpha) = 0$ we see that

$$v(x) - cu(x) \geqq 0 \quad \text{for all} \quad x \in \langle\alpha, \beta\rangle$$

$$v(\tau) - cu(\tau) = 0 \quad \text{for} \quad \tau \in (\alpha, \beta) .$$

Thus the function $f(x) = v(x) - cu(x)$ has a minimum in the interval $\langle\alpha, \beta$ at the point $\tau \in (\alpha, \beta)$, and for all $x \in \langle\alpha, \beta\rangle$ is $f(x) \geqq 0$. Since the functions $u(x)$ and $v(x)$ have a derivative of the second order in $\langle\alpha, \beta\rangle$, there results $f'(\tau) = 0$ and $f''(\tau) \geqq 0$.

Theorem 4. *Consider the differential equations*

$$(A) \qquad y''' + B(x)y' + C(x)y = 0$$

$$(a) \qquad z''' + b(x)z' + c(x)z = 0 .$$

Suppose that the coefficients of (A) *and* (a) *satisfy the assumptions*

$$(4) \qquad B(x) \geqq b(x),\ 2C(x) - B'(x) \geqq 2c(x) - b'(x),\ 2C(x) - B'(x) \geqq 0 .$$

Let α, β *be two consecutive zeros of a solution* $z(x)$ *of* (a). *Let* α *be the double zero of* $z(x)$. *Then the solution* $y(x)$ *of* (A) *with the single zero at the point* α *has a zero in the interval* $(\alpha, \beta\rangle$.

Proof. Let $\bar{z}(x)$ be a solution of (a) such that $\bar{z}(\alpha) = \bar{z}'(\alpha) = 0$, $\bar{z}(\beta) = 0$. Suppose that $\bar{z}'(\alpha) > 0$ and $\bar{z}(x) > 0$ in (α, β). Let $y(x)$ be a solution of (A) such that $y(\alpha) = 0$, $y'(\alpha) > 0$. Suppose on the contrary that $y(x) > 0$ in $(\alpha, \beta\rangle$.

Therefore, by Lemma 2, there exists such a constant $c > 0$ that the function $\bar{f}(x) = y(x) - c\bar{z}(x)$ has a double zero at a point $\tau \in (\alpha, \beta)$ and at the same time $\bar{f}(x) \geqq 0$ in $\langle \alpha, \tau \rangle$ and $\bar{f}''(\tau) \geqq 0$. If we denote $c\bar{z}(x) = z(x)$, then the function $f(x) = y(x) - z(x)$ satisfies the conditions

$$(5) \qquad f(\tau) = f'(\tau) = 0, \quad f''(\tau) \geqq 0, \quad f(x) \geqq 0 \quad \text{in} \quad \langle \alpha, \tau \rangle .$$

Multiplying (A) by $y(x)$ and (a) by $z(x)$ we obtain the following identities

$$\left[yy'' - \frac{1}{2} y'^2 + \frac{1}{2} B(x) y^2 \right]' = -\frac{1}{2} [2C(x) - B'(x)] y^2$$

$$\left[zz'' - \frac{1}{2} z'^2 + \frac{1}{2} b(x) z^2 \right]' = -\frac{1}{2} [2c(x) - b'(x)] z^2 .$$

Subtraction of these identities and integration over (α, τ) yields

$$(6) \qquad \left[\left(zz'' - \frac{1}{2} z'^2 + \frac{1}{2} b(x) z^2 \right) - \left(yy'' - \frac{1}{2} y^2 + \frac{1}{2} B(x) y^2 \right) \right]_a^\tau =$$

$$= \frac{1}{2} \int_a^\tau [(2C(x) - B'(x)) y^2 - (2c(x) - b'(x)) z^2] \, dx .$$

Since $f(x) \geqq 0$ in $\langle \alpha, \tau \rangle$, $y(x) \geqq z(x) \geqq 0$. Then $y^2(x) \geqq z^2(x)$ for all $x \in \langle \alpha, \tau \rangle$. From this inequality and from (4) we obtain that for all $x \in \langle \alpha, \tau \rangle$

$$[2C(x) - B'(x)] y^2(x) \geqq [2c(x) - b'(x)] z^2(x) .$$

Hence the right-hand side of (6) is nonnegative, and therefore

$$\left[\left(zz'' - \frac{1}{2} z'^2 + \frac{1}{2} b(x) z^2 \right) - \left(yy'' - \frac{1}{2} y'^2 + \frac{1}{2} B(x) y^2 \right) \right]_\alpha^\tau \geqq 0 .$$

Since $z(x) = cz(x)$, then $z(\alpha) = z'(\alpha) = 0$, $z''(\alpha) > 0$. Further $y(\alpha) = 0$, $y(\tau) = z(\tau)$ and $y'(\tau) = z'(\tau)$. After the substitution in the last inequality we obtain

$$- z(\tau) [y''(\tau) - z''(\tau)] - \frac{1}{2} [z'^2(\tau) - y'^2(\tau)] -$$

$$- \frac{1}{2} z^2(\tau) [B(\tau) - b(\tau)] - \frac{1}{2} y'^2(\alpha) \geqq 0,$$

i. e.

$$-z(\tau) \, . f''(\tau) - z'(\tau) f'(\tau) - \frac{1}{2} z^2(\tau) [B(\tau) - b(\tau)] - \frac{1}{2} y'^2(\tau) \geqq 0.$$

From the assumptions (4) and from the inequalities (5) it follows that the left-hand side of the last inequality is negative, which is a contradiction. Consequently the solution $y(x)$ of (A) has a zero in the interval $(\alpha, \beta\rangle$.

Theorem 5. *Suppose that the coefficients of* (A) *and* (a) *satisfy* $B(x) \geqq b(x)$, $2C(x) - B'(x) \geqq 2c(x) - b'(x)$, $2C(x) - B'(x) \geqq 0$. *Let the equation* (a) *be such that any of its solutions with the double zero has another zero. Then the equation* (A) *is oscillatory.*

Proof. Let $z(x)$ be a solution of (a) which satisfies the conditions $z(\alpha) = z'(\alpha) = 0$, $z''(\alpha) > 0$ and $z(\beta) = 0$, where $0 < \alpha < \beta$. Let $y(x)$ be a solution of (A) which has a simple zero at the point α. By Theorem 4, the solution $y(x)$ has a zero, denote it as α_1, in $(\alpha, \beta\rangle$. The zero α_1 of $y(x)$ is simple. (From Remark 1 it follows that if α_1 were the double zero of $y(x)$, then there α would not be the simple zero of $y(x)$). Now consider the solution $z(x)$ of (a) which has the double zero at the point α_1 and let $z(\beta_1) = 0$, where $\beta_1 > \alpha_1$. Hence, by Theorem 4, the solution $y(x)$ of (A) has another simple zero in the interval $(\alpha_1, \beta_1\rangle$. By induction we obtain that $y(x)$ has an infinite number of zeros in $(0, \infty)$. These zeros cannot have a finite limit point, otherwise $y(x) \equiv 0$. Consequently equation (A) is oscillatory.

Corollary 1. *Suppose that the coefficients of* (A) *and* (a) *satisfy* (4). *Suppose that the equation* (a) *is of class* V_1 *and is oscillatory. Then equation* (A) *is oscillatory.*

Proof. If equation (a) is of class V_1 and is oscillatory then, by Theorem 3.4 in [1], every solution with a double zero has another simple zero. Hence, by Theorem 5, equation (A) is oscillatory.

Theorem 6. *Suppose that the coefficients of equations* (A) *and* (a) *satisfy*

$$B(x) \geqq b(x), \quad 2C(x) - B'(x) \geqq 2c(x) - b'(x) \geqq 0\,. \tag{7}$$

Then equation (A) *is oscillatory if equation* (a) *is oscillatory.*

Proof. If equation (a) is oscillatory and $2c(x) - b'(x) \geqq 0$, then every solution with a double zeros is oscillatory by Theorem 1. Hence equation (A) is oscillatory by Theorem 5.

Theorem 7. *Suppose the coefficients of equations* (A) *and* (a) *satisfy*

$$B(x) \geqq b(x), \quad 2C(x) - B'(x) \leqq 2c(x) - b'(x) \leqq 0\,. \tag{8}$$

Then equation (A) *is oscillatory if equation* (a) *is oscillatory.*

Proof. The equations adjoint to equations (A) and (a), respectively, are

$$y''' + B(x)y' + [B'(x) - C(x)]y = 0\,, \tag{A'}$$

$$z''' + b(x)z' + [b'(x) - c(x)]z = 0\,. \tag{a'}$$

Let equation (a) be oscillatory and let $2c(x) - b'(x) \leqq 0$. Then equation (a') is oscillatory by Theorem 3. From the assumption (8) it follows that

$$B(x) \geqq b(x)$$

$$2[B'(x) - C(x)] - B'(x) \geqq 2[b'(x) - c(x)] - b'(x) \geqq 0\,,$$

i. e. the coefficients of equations (A') and (a') satisfy the assumptions (7) of Theorem 6 and hence, equation (A') is oscillatory. Again, by Theorem 3, equation (A) is oscillatory too.

Corollary 2. *Let the coefficients of* (A) *and* (a) *satisfy* $B(x) \geqq b(x)$, $2C(x) - B'(x) \leqq 2c(x) - B'(x)$, $2C(x) - B'(x) \leqq 0$. *Let* (a) *be of class* V_2 *and let* (a) *be oscillatory, then* (A) *is oscillatory.*

Proof. From the assumptions of this corollary it follows that the coefficients of the adjoint equations (A') and (a') satisfy the assumptions (4) of Theorem 4. If equation (a) is of class V_2 and is oscillatory, then, by Theorem 4.7 in [1], equation (a') is of class V_1 and is oscillatory. By Corollary 1, equation (A') is oscillatory and by Theorem 3, equation (A) is oscillatory too.

Theorem 8. *Suppose the coefficients* (A) *satisfy the assumption* $2C(x) - B'(x) \geqq 0$, *resp.* $2C(x) - B'(x) \leqq 0$. *Let the function* $B(x)$ *be such that the differential equation of the second order*

$$y'' + \frac{1}{4}B(x)y = 0 \tag{9}$$

is oscillatory. Then equation (A) *is oscillatory.*

Proof. Consider the self-adjoint equation

$$z''' + B(x)z' + \frac{1}{2}B'(x)z = 0\,. \tag{10}$$

Let y_1, y_2 be a fundamental system of (9). Then a fundamental system of (10) is $z_1 = y_1^2$, $z_2 = y_1y_2$, $z_3 = y_2^2$. Since equation (9) is oscillatory, then equation (10) is oscillatory as well. A relation $2c(x) - b'(x)$ in (10) is identical with zero. Therefore, by Theorems 6 and 7, equation (A) is oscillatory.

4. Oscillation criteria

The comparison theorems will lead to oscillation criteria whenever the oscillatory behavior of a given equation is known. As the first example, consider the equation with a constant coefficient

$$z''' + pz' = 0, \quad \text{where} \quad p > 0,$$

or the Euler Equation

$$z''' + \frac{p}{x^2} z' - \frac{p}{x^3} z = 0, \quad \text{where} \quad p > 1,$$

which are oscillatory, and the relation $2c(x) - b'(x)$ is identical with zero. Then from Theorems 6 and 7 the following theorem follows immediately

Theorem 9. *Suppose the coefficients* (A) *satisfy* $2C(x) - B'(x) \geqq 0 \,(\leqq 0)$ *and*

$$B(x) \geqq p \quad \textit{for} \quad p > 0, \quad \textit{or} \quad B(x) \geqq \frac{p}{x^2} \quad \textit{for} \quad p > 1.$$

Then equation (A) *is oscillatory.*

Remark 3. Theorem 9 generalizes Theorem 5.5 in [1] in which the assumptions $2C(x) - B'(x) > 0$ and $C(x) > -p/x^3$ are in addition demanded.

The other oscillation criteria will be obtained by applying the comparison theorems to the differential equation with constant coefficients

$$z''' + pz' \pm \frac{q}{2} z = 0, \tag{11}$$

where $p \leqq 0$, $q > 0$. This equation is oscillatory if and only if

$$q > \frac{4}{3\sqrt{3}} (-p)^{\frac{3}{2}}.$$

Similary, applying the comparison theorems to the Euler Equation

$$z''' + \frac{p}{x^2} z' - \frac{\pm \frac{\varepsilon}{2} - p}{x^3} z = 0, \tag{12}$$

where $p < 1$, $\varepsilon > 0$, and which is oscillatory if and only if

$$\varepsilon > \frac{4}{3\sqrt{3}} (1 - p)^{\frac{3}{2}},$$

we obtain the further oscillation criteria.

Both cases are included in the following theorem

Theorem 10. *Let the coefficients* (A) *satisfy the assumptions*

$$B(x) \geqq p \quad \textit{and} \quad |2C(x) - B'(x)| \geqq q, \tag{13}$$

where $p \leqq 0$ *and* $q > \frac{4}{3\sqrt{3}} (-p)^{\frac{3}{2}}$, p, q *are constants, or the assumptions*

(14) $$B(x) \geqq \frac{p}{x^2} \quad and \quad |2C(x) - B'(x)| \geqq \frac{\varepsilon}{x^3},$$

where $p < 1$ *and* $\varepsilon > \frac{4}{3\sqrt{3}}(1-p)^{\frac{3}{2}}$, p, ε *are constants. Then equation* (A) *is oscillatory.*

Proof. From the assumptions (13) it follows that the number q is positive. If $2C(x) - B'(x) \geqq 0$, then from the assumptions (13) we obtain that the assumptions of Theorem 6 for equations (A) and (11) are valid. Indeed

$$B(x) \geqq p \quad \text{and} \quad 2C(x) - B'(x) \geqq q = 2\frac{q}{2} - p' \geqq 0.$$

Since equation (11) is oscillatory, then, by Theorem 6, equation (A) is oscillatory.

If $2C(x) - B'(x) \leqq 0$, then from the assumptions (13) it follows

$$B(x) \geqq p \quad \text{and} \quad 2C(x) - B'(x) \leqq q = 2\frac{-q}{2} - p' \leqq 0.$$

Since p and q are such that equation (11) is oscillatory, then, by Theorem 7 equation (A) is oscillatory.

Similarly, we prove the second part of Theorem 10 by applying Theorems 6 and 7 to the differential equations (A) and (12).

Corollary 3. *Let in equation* (A) *be* $B(x) \equiv 0$. *Then from Theorem 10 we have that equation* (A) *is oscillatory if*

$$|C(x)| > \frac{2}{3\sqrt{3}}\frac{\varepsilon}{x^3}, \ \varepsilon > 1.$$

Example. Let us consider the differential equation

(15) $$y''' + \sin x y' + \frac{1}{2}\left(\cos x + \frac{2}{3}\sqrt{2}\right) y = 0.$$

Both coefficients $B(x) = \sin x$ and $C(x) = \frac{1}{2}\left[\cos x + \frac{2}{3}\sqrt{2}\right]$ of this equation are not of the constant sign in $(0, \infty)$. Therefore, for finding out whether the differential equation (15) is oscillatory or not, we cannot use the criteria known so far (e. g. Hanan 1], Greguš [5], Lazer [6]). However, by Theorem 10, equation (15) is oscillatory, because $B(x) = \sin x \geqq -1$ and

$$2C(x) - B'(x) = 2\frac{1}{2}\left(\cos x + \frac{2}{3}\sqrt{2}\right) - \cos x = \frac{4}{3\sqrt{2}} > \frac{4}{3\sqrt{3}}(+1)^{\frac{3}{2}}.$$

The following theorem gives sufficient conditions for (A) to be nonoscillatory.

Theorem 11. *Let the differential equation* (A) *be of class* V_1, *or class* V_2, *or the assumption* $2C(x) - B'(x) \geqq 0$, *or the assumption* $2C(x) - B'(x) \leqq 0$ *be valid. Let further the following assumptions be fulfilled*

(*i*) $B(x) \leqq p$ *and* $|2C(x) - B'(x)| \leqq q$,

where $p \leqq 0$ *and* $q \leqq \frac{4}{3\sqrt{3}}(-p)^{\frac{3}{2}}$, p, q *are constants, or*

(*ii*) $B(x) \leqq \frac{p}{x^2}$ *and* $|2C(x) - B'(x) \leqq \frac{\varepsilon}{x^3}$,

where $p \leqq 1$ *and* $\varepsilon \leqq \frac{4}{3\sqrt{3}}(1-p)^{\frac{3}{2}}$, p, ε *are constants. Then equation* (A) *is nonoscillatory.*

Proof. Let, for instance, equation (A) be of class V_1 and let $B(x) \leqq p$ and $|2C(x) - B'(x)| \leqq q$, where $p \leqq 0$ and $q \leqq \frac{4}{3\sqrt{3}}(-p)^{\frac{3}{2}}$. Suppose to the contrary that (A) is oscillatory. From the above assumption it follows that $p \geqq B(x)$ and $q \geqq 2C(x) - B'(x)$. Since equation (A) is of class V_1, then $2C(x) - B'(x) \not\equiv 0$ and hence $q > 0$. Therefore, by Corollary 1, the equation

$$z''' + pz' + \frac{q}{2}z = 0$$

is oscillatory. This contradicts the fact that the above equation is oscillatory if and only if $q > \frac{4}{3\sqrt{3}}(-p)^{\frac{3}{2}}$.

We prove another case included in Theorem 11.

Let $2C(x) - B'(x) \leqq 0$, $B(x) \leqq \frac{p}{x^2}$ and $|2C(x) - B'(x)| \leqq \frac{\varepsilon}{x^3}$, where $p \leqq 1$ and $\varepsilon \leqq \frac{4}{3\sqrt{3}}(1-p)^{\frac{3}{2}}$ are constants. From these assumptions it follows that $\frac{p}{x^2} \geqq B(x)$ and $-\frac{\varepsilon}{x^3} \leqq 2C(x) - B'(x) \leqq 0$. Suppose on the contrary that equation (A) is oscillatory. Then, by Theorem 7, the equation

$$z''' + \frac{p}{x^2}z' + \frac{-\frac{\varepsilon}{2} - p}{x^3}z = 0$$

is oscillatory, which is a contradiction since this equation is oscillatory if and only if $\varepsilon > \frac{4}{3\sqrt{3}}(1-p)^{\frac{3}{2}}$.

Corollary 4. *If in equation (A) there is $B(x) \equiv 0$ and the function $C(x)$ is of one sign, then (A) is nonoscillatory if*

$$|C(x)| \leqq \frac{2}{3\sqrt{3}} \frac{1}{x^3}.$$

Theorem 12. *Let equation (A) be such that $2C(x) - B'(x) \geqq 0$ and $B(x) \geqq \geqq p > 0$, or $B(x) \geqq \frac{p}{x^2}$, $p > 1$. Then equation (A) has a fundamental system which consists of two oscillatory and one nonoscillatory solutions. The nonoscillatory solution is without zeros.*

Proof. Equation (A) is oscillatory by Theorem 9. Then the solution $y_1(x)$ of (A) which satisfies the initial conditions $y_1(\alpha) = y_1'(\alpha) = 0$, $y_1''(\alpha) = 1$ is oscillatory. From Theorem 1 we obtain that the solution $y_2(x)$ of (A) such that $y_2(\alpha) = y_2''(\alpha) = 0$, $y_2'(\alpha) = 1$ is oscillatory too. By Theorem 2, equation (A) has a solution without zeros; i. e., there exists a solution $y_3(x)$ of (A) such that $y_3(\alpha) = a^2$, $a \neq 0$, $y_3'(\alpha) = b$, $y_3''(\alpha) = c$, where a, b, c are some constants. Hence the Wroskian $W[y_1(\alpha), y_2(\alpha), y_3(\alpha)] = a^2 \neq 0$ and therefore the solution $y_1(x)$, $y_2(x)$, $y_3(x)$ of (A) form a fundamental system of (A).

Theorem 13. *Suppose the coefficients of (A) satisfy the assumptions* (13) *or* (14). *Suppose that $2C(x) - B'(x) \geqq 0$ and $B(x) \leqq 0$. Then equation (A) has a fundamental system which consists of two oscillatory and one nonoscillatory solution. The nonoscillatory solution tends monotonically to zero along with its first derivative as $x \to \infty$ and every nonoscillatory solution of (A) is a constant multiple of this solution. At the same time equation (A) has a fundamental system which consists of three oscillatory solutions.*

Proof. In the same way as in Theorem 12 we obtain that equation (A) has a fundamental system $y_1(x)$, $y_2(x)$, $y_3(x)$ such that $y_1(x)$, $y_2(x)$ are oscillatory and $y_3(x)$ is a nonoscillatory positive solution of (A). By Theorem 1 in [7], the solution $y_3(x)$ tends to zero along with its first derivative as $x \to \infty$. Since equation (A) is oscillatory, then every nonoscillatory solution of (A) is a constant multiple of $y_3(x)$.

We still have to show that there exist three linearly independent and oscillatory solutions of (A). From the above it follows that there exist two oscillatory solutions $y_1(x)$, $y_2(x)$ of (A) for which the following conditions are fulfilled $y_1(\alpha) = y_1'(\alpha) = 0$, $y_1''(\alpha) = 1$, $y_2(\alpha) = y_2''(\alpha) = 0$, $y_2'(\alpha) = 1$. Let $y_2(\beta) = 0$ for a point $\beta > \alpha > 0$. Consider a solution $y_0(x)$ of (A) with the properties $y_0(\alpha) = 1$, $y_0'(\alpha) = y_0''(\alpha) = 0$. We shall show that $y_0(x_0) = 0$ for a point $x_0 \in (\alpha, \beta\rangle$. Suppose on the contrary that $y_0(x) > 0$ in $(\alpha, \beta\rangle$. Then by Lemma 2, there exists a constant c and a number $\tau \in (\alpha, \beta)$ such that $y_0(\tau) = cy_2(\tau)$ and $y_0'(\tau) = cy_2'(\tau)$. Hence the function $u(x) = y_0(x) - cy_2(x)$ is a solution of (A) and it has a double zero at the point τ. Substituting $u(x)$ instead of $y(x)$ into the identity (1) and integrating over (α, τ) we obtain

$$[y_0(\alpha) - cy_2(\alpha)]\,[y_0''(\alpha) - cy_2''(\alpha)] - \frac{1}{2}\,[y_0'(\alpha) - cy_0'(\alpha)]^2 +$$

$$+ \frac{1}{2}\,B(\alpha)\,[y_0(\alpha) - cy_2(\alpha)]^2 > 0\,,$$

i. e.

$$-\frac{1}{2}\,c^2 + \frac{1}{2}\,B(\alpha) > 0\,,$$

which is a contradiction since $c \neq 0$ and $B(\alpha) \leqq 0$. Therefore the solution $y_0(x)$ has a zero in $(\alpha, \beta\rangle$. Since any solution of (A), which has a zero is oscillatory, $y_0(x)$ is oscillatory too. The solutions $y_0(x)$, $y_1(x)$, $y_2(x)$ are linearly independent since $W[y_0(\alpha), y_1(\alpha), y_2(\alpha)] = -1$.

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