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ON UNIVERSAL QUASIGROUP IDENTITIES

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Summary. A quasigroup identity is called universal if it is invariant under isotopies. In this paper necessary and sufficient conditions for a quasigroup identity to be universal are found.

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The paper deals with quasigroup identities invariant under isotopies. The terminology is taken from [2], [3] and [4]. Stimulated by geometric illustrations, V. D. Belousov in [2] has presented two important identity properties and posed a question for which identities these properties are necessary and sufficient for the identity to be invariant under isotopies. Inspired by V. D. Belousov, G. Monoszová investigated in [6] one special kind of identities for which both Belousov's properties give necessary and sufficient conditions for the identity to be invariant under isotopies. Our purpose is to amend Belousov's properies to such ones which guarantee the identity invariance under isotopies for general identities. We also show a close connection between quasigroup identities invariant under isotopies and 3-basic quasigroup identities.

1. QUASIGROUPS, ISOTOPIES

Let $(Q, .)$ be a quasigroup and let $\backslash$, $/$, ${}^s(\cdot)$, ${}^s(\backslash)$, ${}^s(/)$ be its parastropic operations. Write all these operations in the form

$$A^k_{ij}(x_i, x_j) = x_k \Leftrightarrow x_1 \cdot x_2 = x_3,$$

where (i,j,k) is an arbitrary permutation of the set $\{1,2,3\}$. In more detail

$$\begin{aligned} A_{12}^3(x,y) &= z \Leftrightarrow x\cdot y = z, & A_{21}^3(y,x) &= z \Leftrightarrow y^s(\cdot)x = z,\\ A_{13}^2(x,z) &= y \Leftrightarrow x\backslash z = y, & A_{31}^2(z,x) &= y \Leftrightarrow z^s(\backslash)x = y,\\ A_{32}^1(z,y) &= x \Leftrightarrow z/y = x, & A_{23}^1(y,z) &= x \Leftrightarrow y^s(/)z = x. \end{aligned}$$

It is clear that $A_{ij}^k(x,y) = A_{ji}^k(y,x)$ for all (i,j,k).

A triple $(\alpha_1,\alpha_2,\alpha_3) = \bar{\alpha}$ of bijective maps $\alpha_i\colon Q \to Q'$, $i \in \{1,2,3\}$, is called an isotopy between quasigroups (Q, A_{12}^3) and (Q', B_{12}^3) if

$$\begin{aligned} &A_{ij}^k(x,y) = \alpha_k^{-1}B_{ij}^k(\alpha_i x, \alpha_j y) \quad \text{for all } x,y \in Q, \text{ or}\\ (1)\qquad &B_{ij}^k(x,y) = \alpha_k A_{ij}^k(\alpha_i^{-1}x, \alpha_j^{-1}y) \quad \text{for all } x,y \in Q'. \end{aligned}$$

2. Universal identities

Let (Q,A) be a quasigroup and let Σ_A be its system of parastropic operations. We define a term on the given quasigroup (Q,A) by:

(i) every individual variable, in short variable, $x \in Q$ is a term,

(ii) if t_1 and t_2 are terms, then $B(t_1,t_2)$ is a term for all $B \in \Sigma_A$.

The subterms of a term are defined by:

(i) the only subterm of an individual variable $x \in Q$ is x,

(ii) the subterms of $B(t_1,t_2)$, $B \in \Sigma_A$, are the term itself and the subterms of t_1 and t_2.

We call t_1 and t_2 the major subterms of the term $B(t_1,t_2)$.

A length of $t = B(t_1,t_2)$, denoted by $l(t)$, is defined by $l(t) = l(B(t_1,t_2)) = l(t_1) + l(t_2)$, where the length of an individual variable is taken as 1. Thus by a length $l(t)$ we mean the number of all individual variables counted with respect to their occourence in t.

Let t_1 and t_2 be two terms on a quasigroup (Q,A) in the variables x_1, x_2, ... Then the identity $w\colon t_1 = t_2$ is said to be satisfied or valid on (Q,A) if for any substitution of elements of Q for $x_1, x_2, \ldots$ in t_1 and t_2, the resulting elements of t_1 and t_2 have the same value in Q.

A length of an identity $w\colon t_1 = t_2$ is defined by $l(w) = l(t_1 = t_2) = l(t_1) + l(t_2)$.

The var(t) means the number of pairwise discint variables of t and the var(w) means the number of pairwise discint variables of both sides of w. We have var$(t) \leqslant l(t)$ for every term t.

A term t is said to be a subterm of an identity $t_1 = t_2$ iff it is a subterm of one of t_1 and t_2 at least.

Let t_n, $n \in \{1,2,\ldots\}$ be a term on any quasigroup. Denote by t_{nA} the term on the quasigroup (Q, A_{12}^3) arising from t_n by replacing of all quasigroup operations of t_n by appurtenant operations A_{ij}^k. Similarly w_A will be an identity on (Q, A_{12}^3) corresponding to the identity w on any quasigroup. Now let (Q', B_{12}^3) be another quasigroup. If $l(t_n) = 2$, then we have

$$t_{nA} = A_{ij}^k(x_1, x_2), \qquad t_{nB} = B_{ij}^k(x_1, x_2)$$

and if $l(t_n) > 2$, then

$$t_{nA} = A_{ij}^k(t_{1A}, t_{2A}), \qquad t_{nB} = B_{ij}^k(t_{1B}, t_{2B}),$$

where at least one of t_1, t_2 has a length $\geqslant 2$.

We say that an identity w is valid on (Q, A_{12}^3) and on (Q', B_{12}^3) if appurtenant w_A is valid on (Q, A_{12}^3) and w_B on (Q', B_{12}^3).

A quasigroup identity w is called universal if the validity of w on any quasigroup (Q, A) implies the validity of w on every quasigroup isotopic to (Q, A).

Example 1. Let

$$w\colon y\backslash(u(y\backslash(ux))) = x$$

be an identity on a quasigroup $(Q, .)$. Using symbols A_{ij}^k instead of $\cdot$, $\backslash$, $/$, we rewrite it as the identity

$$w_A\colon A_{13}^2(y, A_{12}^3(u, A_{13}^2(y, A_{12}^3(u, x)))) = x.$$

Let $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be an isotopy between (Q, A_{12}^3) and (Q, B_{12}^3) such that

$$B_{ij}^k(x, y) = \alpha_k^{-1} A_{ij}^k(\alpha_i x, \alpha_j y). \tag{2}$$

If we use (2), then the transcription

$$w_B\colon B_{13}^2(y, B_{12}^3(u, B_{13}^2(y, B_{12}^3(u, x)))) = x$$

can be expressed on (Q, A_{12}^3) as

$$\bar{\alpha}(w_B)\colon A_{13}^2(\alpha_1 y, A_{12}^3(\alpha_1 u, A_{13}^2(\alpha_1 y, A_{12}^3(\alpha_1 u, \alpha_2 x)))) = \alpha_2 x.$$

It is clear that w_B is valid on (Q, B_{12}^3) iff $\bar{\alpha}(w_B)$ is valid on (Q, A_{12}^3). However, after replacing $\alpha_1 y, \alpha_1 u$ and $\alpha_2 x$ by y, u and x in $\bar{\alpha}(w_B)$ we obtain the identity w_A again. Thus the identities $\bar{\alpha}(w_B)$ and w_A are equivalent, written: $\bar{\alpha}(w_B) \Leftrightarrow w_A$, and consequently the given identity w is universal.

Example 2. Let

$$w\colon (z/y)\backslash x = (x/y)\backslash(z\backslash x),$$

then

$$w_A\colon A_{13}^2(A_{32}^1(z, y), x) = A_{13}^2(A_{32}^1(x, y), A_{13}^2(z, x)).$$

Let $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be an isotopy between (Q, A_{12}^3) and (Q, B_{12}^3) and let w_B be the transcription of w_A as in Example 1. Then using (2) we obtain

$$\bar{\alpha}(w_B)\colon A_{13}^2(A_{32}^1(\alpha_3 z, \alpha_2 y), \alpha_3 x) = A_{13}^2(A_{32}^1(\alpha_3 x, \alpha_2 y), \alpha_3 \alpha_2^{-1} A_{13}^2(\alpha_1 z, \alpha_3 x)).$$

It is clear that $w_A \not\Leftrightarrow \bar{\alpha}(w_B)$ and w is not universal.

Now we shall define a position of individual variables and subterms of a given identity. Let w be an identity on a quasigroup (Q, A_{12}^3) and let $A_{ij}^k(t_1, t_2)$ be a subterm of w. Then we define a position of subterms of $A_{ij}^k(t_1, t_2)$ as follows:

(i) the term t_1 (respectively t_2) is in an inner position i (respectively j) with respect to the operation A_{ij}^k, written: $\mathrm{inpos}_{ij}^k(t_1) = i$, $\mathrm{inpos}_{ij}^k(t_2) = j$;

(ii) the term $A_{ij}^k(t_1, t_2)$ is in an outer position k with respect to A_{ij}^k, written: $\mathrm{outpos}(A_{ij}^k(t_1, t_2)) = k$;

(iii) if $w\colon A_{ij}^k(t_1, t_2) = t_3$, then the term t_3 is also in an outer position k with respect to A_{ij}^k, written: $\mathrm{outpos}_{ij}^k(t_3) = k$.

If the length of any subterm is 1, then we obtain the definition of inner and outer position fof individual variables.

Now we introduce an important property so-called

Position condition (P): Every subterm of a given identity occurs always in the same positon, independently of whether this position is inner or outer.

Returning to our examples, we can observe that every subterm in the identity from Example 1 occurs always in the same position, namely:

$$\mathrm{inpos}_{12}^3(x) = 2 = \mathrm{outpos}_{13}^2(x),$$
$$\mathrm{inpos}_{13}^2(y) = 1, \qquad \mathrm{inpos}_{12}^3(u) = 1,$$
$$\mathrm{outpos}(A_{12}^3(u, x)) = 3 = \mathrm{inpos}_{13}^2(A_{12}^3(u, x)) \text{ and so on.}$$

Thus the identity from Example 1 has the property (P), whereas the identity from Example 2 has not the property (P), namely

$$\mathrm{inpos}_{32}^1(z) = 3 \neq \mathrm{inpos}_{13}^2(z) = 1,$$
$$\mathrm{inpos}_{13}^2(A_{13}^2(z, x)) = 3 \neq \mathrm{outpos}(A_{13}^2(z, x)) = 2.$$

It is clear that an identity having the property (P) consists exactly of the following subterms:

$$A_{ij}^k(t_1, t_2), \tag{3}$$
$$A_{ij}^k(t_1, A_{pq}^j(t_2, t_3)), \quad A_{ij}^k(A_{pq}^i(t_1, t_2), t_3), \tag{4}$$
$$A_{ij}^k(A_{rm}^i(t_1, t_2), A_{pq}^j(t_3, t_4)), \tag{5}$$

where $l(t_n) \geqslant 1$ for all $n \in \{1, 2, 3, 4\}$.

Lemma 1. *Let w be an identity having (P) and let w have a subterm of the type (5). Then among positions r, m, p, q of subterms t_1, t_2, t_3, t_4 there are all three posibilities 1, 2 and 3.*

Proof. As already stated, the triple (a, b, c) of every quasigroup operation A_{ab}^c is always a permutation of the set $\{1, 2, 3\}$. If the positions r, m, p, q, where $r \neq m$ and $p \neq q$, have only two values, then necessarily $i = j$, because remaining third possibilities for i and j in (r, m, i) and (p, q, j) are the same. However this contradicts to the known fact that (i, j, k) must be a permutation of $\{1, 2, 3\}$. □

Consider the known trivial identities in a quasigroup $(Q,.)$ ([5]):

$$(6)\qquad \begin{cases} x\cdot(x\backslash y)=y, & (y/x)\cdot x=y,\\ x\backslash(x\cdot y)=y, & (y\cdot x)/x=y,\\ x/(y\backslash x)=y, & (x/y)\backslash x=y.\end{cases}$$

Using the symbols A_{ij}^{k} instead of $\cdot$, $\backslash$, $/$, these identities can be written by the unique identity

$$(7)\qquad A_{ij}^{k}(x, A_{ik}^{j}(x,y)) = y,$$

from which, using $A_{ij}^{k}(a,b) = A_{ji}^{k}(b,a)$, we obtain all identities (6) and also identities using remaining parastrophic operations ${}^{s}(\cdot)$, ${}^{s}(\backslash)$, ${}^{s}(/)$. The identity (7) can be extended to any terms t_1, t_2 of the length $\geqslant 1$:

$$(8)\qquad A_{ij}^{k}(t_1, A_{ik}^{j}(t_1,t_2)) = t_2.$$

Now take several special cases of identities on (Q, A_{12}^{3}):

(I) If an identity w contains at least one subterm of the type $A_{ij}^{k}(t_1, A_{ik}^{j}(t_1,t_2))$, then it can be replaced by the subterm t_2 and we obtain a new identity w' equivalent to w with $l(w') \leqslant l(w) - 2$.

(II) If $w\colon A_{ij}^{k}(t_1,t_3) = A_{ij}^{k}(t_2,t_3)$, then $w \Leftrightarrow w'$, where $w'\colon t_1 = t_2$ and $l(w') \leqslant l(w) - 2$.

Similarly

(III) if we have two identities

$$\begin{cases} w\colon A_{ij}^{k}(t_1,t_2) = A_{ij}^{k}(t_3,t_4),\\ w_1\colon t_2 = t_4,\end{cases}$$

then we obtain an identity $w'\colon t_1 = t_3$, where $w' \Leftrightarrow w$ and $l(w') \leqslant l(w) - 2$.

(IV) If an identity w contains a subterm $A_{ij}^{k}(x,y)$ two times at least and if at least one of the variables x, y occurs only in this subterm, then subtituting $A_{ij}^{k}(x,y) = z$ we obtain an identity w' equivalent to w with $l(w') \leqslant l(w) - 2$.

An identity w is said to be reducible or a reduction of an identity w' and w' is said to be an expansion of w if one of the cases (I), (II), (III) and (IV) is happened at least.

(V) The definition of a quasigroup implies a mutual equivalence of the identities:

$$A_{ij}^{k}(t_1,t_2) = t_3 \Leftrightarrow t_1 = A_{jk}^{i}(t_2,t_3) \Leftrightarrow t_2 = A_{ik}^{j}(t_1,t_3).$$

Thus the identities

$$w\colon A_{ij}^{k}(t_1,t_2) = A_{pq}^{m}(t_3,t_4),$$
$$w'\colon t_1 = A_{jk}^{i}(t_2, A_{pq}^{m}(t_3,t_4))$$

are equivalent and $l(w) = l(w')$, $\mathrm{var}(w) = \mathrm{var}(w')$.

(VI) Let w be an identity containing a subterm $A^k_{ij}(x,y)$ and at least one of variables x and y in another subterm; choose for example x. Then subtituting $u = A^k_{ij}(x,y)$ and $x = A^i_{jk}(y,u)$ we obtain an identity w' equivalent to w with $\mathrm{var}(w') = \mathrm{var}(w)$, but about $l(w)$ and $l(w')$ we cannot say anything, because this is dependent on a number of occurrences of subterms $A^k_{ij}(x,y)$ and x.

Two identities w and w' is said to be equivalent if we can transform one into the other by a finite sequence of applications of (I), (II), (III), (IV), (V), and (VI).

It is easy to prove the following

Lemma 2. *Let w be a quasigroup identity fulfiling the position condition (P) and let* $\mathrm{var}(w) = 2$. *Then w is equivalent to someone of trivial identities.*

Proof. The positon condition and Lemma 1 imply that w cannot contain a subterm of the type (5), consequently w contains only subterms of the type (4). If we use that $A^k_{ij}(t_1,t_2) = A^k_{ji}(t_2,t_1)$, then we can confine to first of subterms (4). By direct calculation we find that w has only the following subterms

$$A^j_{ik}(x, \underbrace{A^k_{ij}(x, A^j_{ik}(x,y))}_{y}) = A^j_{ik}(x,y),$$

$$A^k_{ij}(x, A^j_{ik}(x, \underbrace{A^k_{ij}(x, A^j_{ik}(x,y))}_{y}))) = y,$$

$$A^i_{kj}(y, A^j_{ik}(x, \underbrace{A^k_{ij}(x, A^j_{ik}(x,y))}_{y}))) = x.$$

After reductions we get one of three trivial possibilities: $x = x$, $y = y$, $A^j_{ik}(x,y) = A^j_{ik}(x,y)$. □

Corollary. *Every no trivial identity w fulfiling (P) has* $\mathrm{var}(w) \geqslant 3$.

An identity w is said to be regular if

(i) w is no trivial,

(ii) w is no reducible,

(iii) w has no isolated variables i.e. every individual variable occurs two times at least.

Now we can conclude

Lemma 3. *Let w be a regular quasigroup identity having the property (P). Then* $l(w) \geqslant 6$ *and* $\mathrm{var}(w) \geqslant 3$.

E x a m p l e 3. By direct calculation we find that all regular identities w_i, $i = 1, 2, \ldots$, having the property (P), $l(w_i) = 6$ and $\mathrm{var}(w_i) = 3$ are the following:

$$
\begin{aligned}
w_1:&\quad A_{ij}^k(x, A_{ik}^j(y,z)) = A_{ij}^k(y, A_{ik}^j(x,z)),\\
w_2:&\quad A_{ik}^j(y,z) = A_{ik}^j(x, A_{ij}^k(y, A_{ik}^j(x,z))),\\
w_3:&\quad y = A_{kj}^i(z, A_{ik}^j(x, A_{ij}^k(y, A_{ik}^j(x,z)))),\\
w_4:&\quad x = A_{jk}^i(A_{ik}^j(y,z), A_{ij}^k(y, A_{ik}^j(x,z))),\\
w_5:&\quad A_{ij}^k(x,y) = A_{ij}^k(A_{jk}^i(y,z), A_{ik}^j(x,z)),\\
w_6:&\quad x = A_{jk}^i(y, A_{ij}^k(A_{jk}^i(y,z), A_{ij}^k(x,z))).
\end{aligned}
$$

It is easy to convince that by (V) we get $w_1 \Leftrightarrow w_2 \Leftrightarrow w_3$, $w_1 \Leftrightarrow w_4$, $w_5 \Leftrightarrow w_6$ and by (VI) we get $w_1 \Leftrightarrow w_5$, where either the substitution $A_{jk}^i(y,z) = u$ is used in w_5 or the substitution $y = A_{ik}^j(u,z)$ in w_1. Thus we have obtained

Lemma 4. *All regular identities having the property (P), $l(w) = 6$ and $\mathrm{var}(w) = 3$ are mutually equivalent.*

From the definition of the identity's regularity we get

Lemma 5. *Let w and w' be two equivalent identities by (V) or (VI) and let w be regular, then w' is also regular and* $\mathrm{var}(w) = \mathrm{var}(w')$.

We come now to the necessary and sufficient conditions for the identity to be universal, i.e. invariant under isotopies.

Let (Q, A_{12}^3) and (Q, B_{12}^3) be two quasigroups and let $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ be an isotopy between them such that

$$B_{ij}^k(x,y) = \alpha_k^{-1} A_{ij}^k(\alpha_i x, \alpha_j y) \text{ for all } x, y \in Q.$$

Let w_A be an identity on (Q, A_{12}^3), w_B its transcription in B_{ij}^k's and $\bar{\alpha}(w_B)$ the image of w_B under $\bar{\alpha}$. We know that w_A is universal if and only if w_A and $\bar{\alpha}(w_B)$ are equivalent on (Q, A_{12}^3).

All subterms of w_A can be divided into three types:

(a) subterms occuring in inner and outer positions,

(b) subterms ocouring only in inner positions,

(c) subterms occuring only in outer positions.

Theorem 1. *Every quasigroup identity having the property (P) is universal.*

P r o o f. Let w_A be the identity on (Q, A_{12}^3) having the property (P).

(a) Let t_n be a subterm of w_A occuring in the same inner and outer position i. If $l(t_n) = 1$, then $t_n = x$ and we have

$$w_A: \quad A_{pq}^i(t_1, t_2) = x,$$

where at least one of the terms t_1, t_2 will have a subterm $A^k_{ij}(x,t_r)$. Then x goes over onto $\alpha_i x$ in $\bar{\alpha}(w_B)$.

If $l(t_n) \geqslant 2$, then $t_n = A^i_{jk}(t_r, t_{r+1})$, $r \neq n \neq r+1$, where $l(t_r) \geqslant 1$, $l(t_{r+1}) \geqslant 1$ and t_n will be a subterm of some subterm t_{n-1} of the type

$$t_{n-1} = A^q_{ip}\,\big(\underbrace{A^i_{jk}(t_r, t_{r+1})}_{t_n}, t_{n+1}\big),$$

where $\mathrm{outpos}^i_{jk}(t_n) = i = \mathrm{inpos}^q_{ip}(t_n)$. Then under $\bar{\alpha}$ we get a subterm

$$\alpha_q t_{n-1} = A^q_{ip}\big(A^i_{jk}(\alpha_j t_r, \alpha_k t_{r+1}), \alpha_p t_{n+1}\big)$$

of $\bar{\alpha}(w_B)$.

(b) It is clear that solely individual variables of w_A can occur only in inner positions. If x is a variable in an inner position i, then x goes onto $\alpha_i x$ in $\bar{\alpha}(w_B)$.

(c) Subterms occuring only in outer positions are just the terms forming the left and right hand sides of w_A. These two terms are equal and therefore they have the same outer position i. Thus

$$\begin{aligned} w_A: &\quad A^i_{jk}(t_1, t_2) = A^i_{jk}(t_3, t_4), \\ \bar{\alpha}(w_B): &\quad A^i_{jk}(\alpha_j t_1, \alpha_k t_2) = A^i_{jk}(\alpha_j t_3, \alpha_k t_4). \end{aligned}$$

Summarizing (a), (b), and (c) we see that $\bar{\alpha}(w_B)$ is of the form w_A, where variables $\alpha_{r_k} x_k$ occur in positions r_k instead of individual variables x_k. Thus w_A and $\bar{\alpha}(w_B)$ are equivalent and therefore w_A is universal. □

Example 4. Choose the following identity on a quasigroup $(Q,.)$:

$$w_1: \quad x\big(u\backslash y(uz)\big) = y\big(x\backslash u(uz)\big).$$

We see immediately that w_1 is reducible. Substituting $uz = v$ we obtain the identity

$$w_2: \quad x(u\backslash yv) = y(x\backslash uv),$$

where $w_1 \Leftrightarrow w_2$. The identity w_1 has not the property (P), whereas w_2 has (P) and then by Theorem 1 we get that w_2 is universal. Now from the equivalence $w_1 \Leftrightarrow w_2$ we obtain that w_1 is also universal.

From this example we can conclude that there are universal identities which have not the property (P). In other words, the property (P) is not invariant under equivalences between identities. From the definition of equivalent identities we obtain

Lemma 6. *Let w_1 be an identity having the property (P) and let w_2 be an identity equivalent to w_1. Then*

(i) w_2 has also the property (P) if w_2 is a reduction of w_1 or if the equivalence

$w_1 \Leftrightarrow w_2$ *is of the types (V) and (VI);*

(ii) w_2 *need not have the property* (P) *if* w_2 *is an expansion of the identity* w_1.

Lemma 7. *Let* w_1 *be a regular identity,* w_2 *an identity such that* $w_1 \Leftrightarrow w_2$ *and* $l(w_1) = l(w_2)$. *Then* w_2 *is also regular and* $\mathrm{var}(w_1) = \mathrm{var}(w_2)$.

Proof. w_2 cannot be a reduction of w_1 because w_1 is regular. Since $l(w_1) = l(w_2)$, it follows that w_2 cannot be an expansion of w_1. Thus the equivalence between w_1 and w_2 can be only of the types (V) and (VI). Now from Lemma 5 we obtain that w_2 is also regular and $\mathrm{var}(w_1) = \mathrm{var}(w_2)$. □

Theorem 1 asserts that the property (P) is a sufficient condition to the identity invariance under isotopies. If we confine to regular identities, then we prove that the property (P) is also a necessary condition to the identity invariance.

Theorem 2. *Every regular universal identity has the property* (P).

Proof. Let w_A be a regular universal identity on a quasigroup (Q, A^3_{12}) and let w_B, $\bar{\alpha}$, $\bar{\alpha}(w_B)$ be as above. Then it is clear that $l(w_A) = l(w_B) = l(\bar{\alpha}(w_B))$ and $w_A \Leftrightarrow \bar{\alpha}(w_B)$ for all isotopies $\bar{\alpha}$. Thus by Lemma 7 we obtain that $\bar{\alpha}(w_B)$ is also regular and $\mathrm{var}(w_A) = \mathrm{var}(\bar{\alpha}(w_B))$.

Now we suppose that w_A have not the property (P). We shall again distinguish three cases:

(a) Let t_n be a subterm of w_A in two different outer positions $r \neq s$. This means that

$$A^r_{ij}(t_1, t_2) = t_n = A^s_{pq}(t_3, t_4).$$

Aplying $\bar{\alpha}$ we get

$$A^r_{ij}(\alpha_i t_1, \alpha_j t_2) = \alpha_r t_n \neq \alpha_s t_n = A^s_{pq}(\alpha_p t_3, \alpha_q t_4)$$

and $w_A \not\Leftrightarrow \bar{\alpha}(w_B)$.

(b) Let t_n be a subterm of w_A in an outer position r and in an inner position s, $r \neq s$.

If $l(t_n) \geqslant 2$, then $t_n = A^r_{ij}(t_m, t_{m+1})$ and t_n is a subterm of some subterm of the type

$$A^q_{sp}\big(\underbrace{A^r_{ij}(t_m, t_{m+1})}_{t_n}, t_{m+2}\big).$$

Applying equivalences of the type (V) we can obtain a regular identity w'_A of the following form

$$A^s_{pq}(t_{m+2}, t_{m+3}) = \underbrace{A^r_{ij}(t_m, t_{m+1})}_{t_n},$$

where $w'_A \Leftrightarrow w_A$. Here t_n is in two different outer positions and under $\bar{\alpha}$ we get

$$\alpha_r t_n \neq \alpha_s t_n.$$

If $l(t_n) = 1$, then $t_n = x$. An individual variable x can be in an outer position only when w_A is of the type

$$A^r_{ij}(t_1, t_2) = x,$$

where at least one of the terms t_1, t_2 must have a subterm $A^q_{sp}(x, t_m)$. Then, under $\bar{\alpha}$, x goes over to $\alpha_s t$ and $\alpha_r x$ with $\alpha_s x \neq \alpha_r x$. This implies the inequality $\operatorname{var}(w_A) < \operatorname{var}(\bar{\alpha}(w_B))$ and $w_A \not\Leftrightarrow \bar{\alpha}(w_B)$.

(c) Let t_n be a subterm of w_A in two different inner positions $r \neq s$.

If $l(t_n) = 1$, then $t_n = x$ and applying $\bar{\alpha}$ we get $\alpha_r x \neq \alpha_s x$. Thus $\operatorname{var}(\bar{\alpha}(w_B)) > \operatorname{var}(w_A)$ and w_A is not universal.

If $l(t_n) \geqslant 2$, then $t_n = A^k_{ij}(t_{n+1}, t_{n+2})$ and t_n must be a subterm of two subterms of w_A

$$A^q_{rp}(A^k_{ij}(t_{n+1}, t_{n+2}), t_{n+3}) \text{ and } A^h_{sm}(A^k_{ij}(t_{n+1}, t_{n+2}), t_{n+4}),$$

where $r \neq s$. If $k = r$, then $k \neq s$ and if $k = s$, then $k \neq r$. This implies that t_n is also in different outer and inner positions. However, we know, by (b), that $w_A \not\Leftrightarrow \bar{\alpha}(w_B)$.

This completes the proof of Theorem 2. □

Corollary. *Any regular universal identity cannot contain a square $A^k_{ij}(x, x)$.*

Let w be an arbitrary identity on (Q, A^3_{12}) and let σ be a permutation of the set $\{1, 2, 3\}$. Now we define $\sigma(w)$ as an identity arising from w by replacing A^k_{ij} by $A^{\sigma k}_{\sigma i, \sigma j}$, where $\sigma(i, j, k) = (\sigma i, \sigma j, \sigma k)$.

Theorem 3. *Let w be a regular universal identity on (Q, A^3_{12}). Then $\sigma(w)$ is also universal for all permutations σ of the set $\{1, 2, 3\}$.*

Proof. Since w is a regular universal identity, w has the property (P). This means that every subterm t_n of w, $l(t_n) \geqslant 1$, occurs only in one position r_n, which under the permutation σ passes to the unique position $\sigma(r_n)$. Thus $\sigma(w)$ has also the property (P) and $\sigma(w)$ is universal. □

3. 3-BASIC QUASIGROUPS

The quadruple $(Q_1, Q_2, Q_3; A^3_{12})$, where Q_1, Q_2, Q_3 are non-void sets with the same cardinality and A^3_{12} is a map of $Q_1 \times Q_2$ onto Q_3, is called a 3-basic quasigroup if in the equation $A^3_{12}(a_1, a_2) = a_3$ any two of the elements $a_1 \in Q_1$, $a_2 \in Q_2$, $a_3 \in Q_3$ uniquely determine the remaining one ([3]). Similarly as in the classical case we define the parastrophic operations

$$A^k_{ij}(a_i, a_j) = a_k \Leftrightarrow A^3_{12}(a_1, a_2) = a_3$$

for all permutations (i, j, k) of the set $\{1,2,3\}$. If $Q_1 = Q_2 = Q_3 = Q$ we get a usual quasigroup.

If in a usual quasigroup (Q, B^3_{12}) we take any word consisting of any number of elements of Q and formed successively by parastrophic operations B^k_{ij}'s, then there always exists an element of Q equal to this word. However, for a 3-basic quasigroup $(Q_1, Q_2, Q_3; A^3_{12})$ we must form words over it carefully respecting the single places belonging to various Q_i's.

In what follows let $\mathsf{A} = (Q_1, Q_2, Q_3; A^3_{12})$ be a 3-basic quasigroup, A^k_{ij}'s its paratrophic operations, a_i, b_i, c_i, ... elements of Q_i and x_i, y_i, ... variables over Q_i, $i \in \{1,2,3\}$.

It is easy to show that only terms of the following form are possible: $A^k_{ij}(x_i, x_j)$, $A^k_{ij}(A^i_{pq}(x_p, x_q), y_j)$,

$$A^k_{pq}(A^p_{ij}(x_i, x_j), A^q_{sr}(y_s, y_r))$$

and so on.

If we put $A^3_{12}(x_1, x_2) = x_3$, then we obtain

$$A^k_{ij}(A^i_{pq}(x_p, x_q), x_j) = A^k_{ij}(x_i, x_j) = x_k,$$
$$A^k_{pq}(A^p_{ij}(x_i, x_j), A^q_{sr}(x_s, x_r)) = A^k_{pq}(x_p, x_q) = x_k.$$

An isotopy between 3-basic quasigroups $\mathsf{A} = (Q_1, Q_2, Q_3; A^3_{12})$ and $\mathsf{C} = (Q'_1, Q'_2, Q'_3; C^3_{12})$ is defined as a triple $(\alpha_1, \alpha_2, \alpha_3)$ of bijections $\alpha_i \colon Q_i \to Q'_i$, $i \in \{1,2,3\}$, such that

$$A^k_{ij}(x_i, x_j) = \alpha_k^{-1} C^k_{ij}(\alpha_i x_i, \alpha_j x_j) \text{ for all } x_i \in Q_i,\ x_j \in Q_j,$$

or,

$$C^k_{ij}(x_i, x_j) = \alpha_k A^k_{ij}(\alpha_i^{-1} x_i, \alpha_j^{-1} x_j) \text{ for all } x_i \in Q'_i, x_j \in Q'_j.$$

Observe that every term over A has the property (P) automatically as a direct consequence of the definition of a 3-basic quasigroup. Thus we can conclude

Theorem 4. *Any identity on a 3-basic quasigroup is invariant under isotopies (briefly, is universal, provided we transfer this notation also to 3-basic quasigroup in the natural way).*

Now let $\mathsf{A} = (Q_1, Q_2, Q_3; A_{12}^3)$ be a 3-basic quasigroup and $\mathsf{B} = (Q, Q, Q; B_{12}^3)$ a usual quasigroup with $\operatorname{card} Q = \operatorname{card} Q_i$, $i \in \{1,2,3\}$. Then the isotopy $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ between A and B with bijections $\alpha_i \colon Q_i \to Q$, $i \in \{1,2,3\}$ is a special case of the 3-basic quasigroup isotopy

$$A_{ij}^k(x_i, x_j) = \alpha_k^{-1} B_{ij}^k(\alpha_i x_i, \alpha_j x_j) \text{ for all } x_i \in Q_i, x_j \in Q_j,$$

or,

$$B_{ij}^k(x, y) = \alpha_k A_{ij}^k(\alpha_i^{-1} x, \alpha_j^{-1} y) \text{ for all } x, y \in Q.$$

Now we investigate the connection between the corresponding identities on A and on B.

Let w_B be an identity on B, w_A an expression arising from w_B be substitutions between subterms:

$$t_{nA} = A_{ij}^k(x_1, x_2) := B_{ij}^k(x_1, x_2) = t_{nB}, \text{ where } l(x_1) = l(x_2) = 1$$

and

$$t_{nA} = A_{ij}^k(t_{1A}, t_{2A}) := B_{ij}^k(t_{1B,2B}) = t_{nB},$$

where at least one of t_{1A}, t_{2A} has $l(t_{sA}) \geqslant 2$, $s \in \{1,2\}$. Every subterm t_{nB} of w_B with $l(t_{nB}) \geqslant 3$ has the form

$$t_{nB} = B_{ij}^k\big(B_{pq}^m(t_{1B}, t_{2B}), t_{3B}\big)$$

and the corresponding t_{nA} is

$$t_{nA} = A_{ij}^k\big(A_{pq}^m(t_{1A}, t_{2A}), t_{3A}\big).$$

If here $m \neq i$, then t_{nA} has no meaning on A and therefore w_A is not a 3-basic quasigroup identity. We are interested in conditions under which w_B is universal. Similarly as in a usual quasigroup we define a trivial identity, a reducible identity and a regular identity on a 3-basic quasigroup.

Theorem 5. *Let w_B and w_A be as above. Then the following statements are held:*

(i) if w_A is a 3-basic quasigroup identity, then w_B is universal,

(ii) w_B is a regular universal identity iff w_A is a regular 3-basic quasigroup identity.

Proof. By the construction w_A and w_B have the same formal writing.

(i) Since every identity w_A an 3-basic quasigroup has always the property (P), thus the corresponding w_B has also the property (P) and by Theorem 1, we get that w_B is universal.

(ii) If w_B is a regular universal identity, then w_B has the property (P) and this implies that the corresponding w_A has also (P) and w_A is a regular 3-basic quasigroup identity.

If w_A is a regular 3-basic quasigroup identity, then w_B is universal by (i) and by the construction regular. □

References

[1] *V. D. Belousov*: Foundations of the theory of quasigroups and loops, Moskva, 1967. (In Russian.)

[2] *V. D. Belousov*: Algebraic nets and quasigroups, Kishinev, 1971. (In Russian.)

[3] *E. Brožíková*: On 3-basic quasigroups and their congruences, Časopis pro pěst. matematiky *115*, no. 1 (1990), 38–47.

[4] *T. Evans*: The word problem for abstract algebras, J. London Math. Soc. *26* (1951), 64–71.

[5] *T. Evans*: On multiplicative systems defined by generators and relations, Proc. Cambridge Philos. Soc. *47* (1951), 637–649.

[6] *G. Monoszová*: Some quasigroup identities invariant under quasigroup isotopies, Matem. issledovanija, Kishinev no. 102 (1988), 80–91. (In Russian.)

Souhrn

O UNIVERZÁLNOSTI KVAZIGRUPOVÝCH IDENTIT

ELENA BROŽÍKOVÁ

Kvazigrupová identita se nazývá univerzální, jestliže je invariantní vzhledem k izotopiím. V tomto článku jsou nalezeny nutné a postačující podmínky k tomu, aby kvazigrupová identita byla univerzální.

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