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BOCHNER PRODUCT INTEGRATION

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Summary. A new definition of the product integral is given. The definition is based on a procedure which is analogous to the sum definition of the Bochner integral given by J. Kurzweil and E.J. McShane. The new definition is shown to be equivalent to the seemingly very different one given by J.D. Dollard and C.N. Friedman in [1] and [2].

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The concept of product integration goes back to V. Volterra [18]. In 1931 L. Schlesinger published the paper [14] where the case of product integration based on exponential factors of the form $\mathrm{e}^{A(t)\,\mathrm{d}t}$ with an $n \times n$ matrix $A(t)$ with Lebesgue integrable entries is extensively studied. The case of Bochner integrable operator valued functions instead Schlesinger's matrix case is treated in the paper [15] of G. Schmidt and also in the known monograph [1] of J. D. Dollard and C. N. Friedman. The product integral for this case is defined via the L^1-approximations of a Bochner integrable bounded operator valued function by step-functions. Here we give an alternative definition using the concept of gauge integration which was created by J. Kurzweil, R. Henstock and E. J. McShane for the case of ordinary integrals. We show that this concept is equivalent to the concept given in [1]. In [1] also an excellent bibliography on the problem is given.

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Let an interval $[a, b] \subset \mathbf{R}$, $-\infty < a < b < +\infty$ be given. A pair (τ, J) of a point $\tau \in \mathbf{R}$ and a compact interval $J \subset \mathbf{R}$ is called a *tagged interval*, τ is the *tag* of J.

A finite collection $\{(\tau_j, J_j),\ j = 1, \ldots, k\}$ of tagged intervals is called an *L-partition* of $[a, b]$ if

$$\mathrm{Int}(J_i) \cap \mathrm{Int}(J_j) = \emptyset \text{ for } i \neq j$$

and

$$\bigcup_{j=1}^{k} J_j = [a, b].$$

($\mathrm{Int}(J)$ denotes the interior of an interval J.)

An L-partition $\{(\tau_j, J_j),\ j = 1, \ldots, k\}$ for which

$$\tau_j \in J_j,\ j = 1, \ldots, k$$

is called a P-*partition* of $[a, b]$.

Clearly every P-partition of $[a, b]$ is also an L-partition of $[a, b]$.

Sometimes it is useful to denote

$$J_i = [\alpha_{i-1}, \alpha_i], \quad i = 1, \ldots, k$$

for a given L-partition of $[a, b]$, where

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b.$$

In other words we will assume in the sequel that the partition $\{(\tau_i, J_i),\ i = 1, \ldots, k\}$ is ordered in such a way that

$$\sup J_i = \inf J_{i+1}, \quad i = 1, \ldots, k-1.$$

Given a positive function $\delta\colon [a, b] \to (0, +\infty)$ called a *gauge* on $[a, b]$, a tagged interval (τ, J) with $\tau \in [a, b]$ is said to be *δ-fine* if

$$J \subset [\tau - \delta(\tau), \tau + \delta(\tau)].$$

Using this concept we can speak about *δ-fine L-partitions* and *δ-fine P-partitions* $\{(\tau_j, J_j),\ j = 1, \ldots, k\}$ of the interval $[a, b]$ whenever (τ_j, J_j) is δ-fine for every $j = 1, \ldots, k$.

It is a well-known fact that given a gauge $\delta\colon [a, b] \to (0, +\infty)$ there exists a δ-fine P-partition of $[a, b]$.

This result is called *Cousin's lemma*, see e.g. [11, Theorem on p. 119].

Assume that Y is a real Banach space with the norm $\|\cdot\|_Y = \|\cdot\|$.

Let us consider a function $f\colon [a, b] \to Y$ and assume that μ is the Lebesgue measure on the real line.

Definition 1. Denote by $L([a,b];Y)$ the set of functions $f\colon [a,b]\to Y$ for which to every $\varepsilon>0$ there is a gauge δ on $[a,b]$ such that

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\|f(t_i)-f(s_j)\|_Y\mu(J_i\cap L_j)<\varepsilon \tag{1}$$

for every δ-fine L-partitions $\{(t_i,J_i),\, i=1,\dots,k\}$ and $\{(s_j,L_j),\, j=1,\dots,l\}$ of $[a,b]$.

The set $L([a,b];Y)$ is introduced in [10, Chap. 14]. The notation in [10] is different from ours.

Proposition 2. *To every $f\in L([a,b];Y)$ there is an element $S_f\in Y$ such that for every $\varepsilon>0$ there exists a gauge δ on $[a,b]$ such that*

$$\left\|\sum_{i=1}^{k}f(t_i)\mu(J_i)-S_f\right\|_Y<\varepsilon \tag{2}$$

for every δ-fine L-partition $\{(t_i,J_i),\, i=1,\dots,k\}$ of $[a,b]$.

Proof. Let $f\in L([a,b];Y)$, for a given $\varepsilon>0$ let δ be the gauge which corresponds to $\frac{\varepsilon}{2}>0$ by Definition 1. Then by this definition we have

$$\begin{aligned}
&\left\|\sum_{i=1}^{k}f(t_i)\mu(J_i)-\sum_{j=1}^{l}f(s_j)\mu(L_j)\right\|_Y\\
&\qquad=\left\|\sum_{i=1}^{k}\sum_{j=1}^{l}f(t_i)\mu(J_i\cap L_j)-\sum_{i=1}^{k}\sum_{j=1}^{l}f(s_j)\mu(J_i\cap L_j)\right\|_Y\\
&\qquad\leqslant\sum_{i=1}^{k}\sum_{j=1}^{l}\|f(t_i)-f(s_j)\|_Y\mu(J_i\cap L_j)<\frac{\varepsilon}{2}.
\end{aligned}\tag{3}$$

for any two δ-fine partitions $\{(t_i,J_i),\, i=1,\dots,k\}$ and $\{(s_j,L_j),\, j=1,\dots,l\}$ of $[a,b]$.

Denote by $S(\varepsilon)\subset Y$ the set of all integral sums

$$S(f,D)=\sum_{i=1}^{k}f(t_i)\mu(J_i)\in Y$$

where $D=\{(t_i,J_i),\, i=1,\dots,k\}$ is an arbitrary δ-fine L-partition of $[a,b]$, i.e.

$$S(\varepsilon)=\{S(f,D)\in Y;\ D\text{ is a }\delta\text{-fine L-partition of }[a,b]\}.$$

Since by Cousin's Lemma the set of δ-fine L-partitions of $[a,b]$ is nonempty, we have $S(\varepsilon) \neq \emptyset$ and clearly also $S(\eta) \subset S(\varepsilon)$ provided $\eta < \varepsilon$. By (3) we get $\operatorname{diam} S(\varepsilon) < \frac{\varepsilon}{2}$, $\operatorname{diam} M$ being the diameter of a set $M \subset Y$. Therefore the intersection $\bigcap_{\varepsilon>0} \overline{S(\varepsilon)}$ of the closures $\overline{S(\varepsilon)}$ of all sets $S(\varepsilon)$ consists of one point

$$S_f = \bigcap_{\varepsilon>0} \overline{S(\varepsilon)} \in Y$$

because Y is complete and therefore

$$\Big\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - S_f \Big\|_Y \leqslant \frac{\varepsilon}{2},$$

i.e. (2) is satisfied. From (3) it is also clear that there is exactly one $S_f \in Y$ satisfying (2). □

Definition 3. The element $S_f \in Y$ given by Proposition 2 for a given function $f \in L([a,b];Y)$ is called the *S-integral* of f over $[a,b]$ and we use the notation $S_f = (S)\int_a^b f(t)\,\mathrm{d}t$.

In [10, 14.7] the following interesting result is shown.

Theorem 4. *A function $f\colon [a,b] \to Y$ is Bochner integrable if and only if $f \in L([a,b];Y)$ and in this case we have*

$$(B)\int_a^b f(t)\,\mathrm{d}t = (S)\int_a^b f(t)\,\mathrm{d}t,$$

where $(B)\int_a^b f(t)\,\mathrm{d}t$ denotes the Bochner integral of f over $[a,b]$.

In the sequel we use the notation $\int_a^b f(t)\,\mathrm{d}t$ instead of $(S)\int_a^b f(t)\,\mathrm{d}t$. For the notion of the Bochner integral see e.g. [3], [7], [20].

Theorem 4 shows that the set $L([a,b];Y)$ of Y-valued functions defined on $[a,b]$ coincides with the set of Bochner integrable functions and Proposition 2 yields the fact that if the Bochner integral $(B)\int_a^b f(t)\,dt$ exists, then it can be approximated by Riemann type integral sums of the form

$$\sum_{i=1}^{k} f(t_i)\mu(J_i).$$

Remark 5. It is well-known that in the case $Y = \mathbb{R}$ the Bochner integral of a function $f\colon [a,b] \to \mathbb{R}$ coincides with the Lebesgue integral. Therefore Definition 1 and Theorem 4 give also a characterization of Lebesgue integrable real functions. More precisely $g\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable over $[a,b]$ if and only if for every $\varepsilon > 0$, there is a gauge δ on $[a,b]$ such that

$$\sum_{i=1}^{k}\sum_{j=1}^{l} |g(t_i) - g(s_j)|\mu(J_i \cap L_j) < \varepsilon$$

for every δ-fine L-partitions $\{(t_i, J_i),\ i = 1,\dots,k\}$ and $\{(s_j, L_j),\ j = 1,\dots,l\}$ of $[a,b]$. See again [10] for more details.

Proposition 6. *If $f \in L([a,b];Y)$ then $\|f\|\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and*

$$\left\| \int_a^b f(t)\,\mathrm{d}t \right\| \leqslant \int_a^b \|f(t)\|\,\mathrm{d}t. \tag{4}$$

Proof. Let to a given $\varepsilon > 0$ the gauge δ be given by Definition 1. Given δ-fine L-partitions $\{(t_i, J_i),\ i = 1,\dots,k\}$ and $\{(s_j, L_j),\ j = 1,\dots,l\}$ of $[a,b]$ we have

$$\big|\|f(t_i)\| - \|f(s_j)\|\big| \leqslant \|f(t_i) - f(s_j)\|$$

for every $i = 1,\dots,k$, $j = 1,\dots,l$. Hence

$$\begin{aligned}\sum_{i=1}^{k}\sum_{j=1}^{l} &\big|\|f(t_i)\| - \|f(s_j)\|\big|\mu(J_i \cap L_j)\\ &\leqslant \sum_{i=1}^{k}\sum_{j=1}^{l} \|f(t_i) - f(s_j)\|\mu(J_i \cap L_j) < \varepsilon\end{aligned}$$

and by Theorem 4 this inequality immediately yields the Lebesgue integrability of $\|f(t)\|$ over $[a,b]$.

To show the inequality (4) assume that $\varepsilon > 0$ is arbitrary. Let δ be a gauge on $[a,b]$ such that—by Proposition 2—

$$\left\| \sum_{i=1}^{k} f(t_i)\mu(J_i) - \int_a^b f(t)\,\mathrm{d}t \right\| < \varepsilon$$

and

$$\left| \sum_{i=1}^{k} \|f(t_i)\|\mu(J_i) - \int_a^b \|f(t)\|\,\mathrm{d}t \right| < \varepsilon$$

for every δ-fine L-partition $\{(t_i, J_i),\ i = 1,\dots,k\}$ of $[a,b]$.

Then we have

$$\begin{aligned}\left\|\int_a^b f(t)\,\mathrm{d}t\right\| &\leqslant \left\|\int_a^b f(t)\,\mathrm{d}t - \sum_{i=1}^k f(t_i)\mu(J_i)\right\| + \left\|\sum_{i=1}^k f(t_i)\mu(J_i)\right\| \\ &< \varepsilon + \sum_{i=1}^k \|f(t_i)\|\mu(J_i) \\ &\leqslant \varepsilon + \left|\sum_{i=1}^k \|f(t_i)\|\mu(J_i) - \int_a^b \|f(t)\|\,\mathrm{d}t\right| + \int_a^b \|f(t)\|\,\mathrm{d}t \\ &< 2\varepsilon + \int_a^b \|f(t)\|\,\mathrm{d}t\end{aligned}$$

and therefore (4) holds because $\varepsilon > 0$ can be taken arbitrarily small. □

Remark. The result given in Proposition 6 is well known for the Bochner integral (see e.g. [7, Theorem 3.7.6]).

The Bochner product integral

Assume now that X is a real Banach space. Denote by $\mathcal{B}(X)$ the Banach space of bounded linear operators on X with the usual operator norm given by

$$\|A\| = \|A\|_{\mathcal{B}(X)} = \sup_{\|x\|=1} \|Ax\|_X$$

for $A \in \mathcal{B}(X)$. The identity operator in $\mathcal{B}(X)$ will be denoted by I.

Let $\mathfrak{J}$ be the set of all compact subintervals in $[a,b]$. Assume that a $\mathcal{B}(X)$-valued point-interval function $V\colon [a,b]\times\mathfrak{J} \to \mathcal{B}(X)$ is given.

For a given L-partition $D = \{(t_i, J_i),\ i = 1,\dots,k\}$ of $[a,b]$ define

$$P(V,D) = V(\tau_k, J_k)V(\tau_{k-1}, J_{k-1})\dots V(\tau_1, J_1)$$

the *ordered* product of elements of $\mathcal{B}(X)$.

Definition 7. A function $V\colon [a,b]\times\mathfrak{J} \to \mathcal{B}(X)$ is called *Bochner product integrable* if there exists $Q \in \mathcal{B}(X)$ such that for every $\varepsilon > 0$ there is a gauge δ: $[a,b] \to (0,+\infty)$ on $[a,b]$ such that

$$\|P(V,D) - Q\| < \varepsilon \tag{5}$$

for every δ-fine L-partition $D = \{(t_i, J_i),\ i = 1,\dots,k\}$ of $[a,b]$.

$Q \in \mathcal{B}(X)$ is called *the Bochner product integral* of V over $[a, b]$ and we use the notation $Q = \prod_a^b V(t, \mathrm{d}t) \in \mathcal{B}(X)$.

Remark 8. A similar concept of product integration was introduced by J. Jarník and J. Kurzweil in [9] (see also [16], [17]) for the case of $n \times n$-matrix valued point-interval functions V when instead of L-partitions in the Definition 7 P-partitions are used. The corresponding product integral is called the *Perron product integral* in [9].

This terminology originates in the well known fact that a real function $g\colon [a, b] \to \mathbb{R}$ is Perron integrable to the value $\int_a^b g(t)\,\mathrm{d}t \in \mathbb{R}$ if and only if to every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that

$$\left| \sum_{i=1}^{k} g(t_i)\mu(J_i) - \int_a^b g(t)\,\mathrm{d}t \right| < \varepsilon$$

for every δ-fine P-partition $D = \{(t_i, J_i),\ i = 1, \ldots, k\}$ of $[a, b]$.

Proposition 9. *Let $V\colon [a, b] \times \mathfrak{J} \to \mathcal{B}(X)$ be given. Then V is Bochner product integrable if and only if for every $\varepsilon > 0$ there is a gauge δ on $[a, b]$ such that*

$$\|P(V, D_1) - P(V, D_2)\| < \varepsilon \tag{6}$$

for every δ-fine L-partitions D_1, D_2 of $[a, b]$.

Proof. If V is Bochner product integrable then the condition (6) is clearly satisfied (see (5) in Definition 7).

Assume that (6) holds. Let $\delta_n\colon [a, b] \to (0, +\infty)$ be the gauge on $[a, b]$ which corresponds to $\varepsilon = \frac{1}{n}$, $n = 1, 2, \ldots$ by (6) and assume that $\delta_{n+1}(t) \leqslant \delta_n(t)$ for every $t \in [a, b]$ and $n = 1, 2, \ldots$.

Denote

$$P_n = \{P(V, D) \in \mathcal{B}(X);\ D \text{ is an } \delta_n\text{-fine L-partition}\}.$$

Clearly $P_{n+1} \subset P_n$ for every n by the choice of δ_n and also

$$\operatorname{diam} P_n = \sup\{\|A - B\|;\ A, B \in P_n\} \leqslant \frac{1}{n}.$$

Since the space $\mathcal{B}(X)$ is complete, the intersection $\bigcap_{n=1}^{\infty} \overline{P_n}$ consists of exactly one point $Q \in \mathcal{B}(X)$ ($\overline{P_n}$ is the closure of the set P_n in $\mathcal{B}(X)$) and $\|P(V, D) - Q\| \leqslant \frac{1}{n}$ for every δ_n-fine L-partition D of $[a, b]$. This proves the statement. □

The following result holds.

Theorem 10. *Let $V\colon [a,b]\times\mathfrak{J}\to\mathcal{B}(X)$ be Bochner product integrable over $[a,b]$, let $\prod_a^b V(t,\mathrm{d}t) = Q \in \mathcal{B}(X)$ where $Q\in\mathcal{B}(X)$ be an invertible operator. Assume that V satisfies the following*

Condition (C_0). *There exist $B<\infty$ and a gauge $\widehat{\delta}\colon [a,b]\to(0,+\infty)$ such that $V(t,J)\in\mathcal{B}(X)$ is invertible for every $\widehat{\delta}$-fine tagged interval (t,J) and*

$$\max(\|V(t,J)\|, \|(V(t,J))^{-1}\|) \leqslant B. \tag{7}$$

Then for every $s\in[a,b]$ the Bochner product integrals $\prod_a^s V(t,\mathrm{d}t)$, $\prod_s^b V(t,\mathrm{d}t)$ exist; the equality

$$\prod_s^b V(t,\mathrm{d}t)\prod_a^s V(t,\mathrm{d}t) = \prod_a^b V(t,\mathrm{d}t) \tag{8}$$

holds and there exists a constant $M>0$ such that

$$\Big\|\prod_a^s V(t,\mathrm{d}t)\Big\| \leqslant M,\quad \Big\|\Big(\prod_a^s V(t,\mathrm{d}t)\Big)^{-1}\Big\| \leqslant M,$$
$$\Big\|\prod_s^b V(t,\mathrm{d}t)\Big\| \leqslant M,\quad \Big\|\Big(\prod_s^b V(t,\mathrm{d}t)\Big)^{-1}\Big\| \leqslant M$$

for all $s\in[a,b]$.

Remark. Let us introduce the following condition concerning the point-interval function $V\colon [a,b]\times\mathfrak{J}\to\mathcal{B}(X)$.

Condition (C). *Let there exists $r\in(0,1)$ such that for every $t\in[a,b]$ one can find $\sigma=\sigma(t)>0$ such that*

$$\|V(t,J)-I\| < r \tag{9}$$

for any interval $J\subset[a,b]\cap(t-\sigma,t+\sigma)$.

If the condition (C) is satisfied for $V\colon [a,b]\times\mathfrak{J}\to\mathcal{B}(X)$ then the condition (C_0) holds for V.

Indeed, let $\widehat{\delta}$ be a gauge on $[a,b]$ such that $J\subset(t-\sigma(t),t+\sigma(t))$ for all $\widehat{\delta}$-fine tagged intervals (t,J).

If the tagged interval (τ, J) is $\widehat{\delta}$-fine then by (9) the inverse $[V(\tau, J)]^{-1}$ exists and

$$\|[V(\tau,J)]^{-1}\| = \left\| \sum_{k=0}^{\infty} (I - V(\tau,J))^k \right\|$$
$$\leqslant \sum_{k=0}^{\infty} \|I - V(\tau,J)\|^k < \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} < \infty.$$

Moreover

$$\|V(\tau,J)\| \leqslant \|V(\tau,J) - I\| + \|I\| < 1 + r.$$

Typical cases of V satisfying condition (C) are for example

$$V_1(t,J) = I + A(t)\mu(J)$$

or

$$V_2(t,J) = \mathrm{e}^{A(t)\mu(J)}$$

where $A\colon [a,b] \to \mathcal{B}(X)$, μ being any non-atomic Borel measure on $[a,b]$ (e.g. the Lebesgue measure on $[a,b]$.)

Proof of Theorem 10. Let $\delta_0\colon [a,b] \to (0,\infty)$ be a gauge on $[a,b]$ such that

$$\|P(V,D) - Q\| < \frac{1}{2}\|Q^{-1}\|^{-1} \tag{10}$$

holds for every δ_0-fine L-partition D of the interval $[a,b]$. Assume further that $\delta_0 \leqslant \widehat{\delta}$ on $[a,b]$, $\widehat{\delta}$ being the gauge from the condition (C_0).

The proof of the theorem will be divided into several steps. First we prove the following assertion.

For every $\tau \in [a,b]$ there is a $K_1(\tau) > 0$ such that if $s \in (\tau - \delta_0(\tau), \tau] \cap [a,b]$ and D_1 is a δ_0-fine L-partition of $[a,s]$ then

$$\max\{\|P(V,D_1)\|, \|(P(V,D_1))^{-1}\|\} \leqslant K_1(\tau), \tag{11}$$

and

if $s \in (\tau, \tau + \delta_0(\tau)] \cap [a,b]$ and D_2 is a δ_0-fine L-partition of $[s,b]$ then

$$\max\{\|P(V,D_2)\|, \|(P(V,D_2))^{-1}\|\} \leqslant K_1(\tau). \tag{12}$$

In order to prove e.g. the estimate (11) let D_3 be an arbitrary fixed δ_0-fine L-partition of the interval $[\tau, b]$. Let

$$D_1 = \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{l-1}, \tau_l, \alpha_l\} = \{(\tau_j, [\alpha_{j-1}, \alpha_j]), j = 1, \dots, l\}$$

be a δ_0-fine L-partition of $[a, s]$ and let D_3 has the form

$$D_3 = \{\alpha_{l+1}, \tau_{l+2}, \alpha_{l+2}, \dots, \alpha_{k-1}, \tau_k, \alpha_k\} = \{(\tau_j, [\alpha_{j-1}, \alpha_j]), j = l+2, \dots k\}.$$

Set

$$\begin{aligned} D &= \{\alpha_0, \tau_1, \alpha_1, \dots, \alpha_{l-1}, \tau_l, \alpha_l = s, \tau_{l+1} = \alpha_{l+1} = \tau, \tau_{l+2}, \alpha_{l+2}, \dots, \alpha_{k-1}, \tau_k, \alpha_k\} \\ &= \{(\tau, [\alpha_{j-1}, \alpha_j]), j = 1, \dots, l\} \cup (\tau, [s, \tau]) \cup \{(\tau, [\alpha_{j-1}, \alpha_j]), j = l+2, \dots k\}. \end{aligned}$$

We use the notation $D = D_1 \circ (\tau, [s, \tau]) \circ D_3$ for this construction of a partition of the interval $[a, b]$. It is evident that D is a δ_0-fine partition of $[a, b]$ and that $V(\tau_i, [\alpha_{i-1}, \alpha_i]) \in \mathcal{B}(X)$ is invertible for every $i = 1, \dots, k$. Therefore

$$P(V, D_1) = V(\tau_l, [\alpha_{l-1}, \alpha_l]) V(\tau_{l-1}, [\alpha_{l-2}, \alpha_{l-1}]) \dots V(\tau_1, [\alpha_0, \alpha_1]) \in \mathcal{B}(X)$$

and

$$P(V, D_3) = V(\tau_k, [\alpha_{k-1}, \alpha_k]) V(\tau_{k-1}, [\alpha_{k-2}, \alpha_{k-1}]) \dots V(\tau_{l+2}, [\alpha_{l+1}, \alpha_{l+2}]) \in \mathcal{B}(X)$$

are invertible and also the inequality (10) holds where by definition we have

$$P(V, D) = P(V, D_3) V(\tau, [s, \tau]) P(V, D_1)$$

and by (10), (C_0) we obtain

$$\begin{aligned} &\|P(V, D_1) - (V(\tau, [s, \tau]))^{-1} (P(V, D_3))^{-1} Q\| \\ &= \|(V(\tau, [s, \tau]))^{-1} (P(V, D_3))^{-1} [P(V, D_3) V(\tau, [s, \tau]) P(V, D_1) - Q]\| \\ &\leqslant B \|(P(V, D_3))^{-1}\| \cdot \frac{1}{2} \|Q^{-1}\|^{-1}. \end{aligned}$$

Consequently, using again (C_0) we get

$$\begin{aligned} \|P(V, D_1)\| &\leqslant \|P(V, D_1) - (V(\tau, [s, \tau]))^{-1} (P(V, D_3))^{-1} Q\| \\ &\quad + \|(V(\tau, [s, \tau]))^{-1} (P(V, D_3))^{-1} Q\| \\ &\leqslant \frac{B}{2} \|Q^{-1}\|^{-1} \|(P(V, D_3))^{-1}\| + \|(V(\tau, [s, \tau]))^{-1}\| \, \|(P(V, D_3))^{-1}\| \, \|Q\| \\ &\leqslant \Big(\frac{B}{2} \|Q^{-1}\|^{-1} + B\|Q\|\Big) \|(P(V, D_3))^{-1}\| = K_0(\tau). \end{aligned} \tag{13}$$

On the other hand, we have

$$\begin{aligned}
&\|(P(V,D_1))^{-1} - Q^{-1}P(V,D_3)V(\tau,[s,\tau])\| \\
&= \|Q^{-1}[Q - P(V,D_3)V(\tau,[s,\tau])P(V,D_1)](P(V,D_1))^{-1}\| \\
&\leqslant \|Q^{-1}\|\,\|Q - P(V,D)\|\,\|(P(V,D_1))^{-1}\| \\
&\leqslant \|Q^{-1}\|\cdot\frac{1}{2}\|Q^{-1}\|^{-1}\|(P(V,D_1))^{-1}\| = \frac{1}{2}\|(P(V,D_1))^{-1}\|
\end{aligned}$$

and by (C_0) also

$$\begin{aligned}
\|(P(V,D_1))^{-1}\| &\leqslant \|(P(V,D_1))^{-1} - Q^{-1}P(V,D_3)V(\tau,[s,\tau])\| \\
&\quad + \|Q^{-1}\|\,\|P(V,D_3)\|\,\|V(\tau,[s,\tau])\| \\
&\leqslant \frac{1}{2}\|(P(V,D_1))^{-1}\| + B\|Q^{-1}\|\,\|P(V,D_3)\|
\end{aligned}$$

i.e. we obtain the inequality

$$\|(P(V,D_1))^{-1}\| \leqslant 2B\|Q^{-1}\|\,\|P(V,D_3)\| = K^0(\tau) > 0. \tag{14}$$

Taking $K_-(\tau) = \max(K_0(\tau), K^0(\tau)) > 0$ we conclude by (13) and (14) that

$$\max\{\|P(V,D_1)\|, \|(P(V,D_1))^{-1}\|\} \leqslant K_-(\tau)$$

holds. Analogously we can show also that if $s \in [\tau, \tau + \delta_0(\tau)) \cap [a,b]$ and D_2 is a δ_0-fine L-partition of the interval $[s,b]$ then

$$\max\{\|P(V,D_2)\|, \|(P(V,D_2))^{-1}\|\} \leqslant K_+(\tau)$$

where $K_+(\tau) > 0$. Putting $K_1(\tau) = \max(K_-(\tau), K_+(\tau))$ we obtain (11) and (12).

Now we show that the following holds.

For every $\tau \in [a,b]$ there is a $K_2(\tau) > 0$ such that

$$\max\{\|P(V,D_1)\|, \|(P(V,D_1))^{-1}\|, \|P(V,D_2)\|, \|(P(V,D_2))^{-1}\|\} \leqslant K_2(\tau) \tag{15}$$

if $s \in (\tau - \delta_0(\tau), \tau + \delta_0(\tau)) \cap [a,b]$ and D_1, D_2 are arbitrary δ_0-fine L-partitions of $[a,s]$, $[s,b]$, respectively.

Let us take for example $s \in [\tau, \tau + \delta_0(\tau))$ and set $D = D_1 \circ D_2$. Then evidently

$$P(V,D) = P(V,D_2)P(V,D_1)$$

and $P(V, D_1), P(V, D_2) \in \mathcal{B}(X)$ are invertible because every factor in these products is invertible. Since (7) is assumed we get

$$\|P(V, D_2)P(V, D_1) - Q\| < \frac{1}{2}\|Q^{-1}\|^{-1}$$

and

$$\begin{aligned}\|P(V, D_1) - (P(V, D_2))^{-1}Q\| &= \|(P(V, D_2))^{-1}(P(V, D_2)P(V, D_1) - Q)\| \\ &\leqslant \frac{1}{2}\,\|(P(V, D_2))^{-1}\|\,\|Q^{-1}\|^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}\|P(V, D_1)\| &\leqslant \|P(V, D_1) - (P(V, D_2))^{-1}Q\| + \|(P(V, D_2))^{-1}\|\,\|Q\| \\ &\leqslant \|(P(V, D_2))^{-1}\|\Big(\frac{1}{2}\,\|Q^{-1}\|^{-1} + \|Q\|\Big). \qquad (16)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\|(P(V, D_1))^{-1} - Q^{-1}P(V, D_2)\| = \|Q^{-1}(Q - P(V, D_2)P(V, D_1))(P(V, D_1))^{-1}\| \\ &\leqslant \|Q^{-1}\|\,\|Q - P(V, D_2)P(V, D_1)\|\,\|(P(V, D_1))^{-1}\| < \frac{1}{2}\|(P(V, D_1))^{-1}\|.\end{aligned}$$

Hence

$$\begin{aligned}\|(P(V, D_1))^{-1}\| &\leqslant \|(P(V, D_1))^{-1} - Q^{-1}P(V, D_2)\| + \|Q^{-1}\|\,\|P(V, D_2)\| \\ &\leqslant \frac{1}{2}\,\|(P(V, D_1))^{-1}\| + \|Q^{-1}\|\,\|P(V, D_2)\|\end{aligned}$$

and finally

$$\|(P(V, D_1))^{-1}\| \leqslant 2\|Q^{-1}\|\,\|P(V, D_2)\|. \qquad (17)$$

Since $s \in [\tau, \tau + \delta_0(\tau)]$ we can use (12) for $P(V, D_2)$ and by (16) and (17) we obtain the estimate

$$\begin{aligned}&\max\{\|P(V, D_1)\|, \|(P(V, D_1))^{-1}\|\} \\ &\leqslant K_1(\tau)\Big[2\|Q^{-1}\| + \frac{1}{2}\|Q^{-1}\|^{-1} + \|Q\|\Big] = K_L(\tau) > 0.\end{aligned}$$

If $s < \tau$ then in a similar way it can be proved that

$$\max\{\|P(V, D_2)\|, \|(P(V, D_2))^{-1}\|\} \leqslant K_R(\tau)$$

where $K_R(\tau) > 0$. Putting now $K_2(\tau) = \max\{K_L(\tau), K_R(\tau)\}$ we obtain (15).

Intervals of the form $(\tau-\delta_0(\tau), \tau+\delta_0(\tau))$ with $\tau \in [a,b]$ represent an open covering of the compact interval $[a,b]$. Therefore there is a finite set $\{t_1, \dots, t_l\} \subset [a,b]$ such that

$$[a,b] \subset \bigcup_{j=1}^{l} (t_j - \delta_0(t_j), t_j + \delta_0(t_j)).$$

Define $K = \max\{1, K_2(t_1), K_2(t_2), \dots, K_2(t_l)\}$ where $K_2(\tau)$ is given by (15). Then the estimate (15) implies the following statement.

There exists a constant $K \geqslant 1$ such that

$$\max\{\|P(V,D_1)\|, \|(P(V,D_1))^{-1}\|\} \leqslant K \tag{18}$$

if $s \in (a,b]$ and D_1 is an arbitrary δ_0-fine L-partition of $[a,s]$ and

$$\max\{\|P(V,D_2)\|, \|(P(V,D_2))^{-1}\|\} \leqslant K \tag{19}$$

if $s \in [a,b)$ and D_2, is an arbitrary δ_0-fine L-partition of $[s,b]$.

Now we prove the following statement.

Assume that $\varepsilon > 0$ is given and let δ be a gauge on $[a,b]$ such that $\delta(\tau) \leqslant \delta_0(\tau)$ for $\tau \in [a,b]$ and

$$\|P(V,D) - Q\| < \varepsilon$$

for every δ-fine L-partition D of $[a,b]$.

If $s \in (a,b]$ and D_1, D_3 are arbitrary δ-fine L-partitions of $[a,s]$, then

$$\|P(V,D_1) - P(V,D_3)\| \leqslant 2K\varepsilon. \tag{20}$$

If $s \in [a,b)$ and D_2, D_4 are arbitrary δ-fine L-partitions of $[s,b]$, then

$$\|P(V,D_2) - P(V,D_4)\| \leqslant 2K\varepsilon. \tag{21}$$

K is the constant from (18) *and* (19).

Let us prove (21) only; the proof of (20) is similar. Assume that $s \in [a,b)$. Denote by D_1 an arbitrary δ-fine L-partition of the interval $[a,s]$ and let us put $D_5 = D_1 \circ D_2$ and $D_6 = D_1 \circ D_4$. Evidently D_5 and D_6 are δ-fine L-partitions of the interval $[a,b]$. Hence

$$\begin{aligned} &\|P(V,D_2)P(V,D_1) - P(V,D_4)P(V,D_1)\| \\ &\leqslant \|P(V,D_5) - Q\| + \|P(V,D_6) - Q\| \leqslant 2\varepsilon \end{aligned}$$

and

$$\|P(V,D_2)-P(V,D_4)\| = \|[P(V,D_2)P(V,D_1)-P(V,D_4)P(V,D_1)](P(V,D_1))^{-1}\|$$
$$\leqslant \|P(V,D_2)P(V,D_1)-P(V,D_4)P(V,D_1)\|\,\|(P(V,D_1))^{-1}\| \leqslant 2K\varepsilon$$

by (18). This yields (21).

Using (20), (21) and Proposition 9 we have the following result.

If $s \in (a,b)$ then there exist $Q^-, Q^+ \in \mathcal{B}(X)$ such that for every $\varepsilon > 0$ there is a gauge $\delta_1\colon [a,b] \to (0,+\infty)$ on the interval $[a,b]$ such that

$$\|P(V,D_1) - Q^-\| < \varepsilon \tag{22}$$

for every δ_1-fine L-partition D_1 of $[a,s]$ and

$$\|P(V,D_2) - Q^+\| < \varepsilon \tag{23}$$

for every δ_1-fine L-partition D_2 of $[s,b]$.

This means that the product integrals $\prod_s^b V(t,\mathrm{d}t) = Q^+$, $\prod_a^s V(t,\mathrm{d}t) = Q^-$ exist. By (18) and (19) it is easy to see that the estimates

$$\left\|\prod_s^b V(t,\mathrm{d}t)\right\| = \|Q^+\| \leqslant K, \quad \left\|\prod_a^s V(t,\mathrm{d}t)\right\| = \|Q^-\| \leqslant K$$

hold. Now we are able to complete the proof of the theorem.

Assume that $s \in (a,b)$ and that $\varepsilon > 0$ is given. Let us choose a gauge δ_2 on $[a,b]$ such that $\delta_2(\tau) \leqslant \min(\delta(\tau), \delta_1(\tau))$, where for ε the gauges δ, δ_1 are given as above for the estimates (20), (21) and (22), (23).

By (18) and (19) we have for δ_2-fine L-partitions D of $[a,b]$, D_1 of $[a,s]$ and D_2 of $[s,b]$ the inequality

$$\begin{aligned}
&\|P(V,D) - Q^+Q^-\| \leqslant \|P(V,D) - P(V,D_2)P(V,D_1)\| \\
&+\|P(V,D_2)P(V,D_1) - Q^+Q^-\| \leqslant \|P(V,D) - P(V,D_2)P(V,D_1)\| \\
&+\|P(V,D_2)P(V,D_1) - Q^+P(V,D_1) + Q^+(P(V,D_1) - Q^-)\| \\
&\leqslant \|P(V,D) - P(V,D_2)P(V,D_1)\| + \|P(V,D_2) - Q^+\|\,\|P(V,D_1)\| \\
&+\|Q^+\|\,\|P(V,D_1) - Q^-\| \\
&\leqslant \|P(V,D) - P(V,D_2)P(V,D_1)\| + 2K\varepsilon \\
&= \|P(V,D) - P(V,D_2 \circ D_1)\| + 2K\varepsilon < 2\varepsilon + 2K\varepsilon.
\end{aligned} \tag{24}$$

Because $\varepsilon > 0$ can be chosen arbitrarily small we finally obtain

$$Q = Q^+ Q^-, \tag{25}$$

i.e. the equality

$$\prod_s^b V(t, \mathrm{d}t) \prod_a^s V(t, \mathrm{d}t) = \prod_a^b V(t, \mathrm{d}t)$$

given in the statement of the theorem holds. Since $Q \in \mathcal{B}(X)$ is invertible, we have by (25) the identity

$$Q^{-1} Q^+ Q^- = I$$

and this means that $Q^{-1}Q^+ \in \mathcal{B}(X)$ is the inverse to Q^-. Similarly it can be also shown that $Q^+ \in \mathcal{B}(X)$ is also invertible with $(Q^+)^{-1} = Q^- Q^{-1}$.

Hence the product integrals $\prod_a^s V(t, \mathrm{d}t) = Q^-$, $\prod_s^b V(t, \mathrm{d}t) = Q^+ \in \mathcal{B}(X)$ are invertible operators.

Further by (18) we have

$$\left\| \left(\prod_a^s V(t, \mathrm{d}t) \right)^{-1} \right\| = \left\| \left(\prod_a^b V(t, \mathrm{d}t) \right)^{-1} \prod_s^b V(t, \mathrm{d}t) \right\| \leqslant K \|Q^{-1}\|$$

and similarly by (19) also

$$\left\| \left(\prod_s^b V(t, \mathrm{d}t) \right)^{-1} \right\| \leqslant K \|Q^{-1}\|.$$

Setting $M = \max(K, K\|Q^{-1}\|)$ we obtain the statement of the final part of the theorem. □

Some auxiliary statements

Lemma 11. *Assume that $A_i, B_i \in \mathcal{B}(X)$, $i = 1, 2, \ldots, m$. Then*

$$\prod_{i=1}^m A_i - \prod_{i=1}^m B_i = \sum_{i=1}^m \left(\prod_{j=i+1}^m A_j \right) [A_i - B_i] \left(\prod_{j=1}^{i-1} B_j \right) \tag{26}$$

and

$$\prod_{i=1}^m A_i - \prod_{i=1}^m B_i = \sum_{i=1}^m \left(\prod_{j=i+1}^m B_j \right) [A_i - B_i] \left(\prod_{j=1}^{i-1} A_j \right), \tag{27}$$

where all the products are ordered according to descending indices, i.e. $\prod_{i=1}^{m} A_i = A_m A_{m-1} \dots A_1$ *etc. and where the convention* $\prod_{j=p}^{q} A_j = I$ *for* $p > q$ *is used.*

P r o o f. The equality (26) evidently holds for $m = 1$. We prove it in general by induction. Assume that (26) holds for m. Then

$$\begin{aligned}
\prod_{i=1}^{m+1} A_i - \prod_{i=1}^{m+1} B_i &= A_{m+1} \prod_{i=1}^{m} A_i - B_{m+1} \prod_{i=1}^{m} B_i \\
&= A_{m+1} \prod_{i=1}^{m} A_i - A_{m+1} \prod_{i=1}^{m} B_i + A_{m+1} \prod_{i=1}^{m} B_i - B_{m+1} \prod_{i=1}^{m} B_i \\
&= A_{m+1} \left(\prod_{i=1}^{m} A_i - \prod_{i=1}^{m} B_i \right) + (A_{m+1} - B_{m+1}) \prod_{i=1}^{m} B_i \\
&= A_{m+1} \sum_{i=1}^{m} \left(\prod_{j=i+1}^{m} A_j \right) [A_i - B_i] \left(\prod_{j=1}^{i-1} B_j \right) + (A_{m+1} - B_{m+1}) \prod_{i=1}^{m} B_i \\
&= \sum_{i=1}^{m} \left(A_{m+1} \prod_{j=i+1}^{m} A_j \right) [A_i - B_i] \left(\prod_{j=1}^{i-1} B_j \right) + (A_{m+1} - B_{m+1}) \prod_{i=1}^{m} B_i \\
&= \sum_{i=1}^{m} \left(\prod_{j=i+1}^{m+1} A_j \right) [A_i - B_i] \left(\prod_{j=1}^{i-1} B_j \right) + (A_{m+1} - B_{m+1}) \prod_{i=1}^{m} B_i \\
&= \sum_{i=1}^{m+1} \left(\prod_{j=i+1}^{m+1} A_j \right) [A_i - B_i] \left(\prod_{j=1}^{i-1} B_j \right).
\end{aligned}$$

This shows that (26) is true for $m + 1$ and the formula (26) is proved.

The equality (27) can be proved analogously. □

R e m a r k. Lemma 11 can be found e.g. in [1] or [4].

Corollary 12. *If* $A, B \in \mathcal{B}(X)$, *then*

$$A^m - B^m = \sum_{k=0}^{m-1} A^{m-k-1} [A - B] B^k \tag{28}$$

and

$$A^m - B^m = \sum_{k=0}^{m-1} B^{m-k-1} [A - B] A^k. \tag{29}$$

Proof. Using (26) we have

$$A^m - B^m = \sum_{i=1}^{m} \Big(\prod_{j=i+1}^{m} A \Big) [A-B] \Big(\prod_{j=1}^{i-1} B \Big)$$
$$= \sum_{i=1}^{m} A^{m-i}[A-B]B^{i-1} = \sum_{k=0}^{m-1} A^{m-k-1}[A-B]B^{k}$$

and (28) is proved. The equality (29) can be shown similarly from (27). □

Lemma 13. *If $A, B \in \mathcal{B}(X)$, then*

$$\|e^A - e^B\| \leqslant \|A-B\| e^{\max(\|A\|,\|B\|)} \leqslant \|A-B\| e^{\|A\|+\|B\|}. \tag{30}$$

Proof. We have
$$e^A - e^B = \sum_{q=1}^{\infty} \frac{1}{q!}(A^q - B^q).$$

Hence by (28) we get

$$\|e^A - e^B\| \leqslant \sum_{q=1}^{\infty} \frac{1}{q!} \|A^q - B^q\| = \sum_{q=1}^{\infty} \frac{1}{q!} \Big\| \sum_{k=0}^{q-1} A^{q-k-1}[A-B]B^k \Big\|$$
$$\leqslant \|A-B\| \sum_{q=1}^{\infty} \frac{1}{q!} \sum_{k=0}^{q-1} \|A\|^{q-k-1} \|B\|^k. \tag{31}$$

Clearly

$$\sum_{k=0}^{q-1} \|A\|^{q-k-1}\|B\|^k \leqslant \sum_{k=0}^{q-1} \max(\|A\|, \|B\|)^{q-1} = q \max(\|A\|, \|B\|)^{q-1}$$

and by (31)

$$\|e^A - e^B\| \leqslant \|A-B\| \sum_{q=1}^{\infty} \frac{1}{(q-1)!} \max(\|A\|, \|B\|)^{q-1} = \|A-B\| e^{\max(\|A\|,\|B\|)}.$$

□

Theorem 14. *Assume that $A\colon [a,b] \to \mathcal{B}(X)$ is Bochner integrable, i.e. $A \in L([a,b];\mathcal{B}(X))$. Let us set $V(t,J) = \mathrm{e}^{A(t)\mu(J)}$ for a tagged interval (t,J) where μ is the Lebesgue measure on $[a,b]$. Then the Bochner product integral*

$$\prod_a^b \mathrm{e}^{A(t)\,\mathrm{d}t} = \prod_a^b V(t,\mathrm{d}t) \in \mathcal{B}(X)$$

exists and is an invertible operator in $\mathcal{B}(X)$.

Remark. It should be mentioned that the result given in Theorem 14 holds also for the case when μ is an arbitrary non-atomic measure on $[a,b]$.

Proof. Assume that to a $\varepsilon > 0$ the gauge δ is given such that (by the Definition 1) we have

$$\sum_{i=1}^{k}\sum_{j=1}^{l} \|A(t_i) - A(s_j)\|_{\mathcal{B}(X)}\mu(J_i \cap L_j) < \varepsilon$$

for every δ-fine L-partitions $D_1 = \{(t_i, J_i), i = 1,\dots,k\}$ and $D_2 = \{(s_j, L_j), j = 1,\dots,l\}$ of $[a,b]$ and that

$$\left|\sum_{i=1}^{k} \|A(t_i)\|_{\mathcal{B}(X)}\mu(J_i) - \int_a^b \|A(t)\|\,\mathrm{d}t\right| < 1$$

and consequently also

$$\sum_{i=1}^{k} \|A(t_i)\|_{\mathcal{B}(X)}\mu(J_i) < 1 + \int_a^b \|A(t)\|\,\mathrm{d}t$$

for every δ-fine L-partition $D_1 = \{(t_i, J_i), i = 1,\dots,k\}$. Then for $V(\tau, J) = \mathrm{e}^{A(\tau)\mu(J)}$ we have

$$P(V, D_1) = \prod_{i=1}^{k} \mathrm{e}^{A(t_i)\mu(J_i)} = \prod_{i=1}^{k}\prod_{j=1}^{l} \mathrm{e}^{A(t_i)\mu(J_i \cap L_j)}$$

because clearly

$$\mathrm{e}^{A(t_i)\mu(J_i)} = \prod_{j=1}^{l} \mathrm{e}^{A(t_i)\mu(J_i \cap L_j)}$$

and similarly also

$$P(V, D_2) = \prod_{j=1}^{l}\prod_{i=1}^{k} \mathrm{e}^{A(s_j)\mu(J_i \cap L_j)}.$$

Assume that K_q, $q = 1, \dots, m$ is the ordered system of intervals which consists of k groups of ordered systems of intervals

$$J_1 \cap L_j,\ J_2 \cap L_j,\ \dots,\ J_k \cap L_j, \quad j = 1, \dots, l$$

where the ordering of intervals in each of these groups is induced by the ordering in the system $L_1, L_2, \dots, L_l$. Denote further $\tau_q = t_i$ for $q = 1, \dots, m$ when $K_q = J_i \cap L_j$. It is easy to see that $\{(\tau_q, K_q), q = 1, \dots, m\}$ is a δ-fine L-partition of $[a, b]$ since the L-partition $D_1 = \{(t_i, J_i),\ i = 1, \dots, k\}$ is assumed to be δ-fine. Then we have

$$P(V, D_1) = \prod_{q=1}^{m} \mathrm{e}^{A(\tau_q)\mu(K_q)}.$$

Using the same procedure for l groups of ordered systems of intervals

$$J_i \cap L_1,\ J_i \cap L_2,\ \dots,\ J_i \cap L_l, \quad i = 1, \dots, k$$

where the ordering of intervals in each of these groups is induced by the ordering in the system $J_1, J_2, \dots, J_k$ we get the same ordered system of intervals K_q as before. Taking $\sigma_q = s_j$ for $q = 1, \dots, m$ when $K_q = J_i \cap L_j$, we obtain a δ-fine L-partition $\{(\sigma_q, K_q), q = 1, \dots, m\}$ of $[a, b]$ since the L-partition $D_2 = \{(s_j, L_j),\ j = 1, \dots, l\}$ is assumed to be δ-fine and

$$P(V, D_2) = \prod_{q=1}^{m} \mathrm{e}^{A(\sigma_q)\mu(K_q)}.$$

Using these relations we obtain by Lemma 11

$$\|P(V, D_1) - P(V, D_2)\| = \left\| \prod_{q=1}^{m} \mathrm{e}^{A(\tau_q)\mu(K_q)} - \prod_{q=1}^{m} \mathrm{e}^{A(\sigma_q)\mu(K_q)} \right\|$$

$$= \left\| \sum_{q=1}^{m} \left(\prod_{r=q+1}^{m} \mathrm{e}^{A(\tau_r)\mu(K_r)} \right) [\mathrm{e}^{A(\tau_q)\mu(K_q)} - \mathrm{e}^{A(\sigma_q)\mu(K_q)}] \left(\prod_{r=1}^{q-1} \mathrm{e}^{A(\sigma_r)\mu(K_r)} \right) \right\|$$

$$\leqslant \sum_{q=1}^{m} \left\| \prod_{r=q+1}^{m} \mathrm{e}^{A(\tau_r)\mu(K_r)} \right\| \cdot \left\| \prod_{r=1}^{q-1} \mathrm{e}^{A(\sigma_r)\mu(K_r)} \right\| \cdot \|\mathrm{e}^{A(\tau_q)\mu(K_q)} - \mathrm{e}^{A(\sigma_q)\mu(K_q)}\|.$$

Further we have

$$\left\| \prod_{r=q+1}^{m} \mathrm{e}^{A(\tau_r)\mu(K_r)} \right\| \leqslant \prod_{r=q+1}^{m} \|\mathrm{e}^{A(\tau_r)\mu(K_r)}\|$$

$$\leqslant \prod_{r=q+1}^{m} \mathrm{e}^{\|A(\tau_r)\|\mu(K_r)} \leqslant \mathrm{e}^{\sum_{r=q+1}^{m} \|A(\tau_r)\|\mu(K_r)}$$

$$\leqslant \mathrm{e}^{\sum_{r=1}^{m} \|A(\tau_r)\|\mu(K_r)} \leqslant \mathrm{e}^{1+\int_a^b \|A(t)\|\,\mathrm{d}t} = K$$

and similarly also

$$\left\| \prod_{r=1}^{q-1} e^{A(\sigma_r)\mu(K_r)} \right\| \leqslant e^{1+\int_a^b \|A(t)\|\,dt} = K.$$

Using the estimate from Lemma 13 we have

$$\|e^{A(\tau_q)\mu(K_q)} - e^{A(\sigma_q)\mu(K_q)}\| \leqslant \|(A(\tau_q) - A(\sigma_q))\mu(K_q)\| e^{\max(\|A(\tau_q)\|,\|A(\sigma_q)\|)\mu(K_q)}$$

and therefore (because clearly $e^{\max(\|A(\tau_q)\|,\|A(\sigma_q)\|)\mu(K_q)} \leqslant e^{1+\int_a^b \|A(t)\|\,dt} = K$) we obtain

$$\|P(V, D_1) - P(V, D_2)\| \leqslant K^3 \sum_{i=1}^{k} \sum_{j=1}^{l} \|A(t_i) - A(s_j)\|_{\mathcal{B}(X)} \mu(J_i \cap L_j) < K^3 \varepsilon$$

for every δ-fine L-partitions $D_1 = \{(t_i, J_i), i = 1, \dots, k\}$ and $D_2 = \{(s_j, L_j), j = 1, \dots, l\}$ of $[a, b]$. Using Proposition 9 we conclude that the Bochner product integral $\prod_a^b e^{A(t)\,dt} \in \mathcal{B}(X)$ exists.

Recall again that the Lebesgue integral $\int_a^b \|A(s)\|\,ds = S \in \mathbb{R}$ exists and that

$$\left| \sum_{i=1}^{k} \|A(t_i)\| \mu(J_i) - S \right| < 1$$

for every δ-fine L-partition $D = \{(t_i, J_i), i = 1, \dots, k\}$.

From the existence of the Bochner product integral $\prod_a^b e^{A(t)\,dt} \in \mathcal{B}(X)$ we obtain that there is a gauge δ_1 on $[a, b]$ such that $\delta_1(t) < \delta(t)$ for all $t \in [a, b]$ such that

$$\left\| P(V, D) - \prod_a^b e^{A(t)\,dt} \right\| < e^{-(S+1)} \tag{32}$$

for every δ_1-fine L -partition $D = \{(t_i, J_i), i = 1, \dots, k\}$.

It is evident that $P(V, D)$ is invertible with

$$[P(V, D)]^{-1} = e^{-A(t_1)\mu(J_1)} \dots e^{-A(t_k)\mu(J_k)}$$

and

$$\begin{aligned} \|[P(V, D)]^{-1}\| &\leqslant \|e^{-A(t_1)\mu(J_1)}\| \dots \|e^{-A(t_k)\mu(J_k)}\| \\ &\leqslant \prod_{i=1}^{k} e^{\|A(t_i)\|\mu(J_i)} = e^{\sum_{i=1}^{k} \|A(t_i)\|\mu(J_i)} \leqslant e^{S+1}. \end{aligned}$$

Hence

$$e^{-(S+1)} \leqslant \frac{1}{\|[P(V,D)]^{-1}\|}$$

and by (32) we obtain

$$\left\| P(V,D) - \prod_a^b e^{A(t)\,dt} \right\| < \frac{1}{\|[P(V,D)]^{-1}\|}$$

for a given δ_1-fine L-partition D. Therefore the Bochner product integral $\prod_a^b e^{A(t)\,dt} \in \mathcal{B}(X)$ is an invertible operator (see Lemma VII.6.1 in [3]). □

Alternative descriptions of the exponential Bochner product integral

In the monograph [1] alternative descriptions for the product integrals of the form $\prod_a^b e^{A(t)\,dt}$ are mentioned for continuous $n \times n$-matrix valued functions A (see in [1, p. 51]). These definitions give alternative Bochner product integrals for the case $A \in L([a,b];\mathcal{B}(X))$, too.

The following definition is presented in [1, p. 51].

Definition 15. Let f be a complex-valued function defined on an open disc

$$D_\varrho = \{z \in \mathbb{C};\ |z| < \varrho\},\ \text{for } \varrho > 0$$

in $\mathbb{C}$. f is called a *P-function* if

(i) f is analytic in the disc D_ϱ, $\varrho > 0$,

(ii)

$$f(0) = f'(0) = 1.$$

A P-function f has a series expansion of the form

$$f(z) = 1 + z + \sum_{n=2}^{\infty} c_n z^n,$$

convergent for $|z| < \varrho$.

If $B \in \mathcal{B}(X)$ and $\|B\| < \varrho$ then we define

$$f(B) = I + B + \sum_{n=2}^{\infty} c_n B^n.$$

Assume that $\|B\| \leqslant r_0 < \varrho$. Then

$$f(B) - I = B + \sum_{n=2}^{\infty} c_n B^n = B\Big(I + \sum_{n=1}^{\infty} c_{n+1} B^n\Big)$$

and

$$\begin{aligned}\|f(B) - I\| &\leqslant \|B\| \left\|I + \sum_{n=1}^{\infty} c_{n+1} B^n\right\| \leqslant \|B\| \left(1 + \sum_{n=1}^{\infty} |c_{n+1}|\, \|B\|^n\right) \\ &\leqslant \|B\| \left(1 + \sum_{n=1}^{\infty} |c_{n+1}| r_0^n\right) = \|B\| N,\end{aligned}$$

where we denote $N = 1 + \sum\limits_{n=1}^{\infty} |c_{n+1}| r_0^n$.

Assume that operators $B_1, \ldots, B_m \in \mathcal{B}(X)$ are given such that $\|B_j\| \leqslant r_0$ for $j = 1, \ldots, m$. Then

$$\|f(B_j)\| \leqslant \|f(B_j) - I\| + 1 \leqslant 1 + N\|B_j\|$$

and

$$\left\|\prod_{j=1}^{m} f(B_j)\right\| \leqslant \prod_{j=1}^{m} \|f(B_j)\| \leqslant \prod_{j=1}^{m} (1 + N\|B_j\|) \leqslant \prod_{j=1}^{m} \mathrm{e}^{N\|B_j\|} = \mathrm{e}^{N\sum_{j=1}^{m} \|B_j\|}.$$

Using this inequality we get by Lemma 11

$$\begin{aligned}\left\|\prod_{i=1}^{m} \mathrm{e}^{B_i} - \prod_{i=1}^{m} f(B_i)\right\| &= \left\|\sum_{i=1}^{m} \Big(\prod_{j=i+1}^{m} \mathrm{e}^{B_j}\Big)[\mathrm{e}^{B_i} - f(B_i)]\Big(\prod_{j=1}^{i-1} f(B_j)\Big)\right\| \\ &\leqslant \sum_{i=1}^{m} \left\|\prod_{j=i+1}^{m} \mathrm{e}^{B_j}\right\| \|\mathrm{e}^{B_i} - f(B_i)\| \left\|\prod_{j=1}^{i-1} f(B_j)\right\| \\ &\leqslant \sum_{i=1}^{m} \Big(\prod_{j=i+1}^{m} \|\mathrm{e}^{B_j}\|\Big) \|\mathrm{e}^{B_i} - f(B_i)\| \Big(\prod_{j=1}^{i-1} \|f(B_j)\|\Big) \\ &\leqslant \sum_{i=1}^{m} \Big(\prod_{j=i+1}^{m} \mathrm{e}^{\|B_j\|}\Big) \|\mathrm{e}^{B_i} - f(B_i)\| \Big(\prod_{j=1}^{i-1} \|f(B_j)\|\Big) \\ &\leqslant \sum_{i=1}^{m} \Big(\mathrm{e}^{\sum_{j=i+1}^{m} \|B_j\|}\Big) \|\mathrm{e}^{B_i} - f(B_i)\| \Big(\mathrm{e}^{N\sum_{j=1}^{i-1} \|B_j\|}\Big) \\ &\leqslant \mathrm{e}^{\max(1,N)\sum_{j=1}^{m} \|B_j\|} \sum_{i=1}^{m} \|\mathrm{e}^{B_i} - f(B_i)\|.\end{aligned}$$

Further we clearly have

$$\|e^{B_i} - f(B_i)\| \leqslant \sum_{n=2}^{\infty} \left|\frac{1}{n!} - c_n\right| r_0^{n-2} \|B_i\|^2$$

and therefore

$$\left\| \prod_{i=1}^{m} e^{B_i} - \prod_{i=1}^{m} f(B_i) \right\| \leqslant e^{\max(1,N) \sum_{j=1}^{m} \|B_j\|} M \sum_{j=1}^{m} \|B_j\|^2, \tag{33}$$

where

$$M = \sum_{n=2}^{\infty} \left|\frac{1}{n!} - c_n\right| r_0^{n-2}.$$

Now we are in the position to prove the following result.

Theorem 16. *Assume that $A \in L([a,b]; \mathcal{B}(X))$. Let f be an arbitrary P-function. Then to every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that*

$$\left\| \prod_{i=1}^{k} f(A(t_i)\mu(J_i)) - \prod_{a}^{b} e^{A(t)\,dt} \right\| < \varepsilon$$

provided $D = \{(t_i, J_i)\,;\, i = 1, \dots, k\}$ is a δ-fine L-partition of $[a,b]$.

Remark. The statement given in Theorem 16 leads really to alternative descriptions of the Bochner product integral $\prod_a^b e^{A(t)\,dt}$ because if f is an arbitrary P-function and if we set $V(t,J) = f(A(t)\mu(J))$ for a tagged interval (t,J) then $\prod_a^b V(t, dt) = \prod_a^b e^{A(t)\,dt}$. Since evidently $f(z) = 1 + z$ is a P-function, we have the special formula

$$\prod_{a}^{b} (I + A(t)\,dt) = \prod_{a}^{b} e^{A(t)\,dt}$$

for every $A \in L([a,b]; \mathcal{B}(X))$. The case of product integrals of the form $\prod_a^b (I + A(t)\,dt)$ was extensively studied e.g. in [4]. Theorem 16 shows that even in the Bochner case these product integrals are the same as the exponential product integrals $\prod_a^b e^{A(t)\,dt}$.

Proof of Theorem 16. The existence of $\prod_a^b e^{A(t)\,dt}$ was shown in Theorem 14. Assume that $\varepsilon > 0$ is given. By the definition of the Bochner product integral there is a gauge δ on $[a,b]$ such that

$$\left\| \prod_{i=1}^{k} \mathrm{e}^{A(t_i)\mu(J_i)} - \prod_{a}^{b} \mathrm{e}^{A(t)\,\mathrm{d}t} \right\| < \frac{\varepsilon}{2}$$

for every δ-fine L-partition $\{(t_i, J_i);\ i = 1, \ldots, k\}$ of $[a,b]$.

Fix $r_0 < \varrho$ (ϱ is given by the Definition 15 of the P-function f). Then for any given $\eta \leqslant r_0$ we can assume that the gauge δ satisfies

$$\delta(t) < \frac{\eta}{2(\|A(t)\| + 1)}$$

and that

$$\sum_{i=1}^{k} \|A(t_i)\| \mu(J_i) \leqslant 1 + \int_a^b \|A(t)\|\,\mathrm{d}t$$

for every δ-fine L-partition $\{(t_i, J_i);\ i = 1, \ldots, k\}$ of $[a,b]$.

If the tagged interval (t, J) is δ-fine, then $\mu(J) < \frac{\eta}{\|A(t)\|+1}$ because

$$J \subset (t - \delta(t), t + \delta(t)) \quad \text{and} \quad \|A(t)\|\mu(J) < \frac{\eta\|A(t)\|}{\|A(t)\| + 1} < \eta.$$

The result given in (33) can be used for the following inequality.

$$\begin{aligned}
&\left\| \prod_{i=1}^{k} \mathrm{e}^{A(t_i)\mu(J_i)} - \prod_{i=1}^{k} f(A(t_i)\mu(J_i)) \right\| \\
&\leqslant \mathrm{e}^{\max(1,N)\sum_{j=1}^{k} \|A(t_j)\|\mu(J_j)}\, M \sum_{j=1}^{k} \|A(t_j)\|^2 (\mu(J_j))^2 \\
&\leqslant \eta \mathrm{e}^{\max(1,N)\sum_{j=1}^{k} \|A(t_j)\|\mu(J_j)}\, M \sum_{j=1}^{k} \|A(t_j)\| \mu(J_j) \\
&\leqslant \eta M \mathrm{e}^{\max(1,N)(1+\int_a^b \|A(t)\|\,\mathrm{d}t)} \Big(1 + \int_a^b \|A(t)\|\,\mathrm{d}t\Big), \qquad (34)
\end{aligned}$$

where $M = \sum\limits_{n=2}^{\infty} \left|\frac{1}{n!} - c_n\right| g r_0^{n-2}$ and $N = 1 + \sum\limits_{n=1}^{\infty} |c_{n+1}| r_0^n$. From (34) we obtain finally

$$\begin{aligned}
&\left\| \prod_{i=1}^{k} f(A(t_i)\mu(J_i)) - \prod_{a}^{b} \mathrm{e}^{A(t)\,\mathrm{d}t} \right\| \\
&\leqslant \left\| \prod_{i=1}^{k} f(A(t_i)\mu(J_i)) - \prod_{i=1}^{k} \mathrm{e}^{A(t_i)\mu(J_i)} \right\| + \left\| \prod_{i=1}^{k} \mathrm{e}^{A(t_i)\mu(J_i)} - \prod_{a}^{b} \mathrm{e}^{A(t)\,\mathrm{d}t} \right\| < \varepsilon
\end{aligned}$$

whenever $0 < \eta < r_0$ is chosen sufficiently small, e.g. such that

$$\eta M e^{\max(1,N)(1+\int_a^b \|A(t)\| \, dt)} \left(1 + \int_a^b \|A(t)\| \, dt\right) < \frac{\varepsilon}{2}.$$

□

Equivalence of the Bochner product integral and the classical product integral

Assume that $B\colon [a,b] \to \mathcal{B}(X)$ is a step-function, i.e. that there is a finite system of points

$$a = s_0 < s_1 < \ldots < s_{m-1} < s_m = b$$

such that B is constant on each (s_{k-1}, s_k) with the value $B_k \in \mathcal{B}(X)$, $k = 1, \ldots, m$.

For a given step-function $B\colon [a,b] \to \mathcal{B}(X)$ define

$$E_B = e^{B_m(s_m - s_{m-1})} e^{B_{m-1}(s_{m-1} - s_{m-2})} \ldots e^{B_1(s_1 - s_0)}.$$

In this way the product integral of a step-function is defined. In the monograph [1, p. 54] the following definition of the product integral is given.

Definition 17. Assume that $A \in L([a,b]; \mathcal{B}(X))$ is given. The (Lebesgue type) product integral $(L)\prod_a^b e^{A(t)\,dt}$ is defined by

$$(L)\prod_a^b e^{A(t)\,dt} = \lim_{n\to\infty} E_{A_n} \tag{35}$$

where A_n, $n = 1, 2, \ldots$ is any sequence of step-functions convergent to A in the L^1 sense, i.e.

$$\lim_{n\to\infty} \int_a^b \|A_n(s) - A(s)\| \, ds = 0$$

and E_{A_n} is the product integral of the step-function A_n.

It should be mentioned that if $A \in L([a,b]; \mathcal{B}(X))$ then there exists a sequence of step-functions converging to A in the L^1 sense and therefore $(L)\prod_a^b e^{A(t)\,dt}$ is well defined since in this case the sequence of products E_{A_n} converges for $n \to \infty$ (see [1, pp. 54, 83] for more details).

Now we will show that the product integral given by Definition 17 for $A \in L([a,b];\mathcal{B}(X))$ is equivalent to the Bochner product integral given by Definition 7 for the case

$$V(t,J) = \mathrm{e}^{A(t)\mu(J)}.$$

First let us prove the following result.

Lemma 18. *Assume that $A_1, A_2 \in L([a,b];\mathcal{B}(X))$ are given. Then for every $[c,d] \subset [a,b]$ the inequality*

$$\left\| \prod_c^d \mathrm{e}^{A_2(t)\,\mathrm{d}t} - \prod_c^d \mathrm{e}^{A_1(t)\,\mathrm{d}t} \right\| \leqslant K \int_c^d \|A_2(s) - A_1(s)\|\,\mathrm{d}s \tag{36}$$

holds, where

$$K = \left(\mathrm{e}^{\int_a^b \|A_1(s)\|\,\mathrm{d}s} + 1\right)^2 \left(\mathrm{e}^{\int_a^b \|A_2(s)\|\,\mathrm{d}s} + 1\right)^2.$$

Proof. By Proposition 6 the functions $\|A_1\|, \|A_2\| \colon [a,b] \to \mathbb{R}$ are Lebesgue integrable over $[a,b]$. Since $A_2 - A_1 \in L([a,b];\mathcal{B}(X))$ we get by Proposition 6 that also the function $\|A_2(s) - A_1(s)\|$ is Lebesgue integrable over $[a,b]$.

Let us fix an interval $[c,d] \subset [a,b]$. The functions $\|A_1\|, \|A_2\| \colon [a,b] \to \mathbb{R}$ are Lebesgue integrable over the interval $[c,d] \subset [a,b]$. Therefore by Definition 3 (see also Remark 5) there is a gauge δ_1 on $[a,b]$ such that

$$\left| \sum_{j=1}^m \|A_l(\tau_j)\| \mu(J_j) - \int_c^d \|A_l(s)\|\,\mathrm{d}s \right| < 1$$

for $l = 1,2$ and every δ_1-fine L-partition $\{(\tau_j, J_j), j = 1, \ldots, m\}$ of $[c,d]$. Hence we have

$$\sum_{j=1}^m \|A_l(\tau_j)\| \mu(J_j) < \int_c^d \|A_l(s)\|\,\mathrm{d}s + 1 \leqslant \int_a^b \|A_l(s)\|\,\mathrm{d}s + 1 \tag{37}$$

for $l = 1,2$ and every δ_1-fine L-partition $\{(\tau_j, J_j), j = 1, \ldots, m\}$ of $[c,d]$.

Assume that $\varepsilon > 0$ is given.

Since the Lebesgue integral $\int_a^b \|A_2(s) - A_1(s)\|\,\mathrm{d}s$ exists, there exists a gauge $\delta_2 < \delta_1$ on $[c,d]$ such that

$$\left| \sum_{j=1}^m \|A_2(\tau_j) - A_1(\tau_j)\| \mu(J_j) - \int_c^d \|A_2(s) - A_1(s)\|\,\mathrm{d}s \right| < \varepsilon$$

and therefore also

$$\sum_{j=1}^{m} \|A_2(\tau_j) - A_1(\tau_j)\| \mu(J_j) < \int_c^d \|A_2(s) - A_1(s)\| \, \mathrm{d}s + \varepsilon \tag{38}$$

for every δ_2-fine L-partition $\{(\tau_j, J_j), j = 1, \ldots, m\}$ of $[c, d]$.

By Theorem 14 the Bochner product integral $\prod_c^d \mathrm{e}^{A_l(t)\,\mathrm{d}t} \in \mathcal{B}(X)$ exists for $l = 1, 2$. Hence there is a gauge $\delta < \delta_2$ on $[a, b]$ such that

$$\left\| \prod_{j=1}^{m} \mathrm{e}^{A_l(\tau_j)\mu(J_j)} - \prod_c^d \mathrm{e}^{A_l(t)\,\mathrm{d}t} \right\| < \varepsilon \tag{39}$$

for $l = 1, 2$ and for every δ-fine L-partition $\{(\tau_j, J_j), j = 1, \ldots, m\}$ of $[c, d]$. Hence by (39) we get

$$\begin{aligned} \left\| \prod_c^d \mathrm{e}^{A_2(t)\,\mathrm{d}t} - \prod_c^d \mathrm{e}^{A_1(t)\,\mathrm{d}t} \right\| &\leqslant \left\| \prod_c^d \mathrm{e}^{A_2(t)\,\mathrm{d}t} - \prod_{j=1}^{m} \mathrm{e}^{A_2(\tau_j)\mu(J_j)} \right\| \\ + \left\| \prod_{j=1}^{m} \mathrm{e}^{A_2(\tau_j)\mu(J_j)} - \prod_{j=1}^{m} \mathrm{e}^{A_1(\tau_j)\mu(J_j)} \right\| &+ \left\| \prod_{j=1}^{m} \mathrm{e}^{A_1(\tau_j)\mu(J_j)} - \prod_c^d \mathrm{e}^{A_1(t)\,\mathrm{d}t} \right\| \\ &< \left\| \prod_{j=1}^{m} \mathrm{e}^{A_2(\tau_j)\mu(J_j)} - \prod_{j=1}^{m} \mathrm{e}^{A_1(\tau_j)\mu(J_j)} \right\| + 2\varepsilon \end{aligned} \tag{40}$$

for every δ-fine L-partition $\{(\tau_j, J_j), j = 1, \ldots, m\}$ of $[c, d]$. For the first term on the right hand side of (40) we have by Lemma 11

$$\begin{aligned} &\prod_{j=1}^{m} \mathrm{e}^{A_2(\tau_j)\mu(J_j)} - \prod_{j=1}^{m} \mathrm{e}^{A_1(\tau_j)\mu(J_j)} \\ &= \sum_{i=1}^{m} \left(\prod_{j=i+1}^{m} \mathrm{e}^{A_2(\tau_j)\mu(J_j)} \right) [\mathrm{e}^{A_2(\tau_i)\mu(J_i)} - \mathrm{e}^{A_1(\tau_i)\mu(J_i)}] \left(\prod_{j=1}^{i-1} \mathrm{e}^{A_1(\tau_j)\mu(J_j)} \right). \end{aligned} \tag{41}$$

Further by (37)

$$\begin{aligned} \left\| \prod_{j=1}^{i-1} \mathrm{e}^{A_l(\tau_j)\mu(J_j)} \right\| &\leqslant \prod_{j=1}^{i-1} \mathrm{e}^{\|A_l(\tau_j)\|\mu(J_j)} = \mathrm{e}^{\sum_{j=1}^{i-1} \|A_l(\tau_j)\|\mu(J_j)} \\ &\leqslant \mathrm{e}^{\sum_{j=1}^{m} \|A_l(\tau_j)\|\mu(J_j)} \leqslant \mathrm{e}^{\int_c^d \|A_l(s)\|\,\mathrm{d}s + 1} \leqslant \mathrm{e}^{\int_a^b \|A_l(s)\|\,\mathrm{d}s + 1} \end{aligned}$$

and analogously

$$\left\| \prod_{j=i+1}^{m} \mathrm{e}^{A_l(\tau_j)\mu(J_j)} \right\| \leqslant \mathrm{e}^{\int_a^b \|A_l(s)\|\,\mathrm{d}s + 1}$$

for $l = 1, 2$ and every $i = 1, \ldots, m$ and therefore (41) yields

$$\left\| \prod_{j=1}^{m} e^{A_2(\tau_j)\mu(J_j)} - \prod_{j=1}^{m} e^{A_1(\tau_j)\mu(J_j)} \right\|$$

$$\leqslant e^{\int_a^b \|A_1(s)\| \, ds + 1} e^{\int_a^b \|A_2(s)\| \, ds + 1} \sum_{i=1}^{m} \left\| e^{A_2(\tau_i)\mu(J_i)} - e^{A_1(\tau_i)\mu(J_i)} \right\|. \tag{42}$$

Using the estimate given in Lemma 13 we have

$$\begin{aligned} &\|e^{A_2(\tau_i)\mu(J_i)} - e^{A_1(\tau_i)\mu(J_i)}\| \\ &\leqslant \|A_2(\tau_i)\mu(J_i) - A_1(\tau_i)\mu(J_i)\| e^{(\|A_2(\tau_i)\| + \|A_1(\tau_i)\|)\mu(J_i)} \\ &\leqslant \|A_2(\tau_i) - A_1(\tau_i)\| \mu(J_i) e^{\int_a^b \|A_1(s)\| \, ds + 1} e^{\int_a^b \|A_2(s)\| \, ds + 1} \end{aligned}$$

for every $i = 1, \ldots, m$. Hence by (42) we get

$$\left\| \prod_{j=1}^{m} e^{A_2(\tau_j)\mu(J_j)} - \prod_{j=1}^{m} e^{A_1(\tau_j)\mu(J_j)} \right\|$$

$$\leqslant \left(e^{\int_a^b \|A_1(s)\| \, ds + 1}\right)^2 \left(e^{\int_a^b \|A_2(s)\| \, ds + 1}\right)^2 \sum_{i=1}^{m} \|A_2(\tau_i) - A_1(\tau_i)\| \mu(J_i)$$

$$= K \sum_{i=1}^{m} \|A_2(\tau_i) - A_1(\tau_i)\| \mu(J_i), \tag{43}$$

where

$$K = \left(e^{\int_a^b \|A_1(s)\| \, ds + 1}\right)^2 \left(e^{\int_a^b \|A_2(s)\| \, ds + 1}\right)^2.$$

Finally the relation (38) yields

$$\left\| \prod_{j=1}^{m} e^{A_2(\tau_j)\mu(J_j)} - \prod_{j=1}^{m} e^{A_1(\tau_j)\mu(J_j)} \right\|$$

$$< K \int_c^d \|A_2(s) - A_1(s)\| \, ds + K\varepsilon. \tag{44}$$

Therefore by (40) we get

$$\left\| \prod_c^d e^{A_2(t)\, dt} - \prod_c^d e^{A_1(t)\, dt} \right\| < K \int_c^d \|A_2(s) - A_1(s)\| \, ds + (K+2)\varepsilon$$

and this proves (36) because $\varepsilon > 0$ can be taken arbitrarily small. □

Theorem 19. *If $A \in L([a,b];\mathcal{B}(X))$ then both the product integral $(L)\prod_a^b e^{A(t)\,dt}$ and the Bochner product integral $\prod_a^b e^{A(t)\,dt}$ exist and*

$$(L)\prod_a^b e^{A(t)\,dt} = \prod_a^b e^{A(t)\,dt}.$$

P r o o f. The existence of the product integrals is clear by Definition 17 and by Theorem 14.

Assume that $\varepsilon > 0$ is given and let A_n, $n = 1,2,\dots$ be a sequence of $\mathcal{B}(X)$-valued step-functions such that

$$\lim_{n\to\infty}\int_a^b \|A_n(s) - A(s)\|\,ds = 0$$

and

$$\lim_{n\to\infty} E_{A_n} = (L)\prod_a^b e^{A(t)\,dt}$$

(see Definition 17).

There exists an $n_0 \in \mathbb{N}$ such that

$$\int_a^b \|A_n(s) - A(s)\|\,ds < \varepsilon \tag{45}$$

and

$$\left\|(L)\prod_a^b e^{A(t)\,dt} - E_{A_n}\right\| < \varepsilon$$

for every $n > n_0$. From the definition of E_{A_n} it is easy to observe that

$$E_{A_n} = (L)\prod_a^b e^{A_n(t)\,dt} = \prod_a^b e^{A_n(t)\,dt}$$

for every $n \in \mathbb{N}$.

Then

$$\begin{aligned}\left\|(L)\prod_a^b e^{A(t)\,dt} - \prod_a^b e^{A(t)\,dt}\right\| &\leqslant \left\|(L)\prod_a^b e^{A(t)\,dt} - E_{A_n}\right\| + \left\|E_{A_n} - \prod_a^b e^{A(t)\,dt}\right\| \\ &< \varepsilon + \left\|\prod_a^b e^{A_n(t)\,dt} - \prod_a^b e^{A(t)\,dt}\right\|\end{aligned}$$

for any $n > n_0$. Using Lemma 18 for the second term on the right hand side of this inequality we get

$$\left\|(L)\prod_a^b e^{A(t)\,dt} - \prod_a^b e^{A(t)\,dt}\right\|$$

$$< \varepsilon + K_n \int_a^b \|A_n(s) - A(s)\|\,ds \tag{46}$$

where

$$K_n = \left(e^{\int_a^b \|A_n(s)\|\,ds + 1}\right)^2 \left(e^{\int_a^b \|A(s)\|\,ds + 1}\right)^2.$$

Since clearly

$$\big|\|A_n(s)\| - \|A(s)\|\big| \leqslant \|A_n(s) - A(s)\|$$

for every $s \in [a,b]$ we obtain that by (45)

$$\int_a^b \big|\|A_n(s)\| - \|A(s)\|\big|\,ds \leqslant \int_a^b \|A_n(s) - A(s)\|\,ds < \varepsilon,$$

holds for all $n > n_0$, therefore

$$\int_a^b \|A_n(s)\|\,ds < \int_a^b \|A(s)\|\,ds + \varepsilon$$

and

$$e^{\int_a^b \|A_n(s)\|\,ds + 1} < e^{\int_a^b \|A(s)\|\,ds + 1 + \varepsilon}.$$

This inequality yields

$$K_n \leqslant \left(e^{\int_a^b \|A(s)\|\,ds + 1}\right)^4 e^{2\varepsilon} = L$$

for every $n \in \mathbb{N}$, $n > n_0$. Hence by (45) and (46) we get

$$\left\|(L)\prod_a^b e^{A(t)\,dt} - \prod_a^b e^{A(t)\,dt}\right\|$$

$$< \varepsilon + L \int_a^b \|A_n(s) - A(s)\|\,ds < (L+1)\varepsilon.$$

This inequality leads to the conclusion of the theorem because $\varepsilon > 0$ can be chosen arbitrarily small. $\square$

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