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SEMIGROUP COVERINGS OF GROUPS

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1. Introduction

This note arose out of a problem posed some years ago in this journal by Prof. Štefan Schwarz in his paper [1] on semigroups satisfying certain generalisations of the cancellation law. There it was found necessary to consider semigroups which can be covered by pairwise disjoint proper subsemigroups all satisfying one of the various cancellation laws — right, left, two-sided or a certain generalisation whose precise nature does not concern us here — and the problem naturally arose as to what structures are possible for such semigroups. The class formed by them is very wide, and a complete characterisation seems to be a difficult undertaking; on the other hand, Prof. Schwarz gave characterisations of commutative and of periodic semigroups with these properties. This led to a determination of all abelian *groups* which are unions of pairwise disjoint proper subsemigroups (all the cancellation laws are then automatically satisfied): they are simply the abelian groups which have elements of infinite order. This is certainly not valid in the domain of non-abelian groups, and it is the aim of the present note to discuss the same problem, together with others related to it, in this more general situation.

The author's first contact with the subject was in the New Scottish Book, where he misread the problem to read: what groups can be covered by subsemigroups none of which is a subgroup? It turns out that this class of groups is relevant, in that it properly contains the class just mentioned. Let us define four group-theoretical properties, to be denoted by German capitals — for details see § 2:

$\mathfrak{T}$: G is the union of subsemigroups none of which is a subgroup;

$\mathfrak{I}$: G has at least one aperiodic homomorphic image;

$\mathfrak{D}$: G is the union of pairwise disjoint proper subsemigroups;

$\mathfrak{S}$: G is not periodic, and has the property that whenever two elements a and b have positive powers in common, then ab and a also have positive powers in common.

Obviously, periodic groups satisfy none of these properties, but on the other hand, there exist (§ 2) non-periodic groups which also satisfy none of them. Any group possessing a property denoted by $\mathfrak{P}$ is called a $\mathfrak{P}$-group, and the class of all

$\mathfrak{P}$-groups is denoted by $[\mathfrak{P}]$. One of our main results is the following chain of inclusions:

$$[\mathfrak{J}] \supset [\mathfrak{D}] \supseteq [\mathfrak{T}] \supset [\mathfrak{S}],$$

and that they remain valid when all groups considered are metabelian. It seems likely that the middle inclusion can be strengthened to a strict one, but I have been unable to confirm this point. For many groups the four properties coincide: for instance every non-periodic group in which all the two-generator subgroups are nilpotent or free (this is meant to include the possibility of two-generator free and two-generator nilpotent non-abelian subgroups in one and the same group) is an $\mathfrak{S}$-group.

Apart from the partial characterisation given by this last-mentioned fact, the results are of a sketchy nature, usually taking the form of conditions sufficient (but not necessary) or necessary (but not sufficient) that a group satisfy one or other property. A simple though far-reaching result – and one which perhaps accounts for some of the difficulties encountered – is that if $\mathfrak{Y}$ denotes one of the properties $\mathfrak{T}$, $\mathfrak{J}$, $\mathfrak{D}$, then any group which can be mapped homomorphically onto a $\mathfrak{Y}$-group is likewise a $\mathfrak{Y}$-group. It means, of course, that the classes are exceedingly wide, and for instance that any group can be embedded in a group satisfying all three properties $\mathfrak{T}$, $\mathfrak{J}$, $\mathfrak{D}$ simultaneously.

I thank Dr M. F. Newman for some useful comments.

2. Preliminaries

2.1. Groups will be written multiplicatively, with 1 standing for the unit element and E for the unit subgroup of all groups occurring. If all elements of the group G are of finite order, we say that G is *periodic;* if $G \neq E$ and only the unit element has finite order, that G is *aperiodic;* while *non-periodic* is to imply that some element has infinite order. If g, h are elements of G, the *transform* $h^{-1}gh$ of g by h is denoted by g^h, and the *commutator* $g^{-1}h^{-1}gh$ by $[g, h]$. The n-fold commutator $[x_1, x_2, \ldots, x_n]$ is defined inductively by the rule $[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$: that is, commutators are left-normed. It is easy to see that for any group elements g, h and any integer n,

$$(g^h)^n = (g^n)^h. \tag{2.1}$$

2.2. If A and B are subsets of the group G, the symbol $[A, B]$ denotes the subgroup of G generated by all commutators of the form $[a, b]$, with $a \in A$ and $b \in B$. The *lower central series* of G is

$$G = \gamma_0(G) \supseteq \gamma_1(G) \supseteq \ldots \supseteq \gamma_n(G) \supseteq \ldots,$$

where $\gamma_{n+1}(G) = [G, \gamma_n(G)]$ for $n \geqq 0$; G is nilpotent of class n if the lower central series terminates in E after a finite number of steps, and n is the first integer for which $\gamma_n(G) = E$. In this case $\gamma_{n-1}(G)$ is contained in the centre of G, and as a corollary we conclude by an easy induction argument on k that

$$[x_1^k, x_2, \ldots, x_n] = [x_1, x_2^k, \ldots, x_n] = [x_1, x_2, \ldots, x_n]^k \tag{2.2}$$

for any integer k and any elements $x_1, x_2, \ldots, x_n$ of a nilpotent group of class n.

We shall need the following facts about the lower central series and nilpotent groups. The first two are very simple and well-known, the others, as far as I know, new:

2.3. [2, lemma 1.7] *Suppose that the group G is generated by a subset X of its elements. Then, for each $n \geqq 1$, $\gamma_n(G)$ is generated by all transforms by elements of G of all commutators of the form $[x_1, x_2, \ldots, x_{n+1}]$, where $x_i \in X$ for $i = 1, 2, \ldots, n+1$.*

2.4. [2, lemma 1.2] *For any normal subgroup N of the group G and any integer $n \geq 0$, $\gamma_n(G/N) = \gamma_n(G)N/N$.*

Lemma 2.5. *Let G be a nilpotent group generated by two elements which have commuting non-zero powers. Then $\gamma_1(G)$ is finite.*

Proof. We proceed by induction on the nilpotency class of G. For class 1 the result is trivial, since in that case $\gamma_1(G)$ is the unit group. Suppose then that we know the result for nilpotent groups of class less than n, and let G be nilpotent of class n and generated by two elements a, b such that $[a^\alpha, b^\beta] = 1$ for some non-zero integers α, β. Consider the factor-group $G/\gamma_{n-1}(G)$. By 2.4, it is nilpotent of class $n-1$, and so by the inductive hypothesis its derived group $\gamma_1(G)/\gamma_{n-1}(G)$ is finite. It remains now to show that the subgroup $\gamma_{n-1}(G)$ is finite. By 2.1, it is generated by all transforms of all commutators of the form $[x_1, x_2, \ldots, x_n]$, where each argument is a or b; but these elements are all central, so that we have shown that $\gamma_{n-1}(G)$ is a finitely generated abelian group. Next, equations 2.2 tell us that each of these generators is of finite order, since

$$1 = [a^\alpha, b^\beta, x_3, \ldots, x_n] = [a, b, x_3, \ldots, x_n]^{\alpha\beta}.$$

Consequently $\gamma_{n-1}(G)$, and therefore $\gamma_1(G)$, is finite. This completes the proof of the lemma.

It may be remarked that the conclusion of lemma 2.5 holds for any nilpotent group generated by finitely many elements with commuting non-zero powers.

Lemma 2.6. *If a nilpotent group G is generated by two elements a, b which have positive powers in common, then ab and a also have positive powers in common.*

Proof. We again proceed by induction on the nilpotency class of G: when the class is 1, a and b commute so that the result is immediate. Suppose that we know the result for nilpotent groups of class less than n, and let G be nilpotent of class n and generated by two elements a, b such that $a^\alpha = b^\beta$ for some positive integers

α, β. By the inductive hypothesis, if a_1, b_1 are the images of a, b respectively in the factor-group $G/\gamma_{n-1}(G)$, then for some positive integers ρ, σ we have

$$(a_1 b_1)^\rho = a_1^\sigma .$$

This leads to an equation of the form

$$(ab)^\rho = a^\sigma g,$$

for some $g \in \gamma_{n-1}(G)$. Now g is central in G and, by the previous lemma, has finite order; hence, for some $\tau \geqq 0$,

$$(ab)^{\rho\tau} = a^{\sigma\tau} g^\tau = a^{\sigma\tau},$$

which completes the induction and the proof of the lemma.

Definitions and results on free groups and free products of groups are to be found in [3, Volume 2].

Group-theoretical properties will be denoted by capital German letters; if G satisfies property $\mathfrak{P}$, we say that G is a $\mathfrak{P}$-group, and denote the class of all such groups by $[\mathfrak{P}]$. A group is said to be locally-$\mathfrak{P}$ if every finitely generated subgroup has property $\mathfrak{P}$, this being more stringent than notion associated with „local systems" of subgroups (see for instance [3]). We shall next define in more detail the four classes of groups mentioned in the introduction, and obtain some of their more elementary properties.

If G is a group, by a *true* subsemigroup of G we mean a subsemigroup which is not a subgroup – that is, one which contains some element but not its inverse. Clearly, a subsemigroup consisting of elements of finite order cannot be true, so that in particular periodic groups have no true subsemigroups. The group G is said to satisfy property $\mathfrak{T}$ if it can be covered by – in other words, is the set-theoretical union of – true subsemigroups. The remark just made shows that a $\mathfrak{T}$-group must be non-periodic; however, not all non-periodic groups have the property $\mathfrak{T}$:

Example 2.7. Let G be the infinite dihedral group generated by two elements a, b with the defining relations $a^2 = b^2 = 1$. It is easy to see that ab has infinite order, and that only powers of this element can have infinite order. We shall show that the only subsemigroups of G which contain the element a are subgroups of G. To that end, let S be such a subsemigroup. Then if $g \in S$ has finite order k, the inverse g^{k-1} also lies in S; while if $g \in S$ has infinite order, then $g = (ab)^m$ for some integer m, and S contains

$$a(ab)^m a = (ba)^m = (ab)^{-m} = g^{-1}.$$

This means that S is a group, as required.

In fact the same reasoning shows that no splitiing extension (see for instance [3]) of an abelian group A by a cycle of order 2 inducing the inverting automorphism of A can be a $\mathfrak{T}$-group.

A proper subsemigroup of a group is one which is non-empty and strictly smaller than the group; G is said to satisfy property $\mathfrak{T}$ if it can be covered by pairwise disjoint proper subsemigroups. This property is more restrictive than the one just mentioned, as we shall see in the next section. The following result of Schwarz characterises all abelian $\mathfrak{T}$-groups:

2.8. [1, Veta 4.2.] *A necessary and sufficient condition that an abelian group be a $\mathfrak{T}$-group is that it contain elements of infinite order.*

If a, b are elements of a group G, then following Schwarz [1] we say that a and b are *equivalent* if there exist positive integers α, β such that $a^\alpha = b^\beta$. This is clearly an equivalence relation on G, and we shall denote the equivalence class containing the element a by T_a.

Lemma 2.9. *If $G = \bigcup_{i \in \mathfrak{A}} S_i$ is a decomposition of the group G into the union of pairwise disjoint subsemigroups S_i, and a is an arbitrary element of the subsemigroup S_p, then the whole equivalence class T_a of a is contained in S_p.*

Proof. Suppose that $a^\alpha = b^\beta$ for some positive integers α, β. Then b lies in some component S_q of the decomposition, and the disjointness condition ensures that $S_p = S_q$.

If G is a group which contains at least one element of infinite order, and in which all the equivalence classes T_a are subsemigroups of G, then we say that G has property $\mathfrak{S}$. Since the elements of finite order form an equivalence class, this is a proper subsemigroup – and therefore normal subgroup – so that in fact any $\mathfrak{S}$-group is also a $\mathfrak{T}$-group; they form an extensive class, as we shall see later.

Next we define by transfinite induction the *periodic series* of an arbitrary group G,

$$\pi_0(G) \subseteq \pi_1(G) \subseteq \ldots \subseteq \pi_\lambda(G) \subseteq \ldots,$$

as follows. The first term $\pi_0(G)$ is to be the unit subgroup, while if λ is a limit ordinal, $\pi_\lambda(G)$ is to be the union of all $\pi_\mu(G)$ with $\mu < \lambda$. If λ has an immediate predecessor μ, $\pi_\lambda(G)$ is the subgroup of G generated by all elements which have finite order modulo $\pi_\mu(G)$. Each term of the series is normal, since the union of normal subgroups is normal, and if g^n lies in the normal subgroup $\pi_\mu(G)$ for some element g and some integer n, then by 2.1,

$$(g^a)^n = (g^n)^a \in \pi_\mu(G)$$

for arbitrary $a \in G$. The limit $\pi_*(G)$ of the periodic series of G is called the *peak* of G. From the definition it follows that $G/\pi_*(G)$ is aperiodic if $\pi_*(G) \subset G$, so that we have the following result in one direction:

Lemma 2.10. *The group G has an aperiodic homomorphic image if and only if $\pi_*(G) \subset G$.*

Proof. We show that the factor-group G/N is aperiodic, then $N \supseteq \pi_*(G)$ – or what is the same thing, that N contains all terms of the periodic series of G. It is clear that $\pi_0(G) \subseteq N$. Suppose we know that for every ordinal $\mu < \lambda$ that $\pi_\mu(G) \subseteq N$. If λ is a limit ordinal, then obviously $\pi_\lambda(G) \subseteq N$. If on the other hand $\lambda = \mu + 1$ then $\pi_\lambda(G)$ is generated by elements which have finite order modulo $\pi_\mu(G)$. Consequently if $\pi_\lambda(G)$ were not contained in N there would be elements of non-trivial finite order in the factor-group G/N, in contradiction to the hypotheses. Hence $\pi_*(G) \subseteq N \subset G$, and we have the result.

Lemma 2.10 shows that $G/\pi_*(G)$ is the unique maximal aperiodic homomorphic image of G, if such a homomorphic image exists. Groups which have aperiodic homomorphic images will be called $\mathfrak{Z}$-groups: they form a class intermediate between $[\mathfrak{I}]$ and $[\mathfrak{T}]$.

3. Main results

All the results to be proved in this section are based on the following simple theorem.

Theorem 3.1. *If $\mathfrak{Y}$ stands for one of the properties $\mathfrak{I}$, $\mathfrak{T}$, $\mathfrak{Z}$, then any group which can be mapped homomorphically onto a $\mathfrak{Y}$-group is likewise a $\mathfrak{Y}$-group.*

Proof. For the property $\mathfrak{Z}$ the proof is immediate. Suppose next that $G = \bigcup_{i \in \mathfrak{A}} S_i$ is a decomposition of the $\mathfrak{I}$-group G into the union of true subsemigroups S_i, and that φ is a homomorphism of the group H onto G. Then the complete inverse image T_i of S_i under φ is clearly a subsemigroup of H, and the T_i together cover H. Finally, each T_i is a true subsemigroup of H, since a homomorphic image of a subgroup is a subgroup.

The proof for the property $\mathfrak{T}$ follows along precisely similar lines.

However, the property $\mathfrak{S}$ is not preserved in this way. To see this it is sufficient to observe that the elements of finite order in an $\mathfrak{S}$-group form a subgroup, and that if H is any group where the elements of finite order do not form a subgroup, then the direct product $G \times H$ is not an $\mathfrak{S}$-group. Again, none of the properties are preserved under the operations of taking homomorphic images — as we shall see, the infinite cycle satisfies all four properties, but none of its proper homomorphic images has any one of them.

Next we have some lemmas which tell us something about possible structure theorems for the various classes of groups considered, but not much; they usually take the form of sufficient conditions which are not also necessary.

Lemma 3.2. *If every element of finite order in the group G is contained in some true subsemigroup, then G is a $\mathfrak{I}$-group.*

Proof. Clearly, every element of infinite order is contained in a true subsemigroup, namely that consisting of its positive powers.

Corollary 3.3. *Every aperiodic group is a $\mathfrak{I}$-group, so that every $\mathfrak{Z}$-group is a $\mathfrak{I}$-group.*

Lemma 3.4. *If every element of finite order in the group G commutes with some element of infinite order, then G is a $\mathfrak{T}$-group.*

Proof. Suppose that the element a of finite order in G commutes with the element b of infinite order. Then the subset consisting of all elements of the form $a^\alpha b^\beta$, where α is an arbitrary integer and β an arbitrary non-negative integer, is a true subsemigroup containing a. That it is a subsemigroup, is obvious; that it is true follows from the remark that b^{-1} cannot be of the form $a^\alpha b^\beta$, for non-negative β. Lemma 3.2 now applies to give the result.

Corollary 3.5. *Every group can be embedded in a group which satisfies the three properties $\mathfrak{T}$, $\mathfrak{Z}$, $\mathfrak{D}$ simultaneously.*

Proof. Let A be an infinite cyclic group. Then A is clearly an $\mathfrak{Z}$-group; by 2.8 it is a $\mathfrak{D}$-group: and by lemma 3.2 it is a $\mathfrak{T}$-group. Thus if G is our given group, theorem 3.1 tells us that the direct product $G \times A$ will suffice as an embedding.

Lemma 3.4 provides a sufficient condition which is not also necessary. For instance, in the free product of an infinite cyclic group with a cyclic group of order 2, the generator of the finite cycle commutes only with itself and with the unit element; however, the free product can be mapped homomorphically onto the infinite cycle.

Lastly on the subject of $\mathfrak{T}$-groups:

Lemma 3.6. *Any group which can be expressed as the set-theoretical union of $\mathfrak{T}$-groups is itself a $\mathfrak{T}$-group.*

The proof is obvious, and is omitted. These few results together display the width of the class $[\mathfrak{T}]$, and indicate that a characterisation of all $\mathfrak{T}$-groups will be far from simple. Possibly easier (since less numerous) are the $\mathfrak{D}$-groups, to which we now turn attention: but even here we come across difficulties arising presumably out of the width of the class, and out of the very different structures that $\mathfrak{D}$-groups may possess.

Lemma 3.7. *Let $G = \bigcup_{i \in \mathfrak{A}} S_i$ be any decomposition of the group G into the union of pairwise disjoint subsemigroups. Then the peak $\pi_*(G)$ of G is contained in that subsemigroup S_p which contains the unit element.*

Proof. As usual, $\pi_0(G) \subseteq S_p$, and we assume inductively that for all ordinals $\mu < \lambda$, $\pi_\mu(G) \subseteq S_p$. The case of a limit ordinal is again trivial, so we assume that $\lambda = \mu + 1$, and consider an arbitrary element g of $\pi_\lambda(G)$. This element can be expressed as a product $g = g_1 g_2 \cdots g_n$, where each g_i has a positive power in $\pi_\mu(G)$, that is, in S_p; consequently, owing to the disjointness of the S_i, each g_j must also lie in S_p, so that $g \in S_p$, as required.

Corollary 3.8. *Every $\mathfrak{D}$-group is an $\mathfrak{Z}$-group.*

Proof. The lemma shows that the peak of G is contained in a proper subsemigroup, so that it is a proper subgroup of G.

We have now established the following inclusions:

$$[\mathfrak{T}] \supseteq [\mathfrak{Z}] \supseteq [\mathfrak{D}] \supseteq [\mathfrak{S}].$$

It is not hard to show that the first and last inclusions may be replaced by strict inclusions, even if we consider only metabelian groups.*

Example 3.9. Let G be the direct product of an infinite cyclic group with an infinite dihedral group. Then G is metabelian, and, since it can be mapped homomorphically onto the infinite cycle, it is a $\mathfrak{D}$-group. However, it is not an $\mathfrak{S}$-group since the elements of finite order do not form a subgroup.

Example 3.10. Let G be the group generated by 3 elements a, b, c subject to the 6 defining relations

$$a^2 = b^2 = c^2 = [a, b, c] = [b, c, a] = [c, a, b] = 1.$$

We first of all observe that $G = \pi_1(G)$ and therefore that G is not an $\mathfrak{Z}$-group. However, we shall show that it is a metabelian $\mathfrak{T}$-group:

(i) *G is metabelian.* By 2.3, $\gamma_1(G)$ is generated by the transforms of the commutators $[a, b]$, $[a, c]$, and $[b, c]$; but the relations

$$[a, b]^a = [a, b]^{-1}, \qquad [a, b]^b = [a, b]^{-1}, \qquad [a, b]^c = [a, b]$$

and the similar ones for the other two commutators show that $\gamma_1(G)$ is generated by the three commutators themselves. Next, the relations

$$[a, b]^{[a, c]} = [a, b]^{acac} = ([a, b]^{-1})^{cac} = ([a, b]^{-1})^{ac} = [a, b]$$

and others like them show that $\gamma_1(G)$ is abelian.

(ii) *$\gamma_1(G)$ is free abelian on the generators $[a, b]$, $[a, c]$, $[b, c]$.* For this we take an auxiliary infinite dihedral group H generated by two elements d, f subject to the defining relations $d^2 = f^2 = 1$, and suppose that

$$[a, b]^{\lambda}[a, c]^{\mu}[b, c]^{\nu} = 1$$

is a relation between the generators of $\gamma_1(G)$. Then, by von Dyck's Theorem (see [3]), the mapping φ given by $a\varphi = d$, $b\varphi = f$, $c\varphi = 1$ extends to a homomorphism of G onto H. The given relation becomes $[d, f]^{\lambda} = 1$, and this clearly means that $\lambda = 0$. Similarly we show that $\mu = \nu = 0$ and the result follows.

(iii) *Every element of finite order in G commutes with an element of infinite order, so that G is a $\mathfrak{T}$-group.* It is routine to check that $G/\gamma_1(G)$ is elementary abelian of order 8 and that we may choose the set $V = \{1, a, b, c, ab, ac, bc, abc\}$ as a set of coset representatives of G modulo $\gamma_1(G)$, so that every element of G is of the form xv for some $x \in \gamma_1(G)$ and some $v \in V$. We look for the elements of finite order.

(I) $v = 1$. Here only the unit element has finite order.

(II) $v = a, b,$ *or* c. Obviously we may take $v = a$ as typical. Now the element xa

* That is, groups with abelian derived groups.

has finite order if and only if its square does; but $(xa)^2 = xx^a \in \gamma_1(G)$ and this has finite order if and only if $x^a = x^{-1}$. Computation shows that only elements of the form $x = [a, b]^\lambda\ [a, c]^\mu$ have this property, and then of course the element $[b, c]$ commutes with xa.

(III) $v = ab, ac,$ or bc. This time we take $v = ab$ as typical. Here

$$(xab)^2 = xabxab = xx^{ba}[a, b].$$

It is easy to see now that $(xab)^2$ must be of the form $[a, b]^{2\lambda+1}$; elements in this category thus have infinite order and do not concern us.

(IV) $v = abc$. Again it turns out that elements of the form $xabc$ are of infinite order.

This completes the example.

It is likely that there exists a group generated by elements of finite order and yet having elements of infinite order in its centre, but I have been unable to construct one. Such a group would of course be a $\mathfrak{T}$-group without being an $\mathfrak{J}$-group; a candidate is the group generated by two elements a, b subject to the relations

$$a^4 = b^4 = 1, \qquad (ab)^4 = (ba)^4.$$

The element $(ab)^4$ is certainly central, and I can see no reason why it should have finite order.

The last remaining inclusion is more difficult, and in fact I have been unable to decide whether there exists an $\mathfrak{J}$-group which is not also a $\mathfrak{D}$-group. The existence of such a group would entail the existence of an aperiodic group with the same property – every homomorphic image of a non-$\mathfrak{D}$-group is itself non-$\mathfrak{D}$. It seems very likely that such groups exist, and I would conjecture that there exist positive integers $m, n, s, \lambda_1, \mu_1, \ldots, \lambda_s, \mu_s$ such that the group K generated by two elements a, b with the defining relations

$$a^m = b^n, \qquad a^{\lambda_1}b^{\mu_1}a^{\lambda_2}b^{\mu_2}\ldots a^{\lambda_s}b^{\mu_s} = 1$$

is aperiodic. Certainly K is non-$\mathfrak{D}$; for in any decomposition $K = \bigcup_{i \in \mathfrak{A}} S_i$ of K into the union of disjoint subsemigroups, a and b lie in the same subsemigroup S, on account of the relation $a^m = b^n$; while the third relation implies that a^{-1} and b^{-1} lie in S so that in fact $S = K$. I know of no method of testing whether K is aperiodic or not.

In any case we have established the following theorem, in which the letter $\mathfrak{M}$ denotes the property of being metabelian:

Theorem 3.11. *The classes* $[\mathfrak{T}]$, $[\mathfrak{J}]$, $[\mathfrak{D}]$, $[\mathfrak{S}]$ *satisfy the inclusions*

$$[\mathfrak{T}] \supset [\mathfrak{J}] \supseteq [\mathfrak{D}] \supset [\mathfrak{S}],$$

$$[\mathfrak{T}] \cap [\mathfrak{M}] \supset [\mathfrak{J}] \cap [\mathfrak{M}] \supseteq [\mathfrak{D}] \cap [\mathfrak{M}] \supset [\mathfrak{S}] \cap [\mathfrak{M}].$$

Finally we come to an extension of Schwarz's theorem 2.8 to a much wider class of groups. We say that G has property $\mathfrak{X}$ if it is not periodic, and if every two-generator subgroup is either nilpotent or free.

Theorem 3.12. *Every $\mathfrak{X}$-group is an $\mathfrak{S}$-group.*

Proof. Let G be any group satisfying property $\mathfrak{X}$; then we have to show that every equivalence class T_a is a subsemigroup of G. To this end let b, c be equivalent to the element a, so that b and c are equivalent and have positive powers in common. If now b and c together generate a nilpotent subgroup, then lemma 2.6 tells us that bc and b have positive powers in common, and hence that $bc \in T_a$. Otherwise b and c generate a free subgroup. In that case the subgroup must have rank not more than 2, being generated by 2 elements; it cannot have rank exactly 2, since b and c are patently not free generators; hence it is of rank 1, and b and c must be powers of one and the same element. Thus again $bc \in T_a$, and in all cases T_a is a subsemigroup of G.

Corollary 3.13. *The various classes of groups considered satisfy the equalities*

$$[\mathfrak{X}] \cap [\mathfrak{L}] = [\mathfrak{X}] \cap [\mathfrak{N}] = [\mathfrak{X}] \cap [\mathfrak{T}] = [\mathfrak{X}] \cap [\mathfrak{S}] = [\mathfrak{X}],$$

in particular: a locally nilpotent group can be covered by pairwise disjoint proper subsemigroups if and only if it contains elements of infinite order.

Another corollary, to the proof this time, is the following:

Corollary 3.14. *Every non-trivial locally free group is the union of pairwise disjoint proper commutative subsemigroups.*

Proof. If two elements of a locally free group have positive powers in common they must commute.

Theorem 3.12 is most useful as a source of counterexamples. As an instance of its use we begin with the observation that any group G for which the factor-group $G/\gamma_1(G)$ is non-periodic has property $\mathfrak{T}$, and ask whether the converse is also true. The answer is no, for B. H. Neumann gives in [4] an example of a locally free group F which coincides with its derived group. This means, of course, that we can make a $\mathfrak{T}$-group in which the factor-group to the derived group has any preassigned (abelian) structure: for instance if A is an abelian group, the direct product $A \times F = H$ has property $\mathfrak{T}$, and yet $H/\gamma_1(H)$ is isomorphic with A.

REFERENCES

[1] Schwarz Š., *O pologrupách splňujúcich zoslabené pravidlá krátenia*, Matematicko-fyzikálny časopis SAV VI (1956), 149—158.

[2] Hall P., *Nilpotent Groups*, Canadian Mathematical Congress Summer Seminar, University of Alberta, 1957.

[3] Kurosh A. G., *The theory of groups*, Chelsea Publishing Company 1955.
[4] Neumann B. H., *Ascending derived series*, Compositio Mathematica 13 (1956), 47–64.

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ПОКРЫТИЕ ГРУПП ПОЛУГРУППАМИ

Джеймс Уайголд

Резюме

Будем говорить, что группа G удовлетворяет соответственно условиям:

$\mathfrak{S}$: если G — объединение частичных подполугрупп, среди которых нет ни одной группы;

$\mathfrak{A}$: если G имеет хотя бы один апериодический гомоморфный образ;

$\mathfrak{T}$: если G — объединение попарно непересекающихся собственных подполугрупп;

$\mathfrak{Z}$: если G не является периодической группой, и $a^m = b^n$ ($a, b \in G$, m, n — натуральные числа) влечет за собой $a^s = (ab)^t$ для некоторых s, t.

Класс групп, удовлетворяющих условию $\mathfrak{P}$ обозначим знаком $(\mathfrak{P})$. В настоящей статье доказывается: $(\mathfrak{S}) \supset (\mathfrak{A}) \supseteq (\mathfrak{T}) \supset (\mathfrak{Z})$. Из результатов статьи следуют некоторые обобщения результатов статьи [1], в которой изучались группы, удовлетворяющие условию $(\mathfrak{T})$.