SEMIGROUP COVERINGS OF GROUPS II

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1. Introduction

The purpose of this note is to confirm two conjectures made in [1]. A group is called* a $\mathfrak{D}$-group if it is the union of non-empty pairwise disjoint proper subsemigroups, and an $\mathfrak{I}$-group if it has at least one aperiodic homomorphic image. It was proved in [1] that every $\mathfrak{D}$-group is an $\mathfrak{I}$-group, and conjectured that the converse is not true. We shall confirm this by showing that the group G generated by two elements a, b with defining relations

$$a^3 = b^2, \qquad ab(a^2b)^5\, ab^2 = 1 \tag{1.1}$$

ia aperiodic — it was pointed out in [1, page 11] that no group with defining relations like (1.1) can be a $\mathfrak{D}$-group: this fact is very easy to establish. The second derived group $\delta_2(G)$ of G is the cycle generated by b^{16}, so that it is central and in particular G is soluble of length 3.

The other conjecture concerned the existence of a group generated by elements of finite order and having non-periodic centre. Here the construction is considerably easier than that of G: in the group J generated by two elements a, b with defining relations

$$a^3 = b^2 = 1, \qquad [[[a^2, b], [a, b]], a] = [[[a^2, b], [a, b]], b] = 1, \tag{1.2}$$

the element $[[a^2, b], [a, b]]$ — obviously central — has infinite order. This element generates the second derived group of J, so that J is also soluble of length 3.

I have been unable to construct metabelian examples.

2. The group G

Let us first examine the defining relations

$$a^3 = b^2, \qquad aba^2ba^2ba^2ba^2ba^2bab^2 = 1$$

* All notation not explained here is to be found in [1].

of G. As a^3 and b^2 are central in G, the second of these can be rewritten as

$$baba^2ba^2ba^2ba^2ba^2bab = 1,$$

$$b^{-1}b^2a^{-2}a^3ba^2b^{-1}b^2a^{-1}a^3baa^{-2}a^3b^{-1}b^2a^2ba^{-1}a^3b^{-1}b^2ab = 1,$$

$$[b, a^2]\,[b, a]\,[a^2, b]\,[a, b]\,b^8a^{12} = 1,$$

and finally as

$$[[a^2, b], [a, b]]\,b^{16} = 1. \tag{2.1}$$

Now the derived group $\delta_1(G)$ of G is generated by all commutators of the form $[a^\lambda, b^\mu]$, where λ and μ are integers; but a^3 and b^2 are central so that λ may be taken modulo 3 and μ modulo 2, which means that $\delta_1(G)$ is generated by $[a^2, b] = u$ and $[a, b] = v$. Relation 2.1 shows that $[u, v]$ is central in G, which gives immediately that $\delta_2(G)$ is central and cyclic, with b^{16} as generator. Further, $\delta_1(G)$ is necessarily second nilpotent – in fact it turns out that is a free second nilpotent group on the generators u, v. We shall show that G is aperiodic by constructing it as a double splitting extension with central amalgamations.

The starting-point is a free second nilpotent group H_0 on two generators, that is, we take a group H_0 by two elements u_0 and v_0 with defining relations

$$[u_0, v_0, u_0] = [u_0, v_0, v_0] = 1. \tag{2.2}$$

The mapping β_0 of the generators of H_0 given by

$$u_0\beta_0 = u_0^{-1}, \qquad v_0\beta_0 = v_0^{-1}$$

clearly respects the defining relations (2.2), so it generates an automorphism (also called β_0) of order 2 of H_0. Let B_0 be the splitting extension* of H_0 by an infinite cycle whose generator b_0 induces β_0 on H_0:

$$B_0 = Gp(u_0, v_0, b_0;\, [u_0, v_0, u_0] = [u_0, v_0, v_0] = 1,\, u_0^{b_0} = u_0^{-1},\, v_0^{b_0} = v_0^{-1}). \tag{2.3}$$

Then B_0, as an extension of an aperiodic group by an aperiodic group, is aperiodic: further, $[u_0, v_0]$ and b_0^2 are central, since

$$[u_0, v_0]^{b_0} = [u_0^{-1}, v_0^{-1}] = [u_0, v_0],$$

$$u_0^{b_0{}^2} = (u_0^{-1})^{b_0} = u_0, \qquad v_0^{b_0{}^2} = (v_0^{-1})^{b_0} = v_0.$$

The first of these relations follows from the first of the identities

$$[g^m, h^n] = [g, h]^{mn}$$

$$(gh)^m = g^m h^m [h, g]^{m(m-1)/2}, \tag{2.4}$$

valid for any integers m, n and any elements g, h of a second nilpotent group.

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* In general, we write $Gp(X;\, R)$ for the group generated by a set X of elements with defining relation R, and $Gp(X)$ if R is understood or unimportant.

Clearly, as H_0 intersects the cycle generated by b_0 trivially, the centralcycle generated by $[u_0, v_0]\, b_0^{16}$ intersects H_0 and the cycle $Gp(b_0)$ trivially. Thus, if Φ is the canonic homomorphism of B_0 onto $B = B_0/C_0$, then $H = H_0\Phi \simeq H_0$, and $b = b_0\Phi$ has infinite order, so that:

2.5. In the group B generated by three elements u, v, b with defining relations

$$[u, v, u] = [u, v, v] = 1, \qquad u^b = u^{-1}, \qquad v^b = v^{-1}, \qquad [u, v] = b^{-16},$$

the subgroup $H = Gp(u, v)$ is free second nilpotent and b has infinite order.

Lemma 2.6. (i) *In B, $Gp(b) \cap H = Gp(b^{16}) = Gp([u, v])$.*
(ii) *B is aperiodic.*

Proof. (i) It is clear that $H \cap Gp(b) \supseteq Gp(b^{16})$. Let now $x \in H \cap Gp(b)$ so that $x = h_0\Phi = b_0^{\lambda}\Phi$ where $h_0 \in H_0$ and λ is an integer. Then $(h_0 b_0^{-\lambda})\,\Phi = 1$ so that $h_0 b_0^{-\lambda}$ is a power of $[u_0, v_0]\, b_0^{16}$, say $[u_0, v_0]^{\xi} b_0^{16\xi}$; hence $b_0^{-\lambda} = b_0^{16\xi}$ so that $x = b^{\lambda} = b^{-16\xi} \in Gp(b^{16})$. This completes the proof of the first part.

(ii) It is not difficult to see that every element of B can be written in the form

$$g = u^{\alpha} v^{\beta} [u, v]^{\gamma} b^{\delta}$$

with suitable integers α, β, γ, δ. We distinguish two cases:

(A) If δ is even, b^{δ} is in the centre of B, and, as $[u, v]$ is in the centre of B (both these follow immediately from the fact that b_0^2 is central in B_0), then for any interger n,

$$g^n = (u^{\alpha} v^{\beta})^n\, [u, v]^{n\gamma} b^{n\delta}$$

But $(u^{\alpha} v^{\beta})^n = u^{\alpha n} v^{\beta n} [u, v]^{\Theta}$ for some Θ, by 2.4; hence

$$g^n = u^{\alpha n} v^{\beta n} [u, v]^{\Theta'}\, b^{n\delta}$$

for some Θ'. If $g^n = 1$ for some $n \neq 0$, then, as $[u, v] = b^{-16}$, $u^{\alpha n} v^{\beta n}$ lies in the cycle generated by b. The first part of the lemma then gives that $u^{\alpha n} v^{\beta n}$ lies in the cycle generated by $[u, v]$, so that since H is free second nilpotent on u nad v, $n\alpha = n\beta = 0$. Consequently $\alpha = \beta = 0$, $g = [u, v]^{\gamma} b^{\delta}$, so g lies in the cycle generated by b; and this means it can have finite order if and only if it is the unit element.

(B) If δ is odd, say $\delta = 2\delta' + 1$, then

$$g = u^{\alpha} v^{\beta} b [u, v]^{\gamma} b^{2\delta'}.$$

so that

$$\begin{aligned} g^2 &= u^{\alpha} v^{\beta} b u^{\alpha} v^{\beta} b [u, v]^{2\gamma} b^{4\delta'} \\ &= u^{\alpha} v^{\beta} (u^{\alpha} v^{\beta})^{b^{-1}} b^2 [u, v]^{2\gamma} b^{4\delta'} \\ &= u^{\alpha} v^{\beta} u^{-\alpha} v^{-\beta} [u, v]^{2\gamma} b^{4\delta' + 2}, \end{aligned}$$

using throughout the fact that b^2 is central. Finally

$$\begin{aligned} g^2 &= [u^{-\alpha}, v^{-\beta}]\,[u, v]^{2\gamma} b^{4\delta'+2} \\ &= [u, v]^{2\gamma+\alpha\beta} b^{4\delta'+2} \\ &= b^{-16(2\gamma+\alpha\beta)+4\delta'+2} && \text{(by 2.4)} \\ &= b^{4\delta'+2-16\varepsilon}, \end{aligned}$$

for some integer ε. Now $4\delta' + 2 - 16\varepsilon \neq 0$ since $2\delta' + 1 \neq 8\varepsilon$; hence g^2 has infinite order and some must g.

Thus we have proved that only the unit element of B is of finite order, and B is aperiodic.

It will turn out that B is the normal closure of b in the original group G (we shall not prove this) – hence the use of the letter b in defining B.

The next stage is to form a splitting extension of B by an infinite cycle. For clarity at this point we take an isomorphic copy B_1 of B with generators u_1, v_1, b_1 and defining realtions analagous to those given at 2.5. It is a matter of simple routine to verify that the mapping α_1 of the generators of B_1 defined by

$$u_1\alpha_1 = v_1^{-1}, \qquad v_1\alpha_1 = u_1v_1^{-1}, \qquad b_1\alpha_1 = b_1v_1^{-1}$$

respects the defining relations of B_1. Thus α_1 generates an endomorphism (also called α_1) of B_1. Further, α_1^3 is the identity automorphism, as

$$\left.\begin{aligned} &u_1\alpha_1^2 = (v_1\alpha_1)^{-1} = v_1u_1^{-1}, \quad u_1\alpha_1^3 = v_1\alpha_1(u_1\alpha_1)^{-1} = u_1v_1^{-1}v_1 = u_1; \\ &v_1\alpha_1^2 = (u_1v_1^{-1})\,\alpha_1 = v_1^{-1}v_1u_1^{-1} = u_1^{-1}, \quad v_1\alpha_1^3 = (u_1\alpha_1)^{-1} = v_1; \\ &b_1\alpha_1^2 = b_1\alpha_1(v_1\alpha_1)^{-1} = b_1v_1^{-1}v_1u_1^{-1} = b_1u_1^{-1}, \\ &b_1\alpha_1^3 = b_1\alpha_1(u_1\alpha_1)^{-1} = b_1v_1^{-1}v_1 = b_1. \end{aligned}\right| \tag{2.7}$$

This means that α_1 has a two-sided inverse and is consequently an automorphism of order 3 of B_1. Form the splitting extension of B_1 by an infinite cycle whose generator a_1 induces α_1 on B_1: this is a group G_1 generated by four elements u_1, v_1, b_1, a_1 with defining relations

$$\left.\begin{aligned} &[u_1, v_1, u_1] = [u_1, v_1, v_1] = 1, \quad u_1^{b_1} = u_1^{-1}, \quad v_1^{b_1} = v_1^{-1}, \\ &b_1^{-16} = [u_1, v_1], \quad u_1^{a_1} = v_1^{-1}, \quad v_1^{a_1} = u_1v_1^{-1}, \quad b_1^{a_1} = b_1v_1^{-1}. \end{aligned}\right\} \tag{2.8}$$

Again G_1 is aperiodic and it is routine to verify that b_1^2 and a_1^3 are central in G_1. Let A_1, C_1 be the cycles generated by a_1 and $b_1^{-2}a_1^3$ respectively. Then C_1 is central in G_1 and misses B_1 and A_1 so that if ψ is the canonic homomorphism of G_1 onto $G_0 = G_1/C_1$, then $B_1\psi \simeq B_1$ and $a_1\psi = a$ has infinite order. Thus we have proved the first part of the following lemma – the second is proved in a manner completely analagous to that of Lemma 2.6 (i):

Lemma 2.9. *In the group G_0 generated by elements u, v, b, a with defining relations*

$$[u, v, u] = [u, v, v] = 1, \qquad u^b = u^{-1}, \qquad v^b = v^{-1}, \qquad b^{-16} = [u, v],$$
$$u^a = v^{-1}, \qquad v^a = uv^{-1}, \qquad b^a = bv^{-1}, \qquad a^3 = b^2,$$

the subgroup $B = Gp(u, v, b)$ is aperiodic and $A = Gp(a)$ is infinite.

(ii) $B \cap A = Gp(a^3) = Gp(b^2)$.

We can now prove the main result.

Theorem 2.10. *G_0 is aperiodic.*

Proof. Since B is normal in G_0 and $a^3 \in B$, every element of G_0 can be put in one of the forms h, ha, ha^2 with $h \in B$. Obviously, if h is of finite order, $h = 1$ because B is aperiodic. We shall show first that every element of the form ha is of infinite order. As G/B is of order 3 so that $(ha)^3 \in B$, the element ha can have finite order if and only if $(ha)^3 = 1$. Now

$$\begin{aligned}(ha)^2 &= haha = haha^{-1}a^2 = hh^{a^{-1}}a^2,\\ (ha)^3 &= hh^{a^{-1}}a^2ha = hh^{a^{-1}}a^2ha^{-2}a^3\\ &= hh^{a^{-1}}h^{a^{-2}}a^3\\ &= hh^{a^{-1}}h^{a^{-2}}b^2.\end{aligned}$$

It is easy to check that h can be expressed in the form

$$h = u^\alpha v^\beta [u, v]^\gamma b^\delta$$

for suitable integers α, β, γ, δ. Once again there are two cases, depending on the parity of δ.

(A) If δ is even, b^δ is central so that as a^3 is central,

$$\begin{aligned}hh^{a^{-1}}h^{a^{-2}}b^2 &= hh^{a^2}h^a b^2\\ &= u^\alpha v^\beta (u^\alpha)^{a^2}(v^\beta)^{a^2}(u^\alpha)^a(v^\beta)^a[u, v]^{3\gamma}b^{3\delta+2}\\ &= u^\alpha v^\beta (vu^{-1})^\alpha u^{-\beta}v^{-\alpha}(uv^{-1})^\beta [u, v]^{3\gamma}b^{3\delta+2} \qquad \text{(from 2.7)}\\ &= u^\alpha v^\beta v^\alpha u^{-\alpha}u^{-\beta}v^{-\alpha}u^\beta v^{-\beta}[u, v]^{3\gamma+\Phi(\beta)-\Phi(\alpha)}b^{3\delta+2};\end{aligned}$$

this from 2.4, where for any integer k, $\Phi(k) = k(k-1)/2$. A little computation now gives

$$hh^{a^{-1}}h^{a^{-2}}b^2 = [u, v]^{3\gamma+\Phi(\beta)-\Phi(\alpha)+\alpha^2+2\alpha\beta}b^{3\delta+2}.$$

If this is 1, then as $[u, v] = b^{-16}$ and b has infinite order,

$$3\delta + 2 = 16(3\gamma + \Phi(\beta) - \Phi(\alpha) + \alpha^2 + 2\alpha\beta).$$

But $\delta = 2\delta'$ so that

$$3\delta' + 1 = 8(3\gamma + \Phi(\beta) - \Phi(\alpha) + \alpha^2 + 2\alpha\beta).$$

We shall obtain a contradiction by showing that the right hand side of this equation is never congruent to 1 modulo 3. First,

$$\begin{aligned}8(3\gamma + \Phi(\beta) - \Phi(\alpha) + \alpha^2 + 2\alpha\beta) &\equiv 2(\Phi(\beta) - \Phi(\alpha) + \alpha^2 + 2\alpha\beta) \pmod 3\\ &\equiv \beta(\beta-1) - \alpha(\alpha-1) + 2\alpha^2 + \alpha\beta \pmod 3\\ &= \beta^2 + \alpha^2 + \alpha - \beta + \alpha\beta.\end{aligned}$$

The rest is routine: one verifies that $\beta^2 + \alpha^2 + \alpha - \beta + \alpha\beta$ is always congruent to 0 or 2 modulo 3, in all the 9 cases arising out of the 3 congruences modulo 3 possible for each of α and β.

Hence if δ is even, ha is of infinite order.

(B) If δ is odd, say $\delta = 2\delta' + 1$, then

$$h = u^\alpha v^\beta bz,$$

where $z = [u, v]^\gamma b^{2\delta'}$ is in the centre of G_0. Then

$$hh^{a^2}h^a b^2 = u^\alpha v^\beta b(u^{a^2})^\alpha (v^{a^2})^\beta b^{a^2}(u^a)^\alpha (v^a)^\beta b^a b^2 z^3,$$

and consideration of the image of this element in $G_0/Gp(u, v)$ shows it (the orginal element) to be of the form $v'b^{5+6\delta'}$ for some $v' \in Gp(u, v)$. If it is 1, then by lemma 2.6(i), $b^{5+6\delta'}$ lies in $Gp(b^{16})$. This means that $5 + 6\delta'$ is divisible by 16, which is a manifest contradiction. Thus $hh^{a^{-1}}h^{a^{-2}}b^2$ is never the unit element, and it follows that ha has infinite order.

The proof concludes with the remark that ha^2 must also have infinite order, since

$$\begin{aligned}(ha^2)^2 = ha^2ha^2 &= hh^{a^{-2}}a^4 \\ &= hh^{a^{-2}}a^3a\end{aligned}$$

is of the form $h'a$ with $h' \in B$.

It remains only to show that G_0 is isomorphic with G. The defining relations of G_0 in terms of the generators u, v, a, b are

$$[u, v, u] = [u, v, v] = 1, \qquad u^b = u^{-1}, \qquad v^b = v^{-1},$$
$$u^a = v^{-1}, \qquad v^a = uv^{-1}, \qquad b^a = bv^{-1},$$
$$b^{-16} = [u, v], \qquad a^3 = b^2.$$

From these it follows straight away that

$$\begin{aligned} v &= (b^{-1})^a b = [a, b] \\ u = v^a v &= [a, b]^a[a, b] = [a^2, b]\end{aligned}$$

so that G_0 is generates by a and b. One now readily verifies that the first 8 defining relations of G_0 are consequences of the last two; these are precisely the relations 2.1, so that in fact G and G_0 are isomorphic.

To sum up, define a $\mathfrak{C}$-group to be one whose second derived group is central. With the notation of [1],

Theorem 2.11 $[\mathfrak{K}] \cap [\mathfrak{C}] \supset [\mathfrak{D}] \cap [\mathfrak{C}]$.

3. The group J

Here we shall only state results: the proofs are a matter of simple routine.

We again start with a free second nilpotent group on two generators:

$$H = Gp(u, v; \; [u, v, u] = [u, v, v] = 1),$$

and form the splitting extension of H by a cycle of order 2 whose generator b induces the automorphism of order 2 of H generated by the mappings $u \to u^{-1}$, $v \to v^{-1}$:

$$H = Gp(u, v, b;\ [u, v, u] = [u, v, v] = 1,\ b^2 = 1,\ u^b = u^{-1},\ v^b = v^{-1}).$$

In this u and v still generate a free second nilpotent group, so in particular $[u, v]$ is of infinite order. Next form the splitting extension J of B be a cycle of order 3 whose generator a induces the automorphism of B defined by the mappings

$$u \to v^{-1}, \qquad v \to uv^{-1}, \qquad b \to bv^{-1}:$$

J is generated by u, v, a, b with the defining relations of B together with

$$a^3 = 1, \qquad u^a = v^{-1}, \qquad v^a = uv^{-1}, \qquad b^a = bv^{-1}.$$

Note (as with G) that $v = [a, b]$, $u = [a^2, b]$ so that J is generated by a and b, and that $[u, v]$ is central, as

$$[u, v]^b = [u^{-1}, v^{-1}] = [u, v]$$
$$[u, v]^a = [v^{-1}, uv^{-1}] = [v^{-1}, u] = [u, v].$$

This completes the example, except for the simple verification that the group J constructed here is in fact that mentioned in the introduction.

REFERENCES

[1] Wiegold James, Semigroup coverings of groups, Matematicko-fyzikálny časopis SAV 11 (1961), 3–13.

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ПОКРЫТИЕ ГРУПП ПОЛУГРУППАМИ

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Резюме

Настоящая статья является продолжением статьи (1). В статье (1) введено понятие $\mathfrak{T}$-группы (это группа, являющаяся объединением попарно непересекающихся собственных подполугрупп) и понятие $\mathfrak{J}$-группы (это группа, которая имеет хотя бы один апериодический гомоморфный образ) и доказывается, что всякая $\mathfrak{T}$-группа является $\mathfrak{J}$-группой. В настоящей статье построена группа G, которая является $\mathfrak{J}$-группой, но не является $\mathfrak{T}$-группой. Кроме того, построен пример группы с образующими конечного порядка, центр которой не является периодической группой.