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ON SOME CONDITIONS WHICH IMPLY THE CONTINUITY OF ALMOST ALL SECTIONS $x \to f(t,x)$

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Summary. Let I be an open interval, X a topological space and Y a metric space. Some local conditions implying continuity and quasicontinuity of almost all sections $x \to f(t,x)$ of a function $f\colon I \times X \to Y$ are shown.

Keywords: measure, density, category, Baire property, continuity, section

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Let $\mathbf{R}$ be the set of reals and let μ (resp. μ^*) be the Lebesgue measure (resp. the outer Lebesgue measure) in $\mathbf{R}$. The upper outer density $d_{u,e}(A,x)$ of a set $A \subset \mathbf{R}$ at a point $x \in \mathbf{R}$ is defined as $\limsup\limits_{h\to 0} \mu^*(A \cap [x-h, x+h])/2h$. If the set A is measurable (in the Lebesgue sense) then upper outer density of A at x is called the upper density of A at x and it is denoted as $d_u(A,x)$. The corresponding lower limits are called lower outer density and lower density of A at x and denoted by $d_{l,e}(A,x)$ and $d_l(A,x)$ respectively. The family of all measurable sets $A \subset \mathbf{R}$ such that if $x \in A$ then $d_l(A,x) = 1$ is a topology called the density topology $\mathcal{T}_d$ [1, 5]. Moreover, the family $\mathcal{T}_{ae}$ of all sets $A \in \mathcal{T}_d$ such that $\mu(A - \operatorname{int} A) = 0$ is a topology [5] ($\operatorname{int} A$ denotes the Euclidean interior of A). Let $I \subset \mathbf{R}$ be an open interval, let $(X, \mathcal{T})$ be a topological space, and let (Y, ϱ) be a metric space. In [2] the following condition (a_0) is introduced for a function $f\colon I \times X \to Y$:

(a_0) f satisfies (a_0) if for every point $(t,x) \in I \times X$ there is a measurable set $A(t,x) \subset I$ such that $d_l(A(t,x),t) = 1$ and the sections $f_s(x) = f(s,x)$, $s \in A(t,x)$, are $\mathcal{T}$-equicontinuous at x, i.e. for every $\varepsilon > 0$ there is a set $U \in \mathcal{T}$ such that $x \in U$ and $f_s(U) \subset K(f_s(x), \varepsilon) = \{u \in Y;\ \varrho(f(s,x),u) < \varepsilon\}$ for every $s \in A(t,x)$.

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In [2] this condition is used to investigate Carathéodory's superposition $h(t) = f(t, g(t))$ and it is proved that if $X = Y$ is a separable Banach space and if f satisfies the condition (a_0) then almost all sections f_t are $\mathcal{T}$-continuous. Moreover, if f is a bounded function and all its sections $f^x(t) = f(t, x)$ are derivatives then all sections f_t are continuous. In this article I examine some analogous conditions as (a_0).

A function $f: I \times X \to Y$ satisfies the condition:

(a_1) if for every point $(t, x) \in I \times X$ there is a measurable set $A(t, x) \subset I$ such that $d_u(A(t, x), t) > 0$ and the sections f_s, $s \in A(t, x)$, are $\mathcal{T}$-equicontinuous at x;

(a_2) if for every point (t, x) there is a measurable set $A(t, x) \subset I$ such that $d_u(A(t, x), t) > 0$ and the sections f_s, $s \in A(t, x)$, are $\mathcal{T}$-continuous at x;

(a_3) if for every point (t, x) there is a measurable set $A(t, x) \subset I$ such that $d_u(A(t, x), t) > 0$ and the sections f_s, $s \in A(t, x)$, are $\mathcal{T}$-quasi-equicontinuous at x, i.e. for every $\varepsilon > 0$ and for every $\mathcal{T}$-open set $U \ni x$ there is a nonempty $\mathcal{T}$-open set $V \subset U$ such that $f_s(V) \subset K(f(s, x), \varepsilon)$ for every $s \in A(t, x)$;

(b_1) if for every point (t, x) there is a set $A(t, x) \subset I$ having the Baire property and of the second category at t such that the sections f_s, $s \in A(t, x)$, are $\mathcal{T}$-equicontinuous at x;

(b_2) if for every point (t, x) there is a set $A(t, x) \subset I$ having the Baire property and of the second category at t such that the sections f_s, $s \in A(t, x)$, are $\mathcal{T}$-continuous at x;

(b_3) if for every point (t, x) there is a set $A(t, x) \subset I$ having the Baire property and of the second category at x such that the sections f_s, $s \in A(t, x)$, are $\mathcal{T}$-quasi-equicontinuous at x.

Theorem 1. *Suppose that $(X, \mathcal{T})$ is a topological space having a countable basis of open sets. If the function $f: I \times X \to Y$ satisfies the condition (a_1) then there is a set $Z \subset I$ of measure zero such that all sections f_t, $t \in I - Z$, are $\mathcal{T}$-continuous.*

Proof. Assume that the set $B = \{t \in I;\ f_t$ is not continuous at some point $x(t) \in X\}$ is of positive outer measure. Then there are a set $C \subset B$ of positive outer measure and a positive number s such that for every $t \in C$ the oscillation $\operatorname{osc} f_t(x(t)) = \inf\{\sup\{\varrho(f(t, u), f(t, v));\ u, v \in U\};\ U \in \mathcal{T}, x(t) \in U\} > s$. Let $U_1, \ldots, U_n, \ldots$ be an enumeration of all open sets of a basis of the topology $\mathcal{T}$ and let $C_n = \{t \in C;\ x(t) \in U_n\}$ and $D_n = \{t \in C_n;\ d_{l,e}(C_n, t) < 1\}$, $n = 1, 2, \ldots$. Evidently, $\mu(D_n) = 0$ for every $n = 1, 2, \ldots$. Let $D = C - (D_1 \cup D_2 \cup \ldots)$. Then $\mu(C - D) = 0$ and $D \subset C$ is a set of positive outer measure. Let $t \in D$ be a point such that $d_{l,e}(D, t) = 1$. Since f satisfies the condition (a_1), there is a measurable set $A(t,x(t)) \subset I$ such that $d_u(A(t,x(t)),t) > 0$ and the sections f_r, $r \in A(t,x(t))$, are equicontinuous at $x(t)$. Consequently, there is an integer n such that $x(t) \in U_n$ and $\operatorname{osc} f_r < \frac{1}{2}s$ on U_n for every $r \in A(t,x(t))$. Since $t \in D = C - (D_1 \cup D_2 \cup \ldots) = (C - D_1) \cap (C - D_2) \cap \ldots$, we have $d_{l,e}(\{r \in C; x(r) \in U_n\},t) = 1$. Observe that the set $E = A(t,x(t)) \cap \{r \in C; x(r) \in U_n\} \neq \emptyset$. If $p \in E$ then $x(p) \in U_n$ and $\operatorname{osc} f_p(x_p) > s$, in a contradiction with the fact that $\operatorname{osc} f_p < \frac{1}{2}s$ on U_n. This completes the proof. □

Theorem 2. *Suppose that a topological space $(X,\mathcal{T})$ has a countable basis of open sets. If the function $f\colon I \times X \to Y$ satisfies the condition (a_3) then there is a set $Z \subset I$ of measure zero such that all sections f_t, $t \in I - Z$, are $\mathcal{T}$-quasicontinuous, i.e. for every $\varepsilon > 0$, for every $x \in X$ and for every set $U \in \mathcal{T}$ with $x \in U$ there is a nonempty set $V \subset U$ such that $V \in \mathcal{T}$ and $f_t(V) \subset K(f(t,x),\varepsilon)$* [6].

Proof. Let $U_1,\ldots,U_n,\ldots$ be an enumeration of all open sets of a basis in X. Assume that the set $B = \{t \in I;\ f_t$ is not $\mathcal{T}$-quasicontinuous at some point $x(t) \in X\}$ is of positive outer measure. Consequently, there are a positive number s and a set U_k such that the set $C = \{t \in B;\ x(t) \in U_k$ and $\operatorname{osc} f_t > s$ on $V \cup \{x(t)\}$ for every nonempty set $V \in \mathcal{T}$ such that $V \subset U\}$ is of positive outer measure. For $n = 1, 2, \ldots$, let $C_n = \{t \in C; x(t) \in U_n\}$, $D_n = \{t \in C_n; d_{l,e}(C_n,t) < 1\}$, and $D = C - (D_1 \cup D_2 \cup \ldots)$. Evidently, $D \subset C$ is of positive outer measure. Let $t \in D$ be such that $d_{l,e}(D,t) = 1$. Since f satisfies the condition (a_3) there are a measurable set $A(t,x(t))$ and a set $U_n \subset U_k$ such that $d_l\big(A(t,x(t)),t\big) = 1$ and $\operatorname{osc} f_r < \frac{1}{2}s$ on $U_n \cup \{x(t)\}$ for every $r \in A(t,x(t))$. Observe that $d_{l,e}(C_n,t) = 1$. So, $A(t,x(t)) \cap C_n \neq \emptyset$. If $p \in A(t,x(t)) \cap C_n$ then $x(p) \in U_n \subset U_k$ and $\operatorname{osc} f_p < \frac{1}{2}s$ on U_n, in a contradiction with the fact that $\operatorname{osc} f_p > s$ on $V \cup \{x(p)\}$ for every nonempty set $V \in \mathcal{T}$ such that $V \subset U_k$. This contradiction completes the proof. □

Theorem 3. *Suppose that $(X,\mathcal{T})$ is a topological space having a countable basis of open sets. If $f\colon I \times X \to Y$ satisfies the condition (b_1) then there is a set $Z \subset I$ of the first category such that all sections f_t, $t \in I - Z$, are $\mathcal{T}$-continuous.*

Proof. Assume that the set $B = \{t \in I;\ f_t$ is not continuous at some point $x(t) \in X\}$ is of the second category. Then there are a set $C \subset B$ of the second category and a positive number s such that $\operatorname{osc} f_t(x(t)) > s$ for each $t \in C$. Let $U_1, \ldots, U_n, \ldots$ be an enumeration of all open sets of a basis in $(X,\mathcal{T})$ and let $C_n = \{t \in C; x(t) \in U_n\}$, and $D_n = \{t \in C_n;\ C_n$ is of the first category at $t\}$, $n = 1, 2, \ldots$. Every set D_n, $n = 1,2,\ldots$, is of the first category. Put $D = C - (D_1 \cup D_2 \cup \ldots)$. Let $t \in D$ be a point. There is an open interval $J \subset I$ such that $t \in J$ and every set $K \subset J - D$ having the Baire property is of the first category. Since f satisfies the condition (b_1), there is a set $A(t, x(t)) \subset J$ having the Baire property and of the second category at t and such that all sections f_r, $r \in A(t, x(t))$, are $\mathcal{T}$-equicontinuous at $x(t)$. Consequently, there is an integer n such that $x(t) \in U_n$ and for every $r \in A(t, x(t))$ we have $\operatorname{osc} f_r < \frac{1}{2}s$ on U_n. Since $t \in D = C - (D_1 \cup D_2 \cup \ldots)$, there is an open interval $L \subset J$ such that $t \in L$ and every set $K \subset L - \{r \in C;\, x(r) \in U_n\}$ with the Baire property is of the first category. So the set $E = A(t, x(t)) \cap \{r \in C \cap L;\, x(r) \in U_n\}$ is nonempty. If $p \in E$ then $x(p) \in U_n$ and $\operatorname{osc} f_p(x(p)) > s$, in a contradiction with the fact $\operatorname{osc} f_p < \frac{1}{2}s$ on U_n. This contradiction finishes the proof. □

Theorem 4. *Suppose that in a topological space $(X, \mathcal{T})$ there is a countable basis of open sets. If a function $f\colon I \times X \to Y$ satisfies the condition (b_3) then there is a set $Z \subset I$ of the first category such that all sections f_t, $t \in I - Z$, are $\mathcal{T}$-quasicontinuous.*

P r o o f. Assume that the set $B = \{t \in I;\, f_t$ is not $\mathcal{T}$-quasicontinuous at some point $x(t)\}$ is of the second category. Then there are a positive number s and a nonempty set $U \in \mathcal{T}$ such that the set $C = \{t \in B;\, x(t) \in U$ and $\operatorname{osc} f_t > s$ on $V \cup \{x(t)\}$ for every nonempty set $V \subset U$ such that $V \in \mathcal{T}\}$ is of the second category. Let $U_1, \ldots, U_n, \ldots$ be an enumeration of all open sets of a basis of the space $(X, \mathcal{T})$. For $n = 1, 2, \ldots$, put $C_n = \{t \in C;\, x(t) \in U_n\}$, $D_n = \{t \in C_n;\, C_n$ is of the first category at $t\}$, and $D = C - (D_1 \cup D_2 \cup \ldots)$. Since every set D_n is of the first category, the set $D \subset C$ is of the second category. Let $t \in D$ be a point. There is an open interval $J \subset I$ such that $t \in J$ and every set $K \subset J - D$ having the Baire property is of the first category. Since f satisfies the condition (b_3), there are a set $A(t, x(t)) \subset J$ having the Baire property and of the second category at t and a set $U_n \subset U$ such that $\operatorname{osc} f_r < \frac{1}{2}s$ on $U_n \cup \{x(t)\}$ for every $r \in A(t, x(t))$. Since $t \in D$, there is an open interval $L \subset J$ such that $t \in L$ and every set $K \subset L - \{r \in C;\, x(r) \in U_n\}$ with the Baire property is of the first category. Thus, the set $E = A(t, x(t)) \cap \{r \in C \cap L;\, x(r) \in U_n\}$ is nonempty. If $p \in E$ then $x(p) \in U_n$ and $\operatorname{osc} f_p < \frac{1}{2}s$ on U_n, in a contradiction with the fact that $\operatorname{osc} f_p > s$ on $V \cup \{x(p)\}$ for every nonempty set $V \subset U$ such that $V \in \mathcal{T}$. This contradiction completes the proof. □

R e m a r k 1. The Continuum Hypothesis CH implies that there is a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ satisfying the conditions (a_2), (b_2) (with respect to the Euclidean metric in $\mathbf{R} = X = Y$) and such that all its sections f_t are not quasicontinuous. Really, there is a nonmeasurable set $D \subset \mathbf{R}^2$ which has not the Baire property and which is such that all its sections $D_t = \{x \in \mathbf{R};\, (t, x) \in D\}$ are singletons or contain two points. The construction of such set D is analogous to the construction of Sierpinski's set in [7]. Then the function $f(t,x) = 1$ for $(t,x) \in D$ and $f(t,x) = 0$ otherwise satisfies the conditions (a_2), (b_2), but all its sections f_t are not quasicontinuous.

R e m a r k 2. Observe that all sections f_t of the function f from Remark 1 are almost everywhere (with respect to the Lebesgue measure) continuous. CH implies that there exists a function $g\colon \mathbf{R}^2 \to R$ satisfying the conditions $(a_2), (b_2)$ such that all its sections g_t are not quasicontinuous at all points of sets of positive measure. Really, let $a_1, \ldots, a_\alpha, \ldots, \alpha < \Omega$, be a transfinite sequence of all reals such that $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$ ($\alpha, \beta < \Omega$ and Ω denotes the first uncountable ordinal number). For every $\alpha < \Omega$ there is a nowhere dense closed set A_α of positive measure such that a_β is not in A_α for $\beta < \alpha$. Let $g(t,x) = 1$ for $t = a_\alpha$ and $x \in A_\alpha$, $\alpha < \Omega$, and $g(t,x) = 0$ otherwise. Then g satisfies the conditions (a_2), (b_2) and any section g_t is not quasicontinuous at a point $x \in A_\alpha$, where α is such that $t = a_\alpha$.

R e m a r k 3. Suppose that $X = Y = \mathbf{R}$ and consider X with the topology $\mathscr{T}_{ae}$ and Y with the Euclidean metric. There is a function $f\colon \mathbf{R}^2 \to R$ satisfying the conditions (a_1), (b_1) (with respect to the topology $\mathscr{T}_{ae}$ in X) and such that any section f_t, $t \in \mathbf{R}$, is not $\mathscr{T}_d$-continuous. Really, let $C \subset \mathbf{R}$ be a Cantor set of measure zero and let $g\colon \mathbf{R} \to C$ be an one-to-one function. Put $f(t,x) = 1$ if $t \in \mathbf{R}$ and $x = g(t)$ and $f(t,x) = 0$ otherwise. Since $f/(\mathbf{R}^2 - (\mathbf{R} \times C)) = 0$, for every $(t,x) \in \mathbf{R}^2$ we can take the set $\mathbf{R} - \{t\}$ as $A(t,x)$. So, f satisfies the conditions (a_1), (b_1), but any section f_t, $t \in \mathbf{R}$, is not $\mathscr{T}_d$-continuous at the point $g(t)$.

In connection with Remarks 1, 2, 3 we will prove the following:

Theorem 5. *Let $J \subset \mathbf{R}$ be an open interval and let $\mathscr{T}$ be a topology in J such that every set $Z \in \mathscr{T}$ is measurable and if $x \in Z$ then $d_u(Z,x) > 0$. Then for every function $f\colon I \times J \to Y$ satisfying the condition (a_1) there is a set $U \subset I$ of measure zero such that for every $t \in I - U$ the section f_t is almost everywhere (with respect to the Lebesgue measure) $\mathscr{T}$-continuous.*

P r o o f. We may assume that I and J are of finite measure. Assume that Theorem 5 does not hold. Then there are a set $B \subset I$ of positive outer measure and a positive number s such that for every $t \in B$ the set $C(t) = \{x \in J\,;\ \operatorname{osc} f_t(x) > s\}$ is of positive outer measure. Observe that the set $D = \bigcup_{t \in B} (\{t\} \times C(t))$ is of positive outer measure in $I \times J$. Let Φ_1 be the family of all sets $K \times L$ such that $K \subset I$ is a measurable set of positive measure and $L \in \mathscr{T}$ is a nonempty set such that $\operatorname{osc} f_t < \frac{1}{2}s$ on L for every $t \in K$. Since f satisfies the condition (a_1), the family Φ_1 is nonempty. Let $s_1 = \sup\{\mu_2(K \times L)\,;\ K \times L \in \Phi_1\}$, where μ_2 denotes the Lebesgue measure in $\mathbf{R}^2$. Evidently, $0 < s_1 \leqslant \mu_2(I \times J)$. Let $K_1 \times L_1 \in \Phi_1$ be such that $\mu_2(K_1 \times L_1) > \frac{1}{2}s_1$. If $\mu_2((I \times J) - (K_1 \times L_1)) > 0$ then we denote by

Φ_2 the family of all sets $(K \times L) \in \Phi_1$ such that $\mu_2((K \times L) - (K_1 \times L_1)) > 0$. The family Φ_2 is nonempty. Really, for this let $E \subset (I \times J) - (I_1 \times J_1)$ be an F_σ set such that $\mu_2((I \times J) - (K_1 \times L_1) - E) = 0$ and for every $(t, x) \in E$ we have $d_l(E_t, x) = 1$, $d_l(E^x, t) = 1$ $(E^x = \{r \in I; (r, x) \in E\})$ [3]. Let $(t, x) \in E$ be a point. Since f satisfies the condition (a_1), there is a measurable set $A(t, x) \subset I$ and a nonempty set $J(t, x) \in \mathcal{T}$ such that $x \in J(t, x)$, $\operatorname{osc} f_r < \frac{1}{2}s$ on $J(t, x)$ for every $r \in A(t, x)$ and $d_u(A(t, x), t) > 0$. Observe that $\mu(J(t, x) \cap E_r) > 0$ for every $r \in A(t, x) \cap E^x$. So, $A(t, x) \times J(t, x) \in \Phi_2$ and the family Φ_2 is nonempty. Let $s_2 = \sup\{\mu_2((K \times L) - (K_1 \times L_1)); (K \times L) \in \Phi_2\}$. Obviously, $0 < s_2$. Let $K_2 \times L_2 \in \Phi_2$ be such that $\mu_2((K_2 \times L_2) - (K_1 \times L_1)) > \frac{1}{2}s_2$. In general, for $n > 2$, if $\mu_2((I \times J) - ((K_1 \times L_1) \cup \ldots \cup (K_{n-1} \times L_{n-1})) > 0$ we find a set $K_n \times L_n \in \Phi_1$ such that

$$\text{(i)} \qquad \mu_2\Big((K_n \times L_n) - \bigcup_{i<n}(K_i \times L_i)\Big) > \tfrac{1}{2}s_n,$$

where $s_n = \sup\big\{\mu_2\big((K \times L) - \bigcup_{i<n}(K_i \times L_i)\big);\ K \times L \in \Phi_1\big\}$. Since $\mu_2(I \times J) < \infty$, $\lim_{n\to\infty} s_n = 0$. From this and from (i) it follows that $\mu_2((I \times J) - \bigcup_n(K_n \times L_n)) = 0$. Since D is of positive outer measure, there are an integer n and a point $(t, x) \in D \cap (K_n \times L_n)$. Consequently, $\operatorname{osc} f_t < \frac{1}{2}s$ on L_n, in a contradiction with the fact that $x \in C(t)$ and $\operatorname{osc} f_t(x) > s$. This contradiction finishes the proof. □

Evidently, the Euclidean topology $\mathcal{T}_e$ in $\mathbf{R}$ and the topology $\mathcal{T}_d$ and the topology $\mathcal{T}_{ae}$ satisfy the hypothesis of Theorem 5.

Problem 1. Let $(J, \mathcal{T})$ be the same as in Theorem 5 and let $f: I \times J \to Y$ satisfies the condition (b_1). If a set $U \subset I$ of the first category and such that for every $t \in I - U$ the section f_t is almost everywhere $\mathcal{T}$-continuous?

Theorem 6. *If $X = Y = \mathbf{R}$ and $\mathcal{T} = \mathcal{T}_d$ $[\mathcal{T} = \mathcal{T}_{ae}]$ and a function $f: I \times \mathbf{R} \to R$ satisfies the condition (a_3) $[(b_3)]$ and all its sections $f^x(t) = f(t, x)$ are measurable [have the Baire property] then f is measurable [has the Baire property] as the function of two variables.*

Proof. For the proof of this theorem see the proofs of Theorems 2 and 4 from [4]. □

Remark 5. In [2] it is proven that if Y is a separable Banach space and a bounded function $f: I \times Y \to Y$ satisfies the condition (a_0) and all its sections f^x are derivatives then all sections f_t are continuous. (f^x is a derivative if for every $t \in I$, $\lim_{h\to 0}(1/h)\int_t^{t+h} f(s)\,\mathrm{ds} = f(t, x)$). Obviously, it is also true for locally bounded

f. We shall show that there is a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ satisfying the condition (a_0) and such that all its sections f^x are derivatives and the section $x \mapsto f(0,x)$ is not continuous. For this, let $a_n = 1/n$, $b_n = a_n - 4^{-n}$, $c_n = a_n + 4^{-n}$, $d_n = 1/n - 1/(n+1)$ and let g_n $(n = 1, 2, \ldots)$ be defined as follows: $g_n(t) = d_k 4^k$ for $t = a_k$, $k > n$, $g_n(t) = 0$ for $t \geqslant c_n$ or $t \in [c_{k+1}, b_k]$, $k \geqslant n$, $g_n(0) = 1$, g_n is linear in the intervals $[b_k, a_k]$ and $[a_k, c_k]$, and $g_n(t) = g_n(-t)$ for $t < 0$. Then the function $f(t,x) = g_n(x)g_n(t)\min(|x - b_n|, |x - c_n|)$ for $x \in [b_n, c_n]$, $n = 1, 2, \ldots$, and $f(t,x) = 0$ otherwise, satisfies required conditions.

In connection with Remark 5 we have also:

Remark 6. Let $X = Y = \mathbf{R}$ and $\mathcal{T} = \mathcal{T}_e$. There is a bounded function $f\colon \mathbf{R}^2 \to \mathbf{R}$ satisfying the condition (a_1), having derivatives as its sections f^x, $x \in \mathbf{R}$, and such that its section $x \mapsto f(0,x)$ is discontinuous. For this, let $a_n = 1/n$, $b_n = \frac{1}{2}(a_{n+1} + a_n)$, $c_n = b_n + 10^{-n}$, $d_n = a_n - 10^{-n}$ and let g_n, $n = 1, 2, \ldots$, be defined as follows: $g_n(t) = 1$ for $t \in [a_{k+1}, b_k]$, $k \geqslant n$, $g_n(t) = 0$ for $t \in [c_k, d_k]$, $k \geqslant n$, or $t \geqslant a_1$, g_n is linear in the intervals $[b_k, c_k]$ and $[d_k, a_k]$, $k \geqslant n$, $g_n(0) = \frac{1}{2}$ and $g_n(t) = g_n(-t)$ for $t < 0$. Then the function $f(t,x) = g_n(t)g_n(x)\min(|x + 4^{-n} - a_n|, |a_n + 4^{-n} - x|)$ for $x \in [a_n - 4^{-n}, a_n + 4^{-n}]$, $n = 1, 2, \ldots$, and $f(t,x) = 0$ otherwise, satisfies all required conditions.

Theorem 7. *Let $J \subset \mathbf{R}$ be an open interval, $\mathcal{T} = \mathcal{T}_e$ and let (Y, ϱ) be a metric space. If a function $f\colon I \times J \to Y$ satisfies the condition (a_1) and all its sections f^x are $\mathcal{T}_d$-continuous then all sections f_t, $t \in \mathbf{R}$, are $\mathcal{T}_e$-continuous.*

Proof. If Theorem 7 does not hold then there are $t \in I$, $x \in J$ and $s > 0$ such that $\operatorname{osc} f_t(x) > 5s$. Consequently, there is a sequence of points $x_n \in J$ such that $\lim\limits_{n\to\infty} x_n = x$ and $\varrho(f(t,x_n), f(t,x)) > 2s$ for $n = 1, 2, \ldots$. Since f satisfies the condition (a_1) there are a measurable set $A(t,x) \subset I$ and an open set $K \subset J$ such that $d_u(A(t,x), t) > 0$, $x \in K$ and $\operatorname{osc} f_t < \frac{1}{2}s$ on K for each $t \in A(t,x)$. Let $x_n \in K$. Since the sections $t \mapsto f(t, x_n)$ and $t \mapsto f(t,x)$ are $\mathcal{T}_d$-continuous, there is a measurable set $B \subset I$ such that $d_l(B,t) = 1$, $\varrho(f(r,x_n), f(t,x_n)) < \frac{1}{2}s$, and $\varrho(f(r,x), f(t,x)) < \frac{1}{2}s$ for each $r \in B$. Evidently, $B \cap A(t,x) \neq \emptyset$. Let $p \in B \cap A(t,x)$. Then $2s < \varrho(f(t,x_n), f(t,x)) \leqslant \varrho(f(t,x_n), f(p,x_n)) + \varrho(f(p,x_n), f(p,x)) + \varrho(f(p,x), f(t,x)) < \frac{1}{2}s + \frac{1}{2}s + \frac{1}{2}s = \frac{3}{2}s$. This contradiction completes the proof. □

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