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DECOMPOSITION OF A DIGRAPH INTO ISOMORPHIC SUBGRAPHS

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The papers [1], [2], [3], 4], [5] deal with decompositions of graphs into isomorphic subgraphs. Everywhere, except in paper [1], only undirected graphs are considered. In paper [1] self-complementary digraphs are discussed, but only their enumeration is done and no properties of such graphs are described.

This paper brings some results concerning decompositions of a digraph into isomorphic subgraphs. These results ase mostly analogous to the results of papers [2], [3], [4], [5], expressed for the case of undirected graphs.

§ 1.

We shall begin with a theorem on self-complementary digraphs.

Theorem 1. *Let a complete digraph $\vec{K}_n$ with n vertices *) be given and let a permutation p on its vertex set U be given such that each of its cycles (except for at most one cycle formed by a fixed vertex) has an even or infinite number of elements. Then there exists a decomposition of the graph $\vec{K}_n$ into two subgraphs G and $\bar{G}$, each of which contains all vertices of the graph $\vec{K}_n$ and which are edge-disjoint and isomorphic to each other, while a corresponding isomorphism f induces the permutation p on the set U.*

Proof. Let a permutation p satisfying the assumption be given. Let u_1, v_1 be two vertices of the graph K_n. Put the directed edge $\overrightarrow{u_1v_1}$ of the graph $\vec{K}_n$ into G. Then for every even number k put the edge $\overrightarrow{p_k(u_1)\,p_k(u_1)}$ into G. On the other hand, for odd k put $\overrightarrow{p_k(u_1)\,p_k(v_1)}$ into $\bar{G}$. We shall prove that no edge is put in this way at the same time into both G and $\bar{G}$. If the edge $\overrightarrow{xy}$ were at the same time in both G and $\bar{G}$, this would mean that $x = p^k(u_1)$, $y = p^k(v_1)$, where k is an even number and at the same time $x = p^l(u_1)$,

*) A complete digraph is a digraph, in which each pair of vertices $\{u, v\}$ is joined by both directed edges $\overrightarrow{vu}$ and $\overrightarrow{uv}$.

$y = p^l(v_1)$, where l is an odd number. As the permutation p has at most one fixed vertex, at least one of the vertices x, y is not fixed; without the loss of generality let x be such a vertex. Therefore there would be $x = p^k(u_1) = p^l(u_1)$, which would imply $u_1 = p^{k-l}(u_1)$ and the vertex u_1 would belong to a cycle of the permutation p, whose number of vertices would be a divisor of $k - l$. But the number $k - l$ is the difference between an even and an odd number, therefore it is odd and each of its divisors is also odd, which is a contradiction with the assumption that the number of vertices of each cycle of the permutation p except for at most one cycle formed by a fixed vertex, is even or infinite.

Now we shall take a pair of vertices u_2, v_2 such that the edge $\overrightarrow{u_2v_2}$ is not yet put into any of the graphs G, $\bar{G}$ and do the same as we have done with the vertices u_1, v_1. If $\overrightarrow{u_2v_2}$ is not yet put into any of the graphs G, $\bar{G}$, then neither the edge $\overrightarrow{p^k(u_2)p^k(v_2)}$ for any k is put yet, because in the reverse case there would have to be $\overrightarrow{p^k(u_2)p^k(v_2)} = \overrightarrow{p^l(u_1)p^l(v_1)}$ for some positive integer l and this would imply $\overrightarrow{u_2v_2} = \overrightarrow{p^{l-k}(u_1)p^{l-k}(v_1)}$ and the edge $\overrightarrow{u_2v_2}$ would be put into G or $\bar{G}$. Then we choose two vertices u_3, v_3 such that the edge $\overrightarrow{u_3v_3}$ is not yet put and again we do the same. In this way we proceed until each edge of the graph $\vec{K}_n$ is put either into G, or into $\bar{G}$. The result is evidently the wanted decomposition.

Now let us prove another theorem, which is properly the inverse of Theorem 1.

Theorem 2. *Let a decomposition of a complete directed graph $\vec{K}_n$ into two isomorphic edge-disjoint subgraphs G, $\bar{G}$ and an isomorphic mapping f of the graph G onto $\bar{G}$ be given. Then each cycle of the permutation p of the vertex set U of the graph $\vec{K}_n$ induced by the mapping f has an even or infinite number of vertices, except for at most one cycle formed by a fixed vertex.*

Proof. Let a cycle $\mathscr{C}$ of the permutation p consist of the vertices u, $p(u)$, $\ldots$, $p^{k-1}(u)$, where k is an odd number greater than one, and let the edge $\overrightarrow{up(u)}$ belong (without the loss of generality) to G. Then the edge $\overrightarrow{p^k(u)p^{k+1}(u)}$ must belong to G, because k is an odd number. But $p^k(u) = u$, $p^{k+1}(u) = p(u)$, because k is the length of the cycle $\mathscr{C}$. Therefore the edge $\overrightarrow{up(u)}$ would belong to both G and $\bar{G}$, which is a contradiction with the assumption that G and $\bar{G}$ are edge-disjoint. Therefore the permutation p has no cycle with an odd number of vertices greater than one. If the permutation p had two fixed vertices, also the edges joining them would be fixed and they would belong at the same time to both G and $\bar{G}$. Thus also this case is excluded and the theorem holds.

Corollary. *Every complete digraph $\vec{K}_n$ can be decomposed into two isomorphic edge-disjoint subgraphs. In the case when n is odd, the corresponding isomorphic* *mapping has exactly one fixed vertex. In the case when n is even, this mapping has no fixed vertex. In the case of an infinite complete digraph both cases can occur.*

If we compare these results with those of papers [2] and [3], concerning undirected graphs, we see that in digraphs it is not necessary to make the assumption that the number of vertices of each cycle of the permutation p is divisible by four (except for a cycle formed by a fixed vertex) and that the number of vertices of the graph must be congruent with 0 or 1 modulo 4.

Theorem 3. *Let a decomposition of the complete digraph $\vec{K}_n$ into two isomorphic subgraphs G and $\bar{G}$ be given so that any two vertices of $\vec{K}_n$ are joined exactly by one edge of G and exactly by one edge of $\bar{G}$. Then each cycle of the permutation p induced on the vertex set U of the graph $\vec{K}_n$ by an isomorphic mapping f of the graph G onto $\bar{G}$ (except for at most one cycle formed by a fixed vertex) has the number of vertices either congruent with 2 modulo 4, or it is infinite.*

Proof. According to Theorem 2 each of such cycles must have either an even, or an infinite number of elements. Therefore if this number is finite, it is congruent either with 0, or with 2 modulo 4. Assume that there exists a cycle $\mathscr{C}$ with the number of vertices divisible by four; let its vertices be u, $p(u)$, $\ldots$, $p^{4k-1}(u)$, where k is a positive integer. Now let us take the pair of vertices u, $p^{2k}(u)$. According to the assumption of the theorem these two vertices are joined exactly by one edge of G and exactly by one edge of $\bar{G}$. Without the loss of generality let $\overrightarrow{up^{2k}(u)}$ belong to G and let $\overrightarrow{p^{2k}(u)u}$ belong to G. The image of the edge $\overrightarrow{up^{2k}(u)}$ in the mapping f^{2k} must belong also to G, because the exponent $2k$ is even:; but this image is the edge $\overrightarrow{p^{2k}(u)u}$, which belongs according to the assumption to $\bar{G}$, which is a contradiction. Thus it is proved that the number of vertices of each cycle, if it is not infinite, or if it is not formed by a fixed vertex, is congruent with 2 modulo 4.

Theorem 4. *Let the complete digraph $\vec{K}_n$ with n vertices and a permutation p, on it each of whose cycles (except for at most one cycle formed by a fixed vertex) has the number of vertices congruent with 2 modulo 4 or infinite, be given. Then there exists a decomposition of the graph $\vec{K}_n$ into two isomorphic and edge-disjoint subgraphs G and $\bar{G}$, each of which contains all vertices of the graph $\vec{K}_n$ and any two vertices of the graph $\overrightarrow{K_n}$ are joined exactly by one edge of G and exactly by one edge of $\bar{G}$.*

The proof is analogous to the proof of Theorem 1 with the difference that for each pair of vertices u_i, v_i we put one of the edges $\overrightarrow{u_iv_i}$, $\overrightarrow{v_iu_i}$ into G and at the same time the other into $\bar{G}$. If there does not exist such k that $p^k(u_i) = v_i$, $p^k(v_i) = u_i$, this can be done without difficulties. If such a k exists, it means that $p^{2k}(u_i) = u_i$ and therefore the number of vertices of the cycle which the vertex u_i (and evidently also the vertex v_i) belongs to is a divisor of the number $2k$. This number of vertices is evidently not infinite, neither can it be equal to one, because we assume that $p^k(u_i) = v_i \neq u_i$, thus u_i is not a fixed vertex. The mentioned number of vertices is therefore congruent with 2 modulo 4, therefore it can be expressed as $2l$, where l is an odd number. This implies that the number $2l$ is a divisor of the number $2k$, therefore the number l is a divisor of the number k. If k is even, it is evidently a multiple of the number $2l$, which implies $p^k(u_i) = u_i$, which is a contradiction with the assumption. The number k must therefore be odd, which is in agreement with the fact that $\overrightarrow{u_iv_i}$ and $\overrightarrow{v_iu_i}$ belong to different graphs of the pair G, $\bar{G}$.

Corollary. *Every complete digraph $\vec{K}_n$ can be decomposed into two isomorphic and edge-disjoint subgraphs G and $\bar{G}$ so that each of the graphs G, $\bar{G}$ contains all vertices of the graph $\vec{K}_n$ and any two vertices of the graph $\vec{K}_n$ are joined exactly by one edge of G and exactly by one edge of $\bar{G}$.*

§ 2.

In this section we shall discuss the results concerning digraphs and analogous to the results of paper [4] for undirected graphs. Similarly as in [4] we understand by the concept of an R_2-digraph such a digraph G which can be decomposed into two isomorphic and edge-disjoint subgraphs G_1 and G_2, while we shall assume again that an isomorphic mapping of the graph G_1 onto G_2 induces the same permutation of the vertex set U of the graph G as some automorphism of the graph G. The vertex set of the graph G will be denoted by U, its edge set by H, the edge sets of the graphs G_1 and G_2 consequently by H_1 and H_2. The complement of the graph G will be denoted by $\bar{G}$, its edge set by $\bar{H}$. The isomorphic mapping f of the graph G_1 onto G_2 induces a permutation p on the vertex set U, about whose cycles we shall state some lemmas. (Now we consider only finite graphs.)

Lemma 1. *If a cycle $\mathscr{C}$ of the permutation p has an odd number of vertices, then any two vertices of this cycle are joined by an edge of $\bar{H}$.*

Lemma 2. *If each of the cycles $\mathscr{C}_1$, $\mathscr{C}_2$ of the permutation p has an odd number of vertices, then no vertex of the cycle $\mathscr{C}_1$ is joined with any vertex of the cycle $\mathscr{C}_2$ by an edge of H.*

Lemma 3. *Let $\hat{p}$ be the permutation of the set $U \times U$ induced by the permutation p of the set U; let $\hat{\mathscr{C}}$ be any of its cycles. If $\hat{\mathscr{C}}$ contains a pair of vertices of U which does not belong to the pairs described in the preceding lemmas (i. e. the pairs of vertices which both belong to cycles with odd numbers of vertices), then $\hat{\mathscr{C}}$ contains an odd number of elements.*

Lemmas 1 and 2 have exactly the same text as Lemmas 1 and 2 of paper [4] expressed for undirected graphs. Also the proofs are exactly the same, therefore we shall not quote them here. On the other hand, Lemma 3 of [4] does not hold for digraphs. Lemma 3 of this paper is similar to Lemma 4 of paper [4]. But these lemmas and their proofs differ in some details, thus we shall prove Lemma 3 here.

Proof of Lemma 3. Let $[u, v] \in U \times U$ and let $[u, v]$ not belong to the pairs described in Lemmas 1 and 2. Let q be the least positive integer such that $p^q([u, v]) = [u, v]$, that means that $p^q(u) = u$, $p^q(v) = v$. As $[u, v]$ does not belong to the pairs described in Lemmas 1 and 2, at least one of the vertices u, v belongs to a cycle of the permutation p with an even number r_1 of vertices; let the other belong to a cycle with r_2 elements. Evidently q is the least common multiple of r_1 and r_2, thus it is even. Now if the vertex set U is given and the permutation p on it, we shall use the construction which will be called the construction (K) and described in the following.

Let U and p be given. If we have two vertices u, v which form a pair described in Lemma 1 or 2, the edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$ belong to $\bar{H}$. Now take an arbitrary pair of vertices u_1, v_1, which is not yet joined, and put the edge $\overrightarrow{u_1v_1}$ into arbitrary one of the sets $\bar{H}, H_1, H_2$. If $\overrightarrow{u_1v_1} \in H_1$ (or $\overrightarrow{u_1v_1} \in H_2$ respectively) we have $f^q(\overrightarrow{u_1v_1}) \in H_2$ (or $f^q(\overrightarrow{u_1v_1}) \in H_1$ respectively) for q odd and $f^q(\overrightarrow{u_1v_1}) \in H_1$ (or $f^q(u_1v_1) \in H_2$ respectively) for q even. For $\overrightarrow{u_1v_1} \in \bar{H}$ there is $f^q(\overrightarrow{u_1v_1}) \in \bar{H}$ for each q. Therefore we shall join all these pairs by corresponding edges. Then again we choose one pair of the pairs not yet joined and go on in the same way until it is decided about each pair, by which edge it has to be joined.

The following theorem is an immediate consequence of Lemma 3.

Theorem 5. *Let a vertex set U and a permutation p on it be given. By the construction (K) always an R_2-digraph with the vertex set U and with the corresponding isomorphism inducing the permutation p will be created.*

It is evident that also every such R_2-digraph can be constructed by the construction (K).

Theorem 6. *If two positive integers m, n are given and m is even, $m \leqq n(n - 1)$, there exists an R_2-digraph with n vertices and m edges.*

Proof. Construct a complete digraph $\vec{K}_n$ with n vertices and choose a decomposition of it into two isomorphic and edge-disjoint subgraphs (see Theorem 1) and determine a corresponding permutation p in such a way that all cycles of the permutation p, except for at most one cycle formed by a fixed vertex, contain two vertices each. (This is evidently possible.) We shall easily verify that in the permutation $\hat{p}$ induced by this permutation on the edge set of the complete digraph $\vec{K}_n$ with the vertex set U, each cycle contains exactly two edges. Let $m' = n(n - 1) - m$. It is evidently an even number. From the graph $\vec{K}_n$ omit $m'/2$ pairs of edges $\{\vec{h}, \vec{k}\}$ such that $p(\vec{h}) = \vec{k}$, $p(\vec{k}) = \vec{h}$, i. e. pairs forming cycles of the permutation $\hat{p}$. The digraph which is created in this way from $\vec{K}_n$ is evidently the wanted digraph.

Now we shall study self-complementary R_2-digraphs. Let G be such a graph, $\bar{G}$ its complement, and let the digraph G be decomposed into two isomorphic edge-disjoint subgraphs G_1 and G_2. By f denote an isomorphic mapping of G onto $\bar{G}$ and by g an isomorphic mapping of G_1 onto G_2 (we assume again that g induces the same permutation on the vertex set of G as some automorphism of G).

Theorem 7. *A finite self-complementary R_2-digraph contains $4k$ or $4k + 1$ vertices, where k is a positive integer.*

Proof. The graphs G and $\bar{G}$ are edge-disjoint and their union is a complete digraph $\vec{K}_n$: it contains $n(n - 1)$ directed edges. Further the graphs G and $\bar{G}$ are isomorphic to each other, thus they contain the same number of edges; therefore each of them contains $\frac{1}{2}n(n-1)$ edges. By a similar consideration we shall prove that each of the graphs G_1 and G_2 contains $\frac{1}{4}n(n - 1)$ edges. This number must be an integer, therefore the number $n(n - 1)$ must be divisible by four. The difference between the numbers n and $n - 1$ is one, thus they are relatively prime. Thus either n, or $n - 1$ is divisible by four.

Theorem 8. *Let a vertex set U with $4k$ or $4k + 1$ elements (k is a positive integer) and a permutation p^* on it be given such that the number of elements of each of its cycles (except for at most one cycle formed by a fixed vertex) is even. Then there exists a self-complementary R_2-digraph G whose vertex set is U and the permutation p^* is induced by an isomorphic mapping g of the graph G onto its complement $\bar{G}$.*

The proof is analogous to the proof of Theorem 4 of [4]. Let the permutation p^* have no fixed vertex and let $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_q$ be its cycles. The cycle $\mathscr{C}_i$ $(i - 1, \ldots, q)$ contains the vertices $u_1^{(i)}, \ldots, u_{r_i}^{(i)}$. The numbers r_i are divisible by two for each $i = 1, \ldots, q$. Let n be the number of elements of the set U; evidently $n = \sum_{i=1}^{q} r_i$. Now for each vertex we shall introduce another notation so that the vertex $u_j^{(i)}$ will be denoted by v_k, where for $j \leqq r_i/2$ there is $k = j + \frac{1}{2} \sum_{z<i} r_z$, for $j > r_i/2$ there is $k = j + \frac{1}{2}(n - r_i + \sum_{z>i} r_z)$. Now by both directed edges (i. e. from the first into the second and from the second into the first vertex) we shall join any two verices with odd subscripts at v. Further for each even α we shall lead directed edges from v_α into all vertices whose subscripts at v are congruent with $\alpha + 1$ modulo 4 and from all vertices whose subscripts at v are congruent with $\alpha - 1$ modulo 4 into v_α. The graph thus constructed can be evidently transformed by an isomorphic mapping g inducing the permutation p^* into its complement. Define the mapping f so that for $k \leqq n - 2$ there is $f(v_k) = v_{k+2}$, $f(v_{n-1}) = v_{n+1}$, $f(v_n) = v_{n+2}$.

Now if U contains a fixed vertex w of the permutation p^*, we shall do the above described construction for $U \dot{-} \{w\}$ and the restriction of p^* on $U \dot{-} \{w\}$. Then we shall join w by both edges with each vertex of the set $U \dot{-} \{w\}$ which has an odd subscript at v. The vertex w will be a fixed vertex also in the mapping f. In this way the wanted graph is constructed (f is an isomorphic mapping of G_1 onto G_2).

§ 3.

In this section we shall study the decomposition of a complete digraph according to a given group. The definition of this decomposition is analogous to the definition for undirected graphs in [5].

Definition. *Let $\vec{K}_n$ be a complete digraph with n vertices, let H be a subgroup of the group of all automorphisms of this graph. By the decomposition of $\vec{K}_n$ according to H we mean such a decomposition $\mathscr{R}$ of $\vec{K}_n$ into pairwise edge-disjoint subgraphs (each of which contains all vertices of $\vec{K}_n$), which satisfies the following condition: To each element $\alpha \in H$ a graph $G(\alpha)$ of the decomposition $\mathscr{R}$ is assigned, in one-to-one manner, while if α, β are two elements of H, then the image of the graph $G(\alpha)$ in the mapping β is the graph $G(\beta\alpha)$.*

Now we shall state a theorem analogous to Theorem 1 of [5].

Theorem 9. *Let $\mathscr{R}$ be a decomposition of the graph $\vec{K}_n$ according to the group H, let K be a normal subgroup of the group H. Then there exists a decomposition $\mathscr{R}'$ of the graph $\vec{K}_n$ according to the factor-group H/K, which is a covering of the decomposition $\mathscr{R}$.*

The proof is the same as the proof of Theorem 1 of [5].

In the following, similarly as in [5], we shall restrict our considerations to Abelian groups.

First we shall state a lemma.

Lemma 4. *Let K_n be a finite complete digraph with n vertices, and let a decomposition $\mathscr{R}$ of the graph $\vec{K}_n$ according to a given Abelian group H exist. If the order m of the group H is a divisor of the number n and is greater than one, then there does not exist any vertex of the graph $\vec{K}_n$ fixed in all mappings of H.*

Proof. Let m be a divisor of n. Assume that there exists a vertex u fixed in all mappings of the group H. Let ε be the unit element of the group H and let $h_1, \ldots, h_r$ be exactly all edges belonging to $G(\varepsilon)$ whose common initial vertex is u. If $\alpha \in H$, then the images $\alpha(h_1), \ldots, \alpha(h_r)$ of the edges $h_1, \ldots, h_r$ in the mapping α belong to $G(\alpha)$. As u is a fixed vertex in all mappings of H, it is a fixed vertex also in the mapping α and therefore also the initial vertex of the edges $\alpha(h_1), \ldots, \alpha(h_r)$. The number of the edges of the graph $G(\alpha)$ with the initial vertex u is therefore greater than or equal to the number of edges of the graph $G(\varepsilon)$ with the initial vertex u. If in the consideration we exchange mutually the graphs $G(\varepsilon)$ and $G(\alpha)$ and instead of the mapping α we consider the mapping α^{-1}, we shall prove that at the same time the number of edges of the graph $G(\varepsilon)$ with the initial vertex u is greater than or equal to the number of edges of the graph $G(\alpha)$ with the initial vertex u; both these numbers are therefore equal to each other and we shall denote them by the symbol r.

Thus it is proved that for each $\alpha \in H$ the graph $G(\alpha)$ contains exactly r edges with the initial vertex u. As from the vertex u exactly $n - 1$ edges go out and each of them belongs exactly to one of the graphs of the decomposition $\mathscr{R}$, we have $n - 1 = mr$, therefore $r = (n - 1)/m$. As r is an integer, the number m is a divisor of $n - 1$. As according to the assumption it is a divisor of n, it must be a divisor of the difference of these numbers, that is of the number 1. But a positive divisor of the number 1 is only the number 1 itself, which is a contradiction with the assumption that $m > 1$.

With the help of this lemma we shall prove a theorem.

Theorem 10. *If n is finite, then it is a necessary and sufficient condition of the existence of the decomposition $\mathscr{R}$ of the complete digraph $\vec{K}_n$ with n vertices according to an Abelian group H that the order m of the group H should be a divisor either of the number n, or of the number $n - 1$.*

Proof. The graph $\vec{K}_n$ contains $n(n - 1)$ edges. Any two graphs of the decomposition $\mathscr{R}$ are isomorphic to each other, therefore they contain the same number of edges, which is a quotient of the number of the edges of the graph $\vec{K}_n$ and of the order of the group H. Therefore m is a divisor of $n(n - 1)$. The numbers n and $n - 1$ differ by one, thus they are relatively prime. Thus the number m can be expressed as the product m_1m_2, where m_1 is a divisor of n and m_2 a divisor of $n - 1$; the numbers m_1 and m_2 are evidently also relatively prime. Then the group H can be expressed as the direct product of the groups H_1 and H_2, where the order of H_1 is m_1, the order of H_2 is m_2. Therefore according to Theorem 9 there exists a decomposition $\mathscr{R}'$ of the graph $\vec{K}_n$ according to the group $H_1 \cong H/H_2$. The proof proceeds analogously to the proof of Theorem 3 of [5], only with the difference that we do not prove the impossibility of the fact that the order of the group H may be even. In the case of a digraph it may be even.

Now if a group H is given, whose order m is a divisor of the number n, then we construct the direct product H^* of the group H with the cyclic group K of the order n/m. The group H^* has evidently the order n. To each $\alpha^* \in H^*$ we assign by a one-to-one manner a vertex $u(\alpha^*)$ of the graph $\vec{K}_n$ and define $\beta^*(u(\alpha^*)) = u(\beta^*\alpha^*)$ for each $\beta^* \in H^*$. Now we shall construct a decomposition

$\mathscr{R}^*$ of the graph $\vec{K}_n$ according to the group H^* in the following way. We choose an edge $\overrightarrow{u_1v_1}$ and put it into $G^*(\varepsilon)$, where ε is the unit element of the group H^* (so as of the group H). For each $\beta^* \in H$ we put the edge $\overrightarrow{\beta^*(u_1)\,\beta^*(v_1)}$ into $G^*(\beta^*)$. Then we shall again choose an edge $\overrightarrow{u_2v_2}$ which is not yet put into any graph of the decomposition $\mathscr{R}^*$ and repeat this procedure until all edges are put into those graphs. Similarly as in the case of the decomposition of $\vec{K}_n$ into two isomorphic subgraphs it can be shown that in this way really a decomposition of the graph $\vec{K}_n$ according to the group H^* is created. As $H \cong H^*/K$, according to Theorem 9 also a decomposition of the graph $\vec{K}_n$ according to the group H exists. If the order m of H is a divisor of $n - 1$, we shall multiply the group H directly by the cyclic group K of the order $(n - 1)/m$, by which we obtain the group H^{**} of the order $n - 1$. We shall choose a vertex v which will be fixed in all mappings of H^{**} and assign remaining vertices again by a one-to-one manner to the elements of the group H^{**}. Then we proceed in the same way as in the preceding case (instead of H^* we take H^{**}).

Theorem 11. *Let K be a complete digraph with $\mathfrak{n}$ vertices, $\mathfrak{n} \geqq \aleph_0$, let H be an Abelian group of the order $\mathfrak{m}$. A necessary and sufficient condition of the existence of the decomposition $\mathscr{R}$ of the graph K according to the group H is that* $\mathfrak{m} \leqq \mathfrak{n}$.

The proof is analogous to the proof of Theorem 4 of [5].

We see that in Theorems 10 and 11, differently from analogous theorems for undirected graphs in [5], the condition that the group should not contain subgroups of even orders is omitted. It is so because in the case of a digraph we may admit involutory pairs of vertices in some mapping of H. In the case of an undirected graph the edge joining such an involutory pair would be fixed, while in the case of a digraph it is transformed into the inversely directed edge.

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