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LAPLACE—STIELTJES TRANSFORMS OF VECTOR-VALUED MEASURES

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1. Introduction. It is well-known that a complex-valued function f on $(0, \infty)$ can be characterized as a Laplace-Stieltjes transform in terms of the maps $L_k(f)$, $k = 1, 2, \ldots$, defined by

$$L_k(f)\,(t) = \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} f^{(k)}\left(\frac{k}{t}\right), \quad t \in (0, \infty).$$

Namely, there exists a complex Borel measure on $[0, \infty)$ such that

$$f(\lambda) = \int_0^\infty e^{-\lambda t}\mu(\mathrm{d}t), \quad \lambda \in (0, \infty), \tag{1}$$

iff f has derivatives of all orders on $(0, \infty)$ and there exists a constant M such that

$$\int_0^\infty |L_k(f)\,(t)|\,\mathrm{d}t \leqq M, \quad k = 1, 2, \ldots; \tag{2}$$

(see for example [4], VII 12a).

Let C_0 denote the space of all continuous complex-valued functions on $[0, \infty)$ which vanish at infinity, equipped with the sup-norm. Then the above condition (2) means that the maps $\Phi_k(f)$, $k = 1, 2, \ldots$, defined by

$$\Phi_k(f)\,(\varphi) = \int_0^\infty \varphi(t) L_k(f)\,(t)\,\mathrm{d}t, \quad \varphi \in C_0,$$

are equibounded linear functionals on C_0; i.e. they take the closed unit ball of C_0 into a bounded set not depending upon k.

In this paper we generalize by letting f take values in a quasi-complete, locally convex space X, whose topology is defined by a system P of seminorms. Defining $L_k(f)$ and $\Phi_k(f)$ as above, but with values now in X, we show that f is the Laplace-Stieltjes transform of a vector measure iff f has derivatives of all orders on $(0, \infty)$ and the maps $\Phi_k(f)$ take the closed unit ball of C_0 into a weakly compact subset of X, not depending upon k. In addition, we show that the vector measure has finite variation iff f has derivatives of all orders on $(0, \infty)$ and, for each $p \in P$, there exists a constant M_p such that

$$(3) \qquad \int_0^\infty p(L_k(f)(t))\, \mathrm{d}t \leqq M_p, \quad k = 1, 2, \ldots .$$

2. Preliminary results. Let C denote the complex number field, and B the σ-ring of all Borel subsets of $[0, \infty)$.

In the following two lemmas, f is a complex-valued function with derivatives of all orders on $(0, \infty)$.

Lemma 1. *If for each* $k = 1, 2, \ldots,$

$$\int_0^v L_k(f)(t)\, \mathrm{d}t = 0(v), \quad v \to \infty,$$

then $f(\infty)$ *exists, and*

$$(4) \qquad \lim_{k\to\infty} \int_0^\infty e^{-\lambda t} L_k(f)(t)\, \mathrm{d}t = f(\lambda) - f(\infty), \quad \lambda \in (0, \infty).$$

Proof. See [4], VII 11b.

Lemma 2. *If there exists a constant* M *such that* (2) *holds, then* $\lim_{k\to\infty} \Phi_k(f)(\varphi)$ *exists, for all* $\varphi \in C_0$.

Proof. By Lemma 1, it follows that (4) holds. Therefore, if A denotes the subalgebra of C_0 consisting of all functions of the form

$$t \to \sum_{i=1}^n \alpha_i e^{-\lambda_i t}, \quad \alpha_i \in C, \quad \lambda_i \in (0, \infty), \quad 1 \leq i \leqq n = 1, 2, \ldots,$$

it is clear that $\lim_{k\to\infty} \Phi_k(f)(\psi)$ exists, for all $\psi \in A$.

Let $\varepsilon > 0$ and $\varphi \in C_0$ be given. Since A is dense in C_0, there exists a function $\psi \in A$ such that $\|\varphi - \psi\|_\infty < \dfrac{\varepsilon}{3M}$. Then

$$|\Phi_k(f)(\varphi) - \Phi_j(f)(\varphi)| \leqq |\Phi_k(f)(\varphi - \psi)| + |\Phi_k(f)(\psi) - \Phi_j(f)(\psi)| +$$
$$+ |\Phi_j(f)(\psi - \varphi)| < \|\varphi - \psi\|_\infty M + \varepsilon/3 + \|\psi - \varphi\|_\infty M < \varepsilon,$$

for k, j sufficiently large.

3. A characterization of the Laplace—Stieltjes transforms of vector-valued measures. Let X' denote the dual of the quasi-complete, locally convex space X. By the weak topology on X we mean the $\sigma(X, X')$ topology.

Theorem 1. *A function* $f:(0,\infty)\to X$ *is the Laplace-Stieltjes transform of a regular measure on* B *iff* f *has derivatives of all orders on* $(0,\infty)$ *and the set*

$$\{\Phi_k(f)(\varphi):\varphi\in C_0,\ \|\varphi\|_\infty\leqq 1,\ k=1,2,\dots\} \tag{5}$$

is relatively weakly compact; i.e. the maps $\Phi_k(f)$, $k=1,2,\dots$, *are weakly equicompact.*

Proof. Suppose firstly that the maps $\Phi_k(f)$, $k=1,2,\dots$, are weakly equicompact.

For fixed $x'\in X'$, define the function $g_{x'}:(0,\infty)\to C$ by

$$g_{x'}(\lambda)=\langle f(\lambda),x'\rangle,\quad \lambda\in(0,\infty).$$

Then it is clear that $g_{x'}$ has derivatives of all orders on $(0,\infty)$ and, for each $k=1,2,\dots$,

$$L_k(g_{x'})(t)=\langle L_k(f)(t),x'\rangle,\quad t\in(0,\infty).$$

Since the set (5) is relatively weakly compact, it is weakly bounded, and so there exists a constant $M_{x'}$ such that

$$|\langle\Phi_k(f)(\varphi),x'\rangle|\leqq M_{x'},\ \varphi\in C_0,\ \|\varphi\|_\infty\leqq 1,\ k=1,2,\dots.$$

Thus, for each $k=1,2,\dots$,

$$\int_0^\infty |L_k(g_{x'})(t)|\,\mathrm{d}t=\sup_{\|\varphi\|_\infty\leqq 1}\Big|\int_0^\infty \varphi(t)L_k(g_{x'})(t)\,\mathrm{d}t\Big| \tag{6}$$

$$=\sup_{\|\varphi\|_\infty\leqq 1}\Big|\Big\langle\int_0^\infty \varphi(t)L_k(f)(t)\,\mathrm{d}t,x'\Big\rangle\Big|\leqq M_{x'}.$$

Hence, by Lemma 2,

$$\lim_{k\to\infty}\Phi_k(g_{x'})(\varphi)=\lim_{k\to\infty}\langle\Phi_k(f)(\varphi),x'\rangle$$

exists, for all $\varphi\in C_0$.

Since $x'\in X'$ was arbitrary, it follows that for fixed $\varphi\in C_0$, the sequence $\{\Phi_k(f)(\varphi)\}_{k=1}^\infty$ is weakly Cauchy. By the weak equicompactness of the maps $\Phi_k(f)$, $k=1,2,\dots$, this sequence is contained in a weakly compact (hence weakly complete) set, and is therefore weakly convergent.

Thus, for each $\varphi\in C_0$, there is a unique $\Phi(f)(\varphi)\in X$ such that

$$\Phi(f)(\varphi)=w\text{-}\lim_{k\to\infty}\Phi_k(f)(\varphi).$$

This defines a linear map $\Phi(f):C_0\to X$ which is clearly weakly compact. In fact, if K is a weakly compact set containing (5), then $\Phi(f)(\varphi)\in K$ whenever $\|\varphi\|_\infty\leqq 1$.

Accordingly, there exists a regular measure $\mu : B \to X$ such that

$$\Phi(f)\,(\varphi) = \int_0^\infty \varphi(t)\mu(\mathrm{d}t), \quad \varphi \in C_0;$$

(see [2], Proposition 1). In particular, since for each $\lambda \in (0, \infty)$ the function $t \to e^{-\lambda t}$ belongs to C_0, we have

$$w\text{-}\lim_{k\to\infty} \int_0^\infty e^{-\lambda t} L_k(f)\,(t)\,\mathrm{d}t = \int_0^\infty e^{-\lambda t}\mu\,(\mathrm{d}t), \quad \lambda \in (0, \infty).$$

Thus, since $f(\infty)$ exists as a weak limit, Lemma 1 implies

$$\langle\int_0^\infty e^{-\lambda t}\mu\,(\mathrm{d}t), x'\rangle = \lim_{k\to\infty} \int_0^\infty e^{-\lambda t} L_k(g_{x'})\,(t)\,\mathrm{d}t = \langle f(\lambda) - f(\infty), x'\rangle,$$

for each $x' \in X'$, so that

$$f(\lambda) - f(\infty) = \int_0^\infty e^{-\lambda t}\mu\,(\mathrm{d}t), \quad \lambda \in (0, \infty).$$

Replacing μ throughout by $\mu - \mu_0$, where $\mu_0 : B \to X$ is the measure taking the value $f(\infty)$ on sets containing $\{0\}$ and zero elsewhere, we obtain (1).

Conversely, suppose that f is the Laplace-Stieltjes transform of $\mu : B \to X$. Clearly, for each $k = 1, 2, \ldots,$ the derivative $f^{(k)}$ is given by

$$f^{(k)}(\lambda) = \int_0^\infty (-s)^k e^{-\lambda s}\mu\,(\mathrm{d}s), \quad \lambda \in (0, \infty),$$

and hence

$$L_k(f)\,(t) = \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty (-s)^k e^{-\frac{ks}{t}}\mu\,(\mathrm{d}s), \quad t \in (0, \infty). \tag{7}$$

Therefore, for fixed $\varphi \in C_0$ with $\|\varphi\|_\infty < 1$, and $k = 1, 2, \ldots,$

$$\Phi_k(f)\,(\varphi) = \int_0^\infty \varphi(t)\left(\int_0^\infty \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} s^k e^{-\frac{ks}{t}}\mu\,(\mathrm{d}s)\right)\mathrm{d}t$$

$$= \int_0^\infty \left(\int_0^\infty \varphi(t)\,\frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} s^k e^{-\frac{ks}{t}}\,\mathrm{d}t\right)\mu\,(\mathrm{d}s),$$

by Fubini 's theorem. Thus

$$\Phi_k\,(f)\,(\varphi) = \int_0^\infty \xi_{k,\varphi}(s)\mu\,(\mathrm{d}s),$$

where

$$\xi_{k,\varphi}(s) \quad \int_0^\infty \varphi(t) \frac{1}{k!} \left(\frac{k}{t}\right)^{k+1} s^k e^{-\frac{ks}{t}} \, dt, \quad s \in [0, \infty).$$

By a simple change of variables,

$$\xi_{k,\varphi}(s) \quad \int_0^\infty \varphi\left(\frac{ks}{u}\right) \frac{u^{k \quad 1}}{(k-1)!} e^{\quad u} \, du,$$

so that

$$\xi_{k,\varphi}(s) \leqq \varphi_\infty \int_0^\infty \frac{u^{k \quad 1}}{(k \quad 1)!} e^{\quad u} \, du \leqq 1, \quad s \in [0, \infty),$$

using the identity

(8) $$\int_0^\infty \frac{u^n}{n!} e^{-u} \, du \quad 1, \quad n \quad 0, 1, 2, \ldots .$$

Thus, for each $\varphi \in C_0$ with $\varphi|_\infty \leqq 1$, and $k \quad 1, 2, \ldots$, $\Phi_k(f)(\varphi) \in co R(\mu)$, the closed absolutely convex hull of the range $R(\mu) \quad \{\mu(E) : E \in B\}$ of μ. Now by [3], $R(\mu)$ is relatively weakly compact, and so by Krein's theorem (see [3]), the set $co\ R(\mu)$ is weakly compact.

Remark. In the case where X is a Banach space, a result equivalent to Theorem 1 has been proved by S. Zaidman (see [5], Theorem 1).

4. Case where the vector-valued measures have finite variation. If the system of seminorms P defines the topology of X, a measure $\mu : B \to X$ has finite variation iff for each $p \in P$ there exists a positive measure ν_p such that $p(\mu(E)) \leqq \nu_p(E)$, for all $E \in B$.

Lemma 3. *A linear map $\Psi : C_0 \to X$ can be represented in the form*

$$\Psi(\varphi) \quad \int_0^\infty \varphi(t) \mu(dt), \quad \varphi \in C_0,$$

for some regular Borel measure μ with finite variation iff for each $p \in P$ there exists a constant M_p such that

$$\varphi_1, \varphi_2, \ldots \varphi_n \in C_0, \quad \sum_{i \quad 1}^{n} |\varphi_i \leqq 1$$

implies

$$\sum_{i=1}^{n} p(\Psi(\varphi_i)) \leqq M_p .$$

Proof. In the case where X is a Banach space, the result follows from [1], III 19.3, Theorems 2 and 3. For our more general space, the proof is essentially the same, and is therefore omitted.

Theorem 2. *A function* $f: (0, \infty) \to X$ *is the Laplace-Stieltjes transform of a regular measure with finite variation on B iff f has derivatives of all orders on* $(0, \infty)$ *and* (3) *holds for each* $p \in P$.

Proof. Suppose (3) holds for each $p \in P$. Then, if $x' \in X'$ is given and $g_{x'}$ is defined as in Theorem 1, it follows that (6) holds for some constant $M_{x'}$. Therefore, by Lemma 1,

$$\lim_{k\to\infty} \int_0^\infty e^{-\lambda t} L_k(g_{x'})(t)\, dt = g_{x'}(\lambda) - g_{x'}(\infty), \quad \lambda \in (0, \infty),$$

so that, since $f(\infty)$ exists (strong limit) and $x' \in X'$ was arbitrary,

$$f(\lambda) - f(\infty) = w\text{—}\lim_{k\to\infty} \int_0^\infty e^{-\lambda t} L_k(f)(t)\, dt, \quad \lambda \in (0, \infty).$$

Similarly, if A denotes the subalgebra of C_0 defined in Lemma 2, $w - \lim_{k\to\infty} \Phi_k(f)(\psi)$ exists, for all $\psi \in A$. Denote it by $\Phi(f)(\psi)$. Since, for each $p \in P$,

$$p(\int_0^\infty \psi(t) L_k(f)(t)\, dt) \leqq M_p \|\psi\|_\infty, \quad k = 1, 2, \ldots,$$

it is clear that

$$p(\Phi(f)(\psi)) \leqq M_p \|\psi\|_\infty, \quad \psi \in A.$$

Since A is dense in C_0 and X is quasi-complete, the uniformly continuous map $\Phi(f): A \to X$ defined above has a unique continuous extension (say $\Phi'(f)$) to C_0. Furthermore, one can easily show that

$$\Phi'(f)(\varphi) = w - \lim_{k\to\infty} \Phi_k(f)(\varphi), \quad \varphi \in C_0. \tag{9}$$

Now let $\varphi_1, \varphi_2, \ldots \varphi_n \in C_0$ be given, with $\sum_{i=1}^{n} |\varphi_i| \leqq 1$. Then, using (3), it follows that for each $p \in P$,

$$\sum_{i=1}^{n} p(\Phi_k(f)(\varphi_i)) \leqq \sum_{i=1}^{n} \int_0^\infty p(\varphi_i(t) L_k(f)(t))\, dt =$$

$$-\int_0^\infty \left(\sum_{i=1}^n |\varphi_i(t)|\right) p(L_k(f)(t))\, dt \leqq M_p, \quad k = 1, 2, \ldots .$$

Also, using (9) for each φ_i, $i = 1, 2, \ldots n$, it is clear that

$$p(\Phi'(f)(\varphi_i)) \leqq \limsup_{k \to \infty} p(\Phi_k(f)(\varphi_i)), \quad p \in P.$$

Thus, for each $p \in P$,

$$\sum_{i=1}^n p(\Phi'(f)(\varphi_i)) \leqq \limsup_{k \to \infty} \sum_{i=1}^n p(\Phi_k(f)(\varphi_i)) \leqq M_p,$$

so that, by Lemma 3, there exists a regular measure $\mu : B \to X$ with finite variation such that

$$\Phi'(f)(\varphi) = \int_0^\infty \varphi(t)\mu(dt), \quad \varphi \in C_0.$$

Proceeding exactly as in Theorem 1, we can now obtain (1).

Conversely, suppose that (1) holds, for some measure μ with finite variation and dominating positive measures ν_p, $p \in P$. Then, as in Theorem 1, we have (7), and hence, for each $p \in P$ and $k = 1, 2, \ldots,$

$$p(L_k(f)(t)) \leqq \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty s^k e^{-\frac{ks}{t}} \nu_p(ds), \quad t \in (0, \infty).$$

Thus, for each $p \in P$ and $k = 1, 2, \ldots,$

$$\int_0^\infty p(L_k(f)(t))\, dt \leqq \int_0^\infty s^k \left(\int_0^\infty \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{-\frac{ks}{t}} dt\right) \nu_p(ds) = \int_0^\infty \nu_p(ds) = M_p,$$

say, using Fubini's theorem and (8).

Remark. A result similar to Theorem 2 has been proved for Banach spaces by S. Zaidman (see [5], Theorem 2), where a certain „weak compactness" condition is imposed on the function f in addition to our condition (3). Except for the case of a weakly sequentially complete Banach space, Zaidman was unable to remove this additional condition.

REFERENCES

[1] DINCULEANU, N.: Vector Measures. New York 1967.
[2] KLUVÁNEK, I.: Fourier transforms of vector-valued functions and measures. Studia math., *37*, 1970, 1—12.
[3] TWEDDLE, I.: Weak compactness in locally convex spaces. Glasgow Math. Journal, *9*, 1968, 123—127.
[4] WIDDER, D. V.: The Laplace Transform. Princeton 1946.
[5] ZAIDMAN, S.: Réprésentation des fonctions vectorielles par des intégrales de Laplace-Stieltjes et compacité faible. C. r. Acad. sci., Paris, *248*, 1959, 1915—1917.

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