Zuzana Bukovská

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CHARACTERIZATION OF FOURIER TRANSFORMS OF VECTOR VALUED FUNCTIONS AND MEASURES

ZUZANA BUKOVSKÁ, Košice

In paper [4] by I. Kluvánek a neccessary and sufficient condition is given for a function with values in a Banach space to be a Fourier transform of some vector valued measure. However, this condition does not allow us to decide whether the given measure is of finite variation or whether it is an identifinite integral of a vector valued function. In this paper a neccessary and sufficient condition for a function to be a Fourier transform of a vector valued measure with finite variation is given. Moreover, we give a neccessary and sufficient condition for a vector valued function to be a Fourier transform of another vector valued function. The given conditions are a generalization of those given by J.S.W. Wong in [6] for the scalar valued case.

Let G be a locally compact Abelian group, Γ its dual group. Let X be a Banach space with a norm $\|\,.\,\|$. In the whole paper, the words „vector valued" mean „with values in X". A vector valued function f defined on G is Bochner integrable, written $f \in L_1(G, X)$, if $\int_G \|f(x)\|\, \mathrm{d}x < \infty$. The Bochner norm is denoted by $\|f\|_{1B}$, i.e. $\|f\|_{1B} = \int_G \|f(x)\|\, \mathrm{d}x$ (for exact definitions see E. Hille, R.S. Phillips [3]). A vector valued measure on G is a σ-additive measure with values in X defined on the set $\mathscr{B}(G)$ of all Borel subsets of G. A vector valued measure μ on G is of finite variation if there exists a real valued finite measure ν such that $\|\mu(E)\| \leqslant \nu(E)$ for every $E \in \mathscr{B}(G)$. The terminology and notations are those of L.H. Loomis [5].

We denote by $A(\Gamma, X)$ the set of all Fourier transforms of the functions belonging to $L_1(G, X)$. Thus, $f \in A(\Gamma, X)$ if $f = \hat{g}$ i.e. $f(\gamma) = \int_G g(x)\,(-x, \gamma)\, \mathrm{d}x$ $\gamma \in \Gamma$, for some $g \in L_1(G, X)$.

The set of all Fourier transforms of vector valued measures on G with finite variation is denoted by $B(\Gamma, X)$. Thus, $f \in B(\Gamma, X)$ if $f = \hat{\mu}$, i.e. $f(\gamma) = \int_G (-x, \gamma)\, \mathrm{d}\mu(x)$, $\gamma \in \Gamma$ for some vector valued measure μ on G with finite variation.

In this paper, a characterization of $A(\Gamma, X)$ and $B(\Gamma, X)$ is presented.

In the next, we assume that there is a sequence of scalar valued functions w_n on Γ with the following properties:

(i) w_n are continuous with compact support and $w_n(\Gamma) \subseteq \langle 0, 1 \rangle$;

(ii) the sequence $\{w_n\}_{n=1}^{\infty}$ is increasing and $\lim_{n\to\infty} w_n(\gamma) = 1$ for each $\gamma \in \Gamma$:

(iii) we denote $\Phi_n(x) = \int_{\Gamma} w_n(\gamma)\ (x, \gamma)\ \mathrm{d}\gamma,\ x \in G$. The sequence $\{\Phi_n\}_{n=1}^{\infty}$ is an approximative unit in $L_1(G)$.

The existence of such a sequence for Γ σ-compact was proved by E. Hewitt in [2].

Let f be a vector valued function defined on Γ. We define a function F_n on G as follows:

$$F_n(x) = \int_{\Gamma} f(\gamma) w_n(\gamma)\ (x, \gamma)\ \mathrm{d}\gamma,\ x \in G.$$

The denotations Φ_n, w_n, F_n are used throughout the whole paper.

Lemma 1. *Let A, B be spaces with σ-additive measures α, β, respectively. Let f be a function from $A \times B$ into X. If f is integrable on $A \times B$ then*

$$\int f(x, y)\ \mathrm{d}(\alpha \times \beta) = \int (\int f(x, y)\ \mathrm{d}\alpha)\ \mathrm{d}\beta = \int (\int f(x, y)\ \mathrm{d}\beta)\ \mathrm{d}\alpha.$$

Proof: By assumption $f \in L_1(A \times B, X)$, i. e. $\int \|f(x, y)\|\ \mathrm{d}(\alpha \times \beta) < \infty$. From the classical Fubini theorem it follows that $f \in L_1(B, X)$ for α-almost all $x \in A$ and $f \in L_1(A, X)$ for β-almost all $y \in B$. Using linear functionals on X the equality follows directly.

Lemma 2. *Φ_n is an approximative unit for* $L_1(G, X)$, i. e.

$$\| \int_G \Phi_n(x - y)\ g(y) \mathrm{d}y - g(x) \|_{1B} \to 0 \text{ when } n \to \infty \tag{1}$$

for every $g \in L_1(G, X)$.

Proof: If E is a measurable subset of G, $x \in X$ then using (iii), one can easily prove (1) for $g = x\,\chi_E$ (χ_E denotes the characteristic function of the set E). As any function $g \in L_1(G, X)$ may be approximated by finite valued functions, the rest of the proof is obvious.

A CHARACTERIZATION OF $B(\Gamma, X)$

Theorem 1. *Let $f \in B(\Gamma, X)$. Then $F_n \in L_1(G, X)$ and there exists a positive number K such that $\|F_n\|_{1B} \leqslant K$ for $n = 1, 2, 3, \ldots$.*

Proof: By assumption there is a vector valued measure μ with finite variation v, μ is defined on $\mathscr{B}(G)$ and $f = \hat{\mu}$. Thus

$$F_n(x) = \int_\Gamma w_n(\gamma)\,(x, \gamma)\,(\int_G (-y, \gamma)\,\mathrm{d}\mu(y))\,\mathrm{d}\gamma.$$

First we prove

$$(2) \qquad \int_\Gamma (\int_G w_n(\gamma)\,(x - y, \gamma)\,\mathrm{d}\mu(y))\,\mathrm{d}\gamma = \int_G (\int_\Gamma w_n(\gamma)\,(x - y, \gamma)\,\mathrm{d}\gamma)\,\mathrm{d}\mu(y).$$

The function $h(y, \gamma) = w_n(\gamma)\,(x - y, \gamma)$ is continuous, bounded and with compact support. ν is a finite measure on G. Hence h is $\nu \times \mathrm{d}\gamma$ -integrable and also $\mathrm{d}\gamma \times \nu$ -integrable. Let $x^* \in X^*$ be a linear continuous functional. Then

$$x^* \int_\Gamma (\int_G w_n(\gamma)\,(x - y, \gamma)\,\mathrm{d}\mu(y))\,\mathrm{d}\gamma = \int_\Gamma (\int_G w_n(\gamma)\,(x - y, \gamma)\,\mathrm{d}x^*\mu(y)\,\mathrm{d}\gamma =$$

$$= \int_G (\int_\Gamma w_n(\gamma)\,(x - y, \gamma)\,\mathrm{d}\gamma)\,\mathrm{d}x^*\mu(y) = x^* \int_G (\int_\Gamma w_n(\gamma)\,(x - y, \gamma)\,\mathrm{d}\gamma)\,\mathrm{d}\mu(y)$$

and (2) holds true.

Now, we have

$$\|F_n\|_{1B} = \int_G \|F_n(x)\|\,\mathrm{d}x = \int_G \|\int_\Gamma w_n(\gamma)\,(x, \gamma)\,(\int_G (-y, \gamma)\,\mathrm{d}\mu(y))\,\mathrm{d}\gamma\|\,\mathrm{d}x =$$

$$\int_G \|\int_G (\int_\Gamma w_n(\gamma)\,(x - y, \gamma)\,\mathrm{d}\gamma)\,\mathrm{d}\mu(y)\|\,\mathrm{d}x = \int_G \|\int_G \Phi_n(x - y)\,\mathrm{d}\mu(y)\|\,\mathrm{d}x \leqslant$$

$$\leqslant \int_G (\int_G |\,\Phi_n(x - y)\,|\,\mathrm{d}\nu(y))\,\mathrm{d}x = \int_G \|\,\Phi_n\|_1\,\mathrm{d}\nu(y) = \int_G \mathrm{d}\nu(y) = \nu(G).$$

Theorem 2. *Let f be a bounded continuous vector valued function on Γ. Let $F_n \in L_1(G, X)$, $\|F_n\|_{1B} \leqslant K$ for $n = 1, 2, 3, \ldots$, where K is positive. Then $f \in B(\Gamma, X)$.*

Proof: Let $\varphi \in [L_1(G) \cap P(G)]$ (i. e. to the linear subspace of $L_1(G)$ generated by $L_1(G) \cap P(G)$, where $P(G)$ is the set of all positive-definite scalar functions on G). As

$$[L_1(G) \cap P(G)]^\wedge = [L_1(\Gamma) \cap P(\Gamma)],$$

we have

$$\varphi(x) = \int_\Gamma \hat{\varphi}(\gamma)\,(x, \gamma)\,\mathrm{d}\gamma, \text{ where}$$

$$\hat{\varphi}(\gamma) = \int_G \varphi(x)\,(-x, \gamma)\,\mathrm{d}x.$$

For $n = 1, 2, \ldots$, we define an operator I_n on $[L_1(G) \cap P(G)]$ with values in X by the equality

$$(3) \qquad I_n(\varphi) = \int_G \varphi(x) F_n(x)\,\mathrm{d}x.$$

The operator I_n is clearly linear and uniformly continuous with respect to $\varphi \in [L_1(G) \cap P(G)]$ and $n = 1, 2, \ldots$, since

$$(4) \qquad \|I_n(\varphi)\| = \|\int_G \varphi(x) F_n(x)\,\mathrm{d}x\| \leqslant \int_G |\varphi(x)|\, \|F_n(x)\|\,\mathrm{d}x \leqslant$$

$$\leqslant \|\varphi\|_\infty \cdot \int_G \|F_n(x)\|\,\mathrm{d}x \leqslant K\, \|\varphi\|_\infty .$$

By definition and Lemma 1, we obtain

$$I_n(\varphi) = \int_G \varphi(x) \left(\int_\Gamma w_n(\gamma)\, f(\gamma)\, (x, \gamma)\,\mathrm{d}\gamma \right) \mathrm{d}x =$$

$$= \int_\Gamma w_n(\gamma)\, f(\gamma)\, \hat{\varphi}(-\gamma)\,\mathrm{d}\gamma.$$

Let $M > 0$ be an upper bound of f, i. e. $\|f(\gamma)\| \leqslant M$ for $\gamma \in \Gamma$. As

$$\text{(i)} \qquad \lim_{n\to\infty} w_n(\gamma)\, f(\gamma)\, \hat{\varphi}(-\gamma) = f(\gamma)\, \hat{\varphi}(-\gamma),$$

$$\text{(ii)} \qquad \|w_n(\gamma)\, f(\gamma)\, \hat{\varphi}(-\gamma)\| \leqslant M\, |\hat{\varphi}(-\gamma)|, \quad |\hat{\varphi}| \in L_1(\Gamma),$$

by the Lebesque theorem, we have $\int_\Gamma f(\gamma)\, \hat{\varphi}(-\gamma)\,\mathrm{d}\gamma = \lim_{n\to\infty} I_n(\gamma)$. Now, we denote $I(\varphi) = \int_\Gamma f(\gamma)\, \hat{\varphi}(-\gamma)\,\mathrm{d}\gamma$. Thus, $I_n(\varphi) \to I(\varphi)$ for every $\varphi \in [L_1(G) \cap P(G)]$. By (4), $\|I(\varphi)\| \leqslant K\, \|\varphi\|_\infty$ for $\varphi \in [L_1(G) \cap P(G)]$.

The operators I_n and I can be continuously extended onto $\overline{[L_1(G) \cap P(G)]}$ (uniform closure of $[L_1(G) \cap P(G)]$). These extensions are denoted also by I_n, I respectively. Evidently, for every $\varphi \in \overline{[L_1(G) \cap P(G)]}$

$$I_n(\varphi) = \int_G \varphi(x)\, F_n(x)\,\mathrm{d}x$$

and
$$\|I_n(\varphi)\| \leqslant K\, \|\varphi\|_\infty, \quad \|I(\varphi)\| \leqslant K\, \|\varphi\|_\infty, \quad I_n(\varphi) \to I(\varphi).$$

Using well known theorems of measure theory, we prove that there exists a vector valued measure μ defined on $\mathscr{B}(G)$ with finite variation ν such that

$$I(\varphi) = \int_\Gamma \hat{\varphi}(-\gamma) \left(\int_G (-x, \gamma)\,\mathrm{d}\mu(x) \right) \mathrm{d}\gamma \qquad \text{for } \varphi \in \overline{[L_1(G) \cap P(G)]}.$$

Let A be a Borel subset of G. Let $\varphi_i \in \overline{[L_1(G) \cap P(G)]}$, $|\varphi_i| \leqslant 1$ and $\sum_{i=1}^{\infty} \varphi_i \leqslant \chi_A$. Let m be a natural number. Then

$$\sum_{i=1}^{m} \|I_n(\varphi_i)\| = \sum_{i=1}^{m} \|\int_G \varphi_i(x)\, F_n(x)\, \mathrm{d}x\| \leqslant \sum_{i=1}^{m} \int_G |\varphi_i(x)|\, \|F_n(x)\|\, \mathrm{d}x \leqslant$$

$$\leqslant \int_G (\sum_{i=1}^{m} |\varphi_i(x)|\, \|F_n(x)\|)\, \mathrm{d}x \leqslant \int_G \|F_n(x)\|\, \mathrm{d}x \leqslant K$$

for $n = 1, 2, \ldots$. Since

$$\sum_{i=1}^{m} \|I(\varphi_i)\| = \lim_{n\to\infty} \sum_{i=1}^{m} \|I_n(\varphi_i)\| \leqslant K,$$

we have

$$\sum_{i=1}^{\infty} \| I(\varphi_i)\| \leqslant K.$$

Hence, the set

$$\{\sum_{i=1}^{\infty} \|I(\varphi_i)\| : \varphi_i \in \overline{[L_1(G) \cap P(G)]},\ |\varphi_i| \leqslant 1,\ \sum_{i=1}^{\infty} |\varphi_i| \leqslant \chi_A\}$$

is bounded. By Theorem 3, § 19 Dinculeanu [1], the operator I is dominated and by Theorem 2 of the same paragraph, there exists a vector valued measure μ defined on $\mathscr{B}(G)$ with finite variation such that

$$I(\varphi) = \int \varphi\, \mathrm{d}\mu \qquad \text{for every} \qquad \varphi \in \overline{[L_1(G) \cap P(G)]}.$$

If $\varphi \in \overline{[L_1(G) \cap P(G)]}$, we have

$$I(\varphi) = \int_G \varphi(x)\, \mathrm{d}\mu(x) = \int_G (\int_\Gamma \hat{\varphi}(\gamma)\, (x, \gamma)\, \mathrm{d}\gamma)\, \mathrm{d}\mu(x) =$$

$$= \int_\Gamma \hat{\varphi}(-\gamma)\, (\int_G (-x, \gamma)\, \mathrm{d}\mu(x))\, \mathrm{d}\gamma.$$

As $\overline{[L_1(G) \cap P(G)]}$ contains $\mathscr{C}_0(G)$ ($\mathscr{C}_0(G)$ denotes the set of all continuous functions vanishing at infinity), for $\varphi \in \mathscr{C}_0(G)$ the following holds true:

$$I(\varphi) = \int_\Gamma \hat{\varphi}(-\gamma) f(\gamma)\, \mathrm{d}\gamma = \int_\Gamma \hat{\varphi}(-\gamma)\, (\int_G (-x, \gamma)\, \mathrm{d}\mu(x))\, \mathrm{d}\gamma.$$

Since f is continuous, we have

$$f(\gamma) = \int_G (-x, \gamma)\, \mathrm{d}\mu(x).$$

A CHARACTERIZATION OF $A(\Gamma, X)$

Theorem 3. *Let $f \in A(\Gamma, X)$. Then $F_n \in L_1(G, X)$ for $n = 1, 2, \ldots$. Moreover, F_n is an L_1-convergent sequence.*

Proof. As $A(\Gamma, X) \subseteq B(\Gamma, X)$, by Theorem 1, $F_n \in L_1(G, X)$. By assumption, there is a $g \in L_1(G, X)$ such that $f = \hat{g}$. Using Lemma 1, we obtain

$$F_n(x) = \int_\Gamma w_n(\gamma)\,(x, \gamma)\,(\int_G (-y, \gamma)\, g(y)\, \mathrm{d}y)\, \mathrm{d}\gamma =$$

$$= \int_G \Phi_n(x - y)\, g(y)\, \mathrm{d}y$$

By Lemma 2, for $g \in L_1(G, X)$ we have

$$\|\int_G \Phi_n(x - y)\, g(y)\, \mathrm{d}y - g(x)\|_{1B} \to 0 \text{ for } n \to \infty,$$

i. e., F_n is an L_1-convergent sequence.

Theorem 4. *Let f be a continuous bounded vector valued function on Γ. If $F_n \in L_1(G, X)$ and $\{F_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the Bochner norm, then $f \in A(\Gamma, X)$.*

Proof. Evidently, the sequence $\{F_n\}_{n=1}^{\infty}$ is bounded. We define an operator I_n on $[L_1(G) \cap P(G)]$ by (3). In the proof of Theorem 2, we have shown that I_n are linear and uniformly continuous with respect to $n = 1, 2, \ldots$ and

$$\lim_{n\to\infty} I_n(\varphi) = I(\varphi) = \int_\Gamma \hat{\varphi}(-\gamma)\, f(\gamma)\, \mathrm{d}\gamma.$$

As $\{F_n\}$ is a Cauchy sequence and $L_1(G, X)$ is a complete space, there exists a function $F \in L_1(G, X)$ such that $\|F_n - F\|_{1B} \to 0$ for $n \to \infty$. Then

$$\int_G F_n(x)\varphi(x)\, \mathrm{d}x \to \int_G F(x)\, \varphi(x)\, \mathrm{d}x.$$

By Lemma 1 we have

$$\int_G F(x)\,(\int_\Gamma \hat{\varphi}(\gamma)\,(x, \gamma)\, \mathrm{d}\gamma)\, \mathrm{d}x = \int_\Gamma \hat{\varphi}(-\gamma)\,(\int_G F(x)\,(-x, \gamma)\, \mathrm{d}x)\, \mathrm{d}\gamma =$$

$$= \int_\Gamma \hat{\varphi}(-\gamma)\, \hat{F}(\gamma)\, \mathrm{d}\gamma).$$

Hence, for every $\varphi \in [L_1(G) \cap P(G)]$, we have proved

$$I_n(\varphi) \to \int_\Gamma \hat{\varphi}(-\gamma)\, f(\gamma)\, \mathrm{d}\gamma,$$

$$I_n(\varphi) \to \int_\Gamma \hat{\varphi}(-\gamma)\, \hat{F}(\gamma)\, \mathrm{d}\gamma.$$

Therefore

$$\int_\Gamma \hat{\varphi}(-\gamma)\, f(\gamma)\, \mathrm{d}\gamma = \int_\Gamma \hat{\varphi}(-\gamma)\, \hat{F}(\gamma)\, \mathrm{d}\gamma.$$

As $[L_1(\Gamma) \cap P(\Gamma)]$ is a dense subset of $L_1(\Gamma)$, we obtain $f = \hat{F}$.

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Katedra matematickej analýzy
Prírodovedeckej fakulty
Šafárikovej univerzity,
Košice