Mieczysław Cichoń

Trichotomy and bounded solutions of nonlinear differential equations

Mathematica Bohemica, Vol. 119 (1994), No. 3, 275–284

Persistent URL: http://dml.cz/dmlcz/126161

119 (1994) MATHEMATICA BOHEMICA No. 3, 275–284

TRICHOTOMY AND BOUNDED SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

MIECZYSŁAW CICHOŃ, Poznań

(Received June 1, 1993)

Summary. The existence of bounded solutions for equations $x' = A(t)x + f(t,x)$ in Banach spaces is proved. We assume that the linear part is trichotomic and the perturbation f satisfies some conditions expressed in terms of measures of noncompactness.

Keywords: bounded solutions, trichotomy, measures of noncompactness, Banach spaces

AMS classification: 34C11, 34G20

1. Introduction

The purpose of this paper is to prove a theorem concerned with the existence of bounded solutions of the nonlinear differential equation

$$x' = A(t)x + f(t,x)$$

on the whole real line $\mathbb{R}$.

This problem was intensively studied by many authors ([2], [4], [5], [6], [8], [9], [11], [13], for instance) and this paper is a continuation of the results mentioned.

In comparision to previous results of this type we assume a more general growth condition (see [3], [5], [11], for example), a more general continuity assumption ([5], [6], [11], [13]) and in our compactness condition a measure of noncompactness is chosen arbitrarily from a class of measures which contains the well-known classical measures (cf. [5], [11], [13]). Let us remark that our compactness-type assumption is more general than the condition (12)–(13) in [11].

We begin by introducing the indispensable notions. Throughout the paper $(E, \|\cdot\|)$ will denote a real Banach space and $B(a,r) = \{y \in E\colon \|y - a\| \leqslant r\}$. By $L(E)$ we will denote the algebra of continuous linear operators from E into itself with induced standard norm $|\cdot|$.

Moreover, by $C(\mathbb{R}, E)$ we will denote the Fréchet space of all continuous functions from $\mathbb{R}$ into E, endowed with the topology of almost uniform convergence on $\mathbb{R}$.

Let $A\colon \mathbb{R} \to L(E)$ be strongly measurable and Bochner integrable on every finite subinterval of $\mathbb{R}$.

Consider the equation

$$x'(t) = A(t)x(t). \tag{1}$$

By $U(t)$ we denote the fundamental solution of $U'(t) = A(t)U(t)$ with $U(0) = \mathrm{Id}$.

Following Elaydi and Hajek we introduce

Definition 1. [6] A linear equation (1) is said to have a trichotomy on $\mathbb{R}$ if there exist linear projections P, Q such that

$$PQ = QP, \quad P + Q - PQ = \mathrm{Id} \tag{2}$$

and constants $\alpha \geqslant 1, \sigma > 0$ such that

$$\begin{aligned} |U(t)PU^{-1}(s)| &\leqslant \alpha e^{-\sigma(t-s)} \quad \text{for} \quad 0 \leqslant s \leqslant t, \\ |U(t)(\mathrm{Id} - P)U^{-1}(s)| &\leqslant \alpha e^{-\sigma(s-t)} \quad \text{for} \quad t \leqslant s, s \geqslant 0, \\ |U(t)QU^{-1}(s)| &\leqslant \alpha e^{-\sigma(s-t)} \quad \text{for} \quad t \leqslant s \leqslant 0, \\ |U(t)(\mathrm{Id} - Q)U^{-1}(s)| &\leqslant \alpha e^{-\sigma(t-s)} \quad \text{for} \quad s \leqslant t, s \leqslant 0. \end{aligned} \tag{3}$$

It is necessary to remark that if (1) has a trichotomy on $\mathbb{R}$, then it has an exponential dichotomy on $\mathbb{R}_+$ and an exponential dichotomy on $\mathbb{R}_-$ (see [4], [5], [6], Lemma 1.2). The converse is not true.

Define the integral kernel $G(t,s) = U(t)L(t,s)U^{-1}(s)$, where

$$L(t,s) = \begin{cases} \mathrm{Id} - Q & \text{for } 0 < s \leqslant \max(t,0), \\ -Q & \text{for } \max(t,0) < s, \\ P & \text{for } s \leqslant \min(t,0), \\ P - \mathrm{Id} & \text{for } \min(t,0) < s \leqslant 0. \end{cases}$$

Then $|G(t,s)| \leqslant \alpha e^{-\sigma|t-s|}$ for $t, s \in \mathbb{R}$ ([7], Lemma 7). More information about this can be found in [6] or [7].

2. Measures of noncompactness

The notion of measure of noncompactness was introduced by K. Kuratowski in 1930. It is a very important notion, for example in the theory of fixed points or in the theory of differential equations. An axiomatic theory of such measures can be found in [1], for instance. For such problems as the theory of linear differential equations it is worth while to introduce a special class of measures of noncompactness μ with the property

$$\mu(KA) \leqslant |K|\mu(A),$$

where $K \in L(E)$ and A is a bounded subset of E.

The Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness have this property ([5]), but unfortunately we must assume additionally this property in the case of axiomatic approach to the above mentioned problems, see [11]. However, each of the so-called $(\mathscr{P}, \mathscr{B}, p)$-measures of noncompactness have this property ([3]), so they are useful in this case and especially in this paper. For the convenience of the reader we repeat without proofs the relevant material from [3], thus making our exposition self-contained.

Let $\mathscr{P}$ be a family of relatively compact subsets of E such that

(P_1) $X \in \mathscr{P} \Rightarrow \overline{X} \in \mathscr{P}$,

(P_2) $X \in \mathscr{P}, Y \neq \emptyset, Y \subset X \Rightarrow Y \in \mathscr{P}$,

(P_3) $X \in \mathscr{P} \Rightarrow \text{conv}\, X \in \mathscr{P}$,

(P_4) the subfamily of all closed sets in $\mathscr{P}$ is closed in the family of all nonempty, bounded and closed subsets of E with respect to the Hausdorff topology.

As in [1], a function $\mu \colon \mathscr{M} \to [0, \infty)$ is said to be a measure of noncompactness with the kernel $\mathscr{P}$ if it is subject to the following conditions:

(M_1) $\mu(X) = 0 \Leftrightarrow X \in \mathscr{P}$,

(M_2) $\mu(\overline{\text{conv}} X) = \mu(X)$,

(M_3) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y), \quad X, Y \in \mathscr{M}$,

where $\mathscr{M}$ denotes the family of all nonempty, bounded subsets of E. Denote by $\mathscr{B}$ a basis of neighbourhoods of zero composed of closed, convex and balanced sets. Let $\mathscr{B}' = \{rB \colon B \in \mathscr{B}, r > 0\}$. Assume that a function $p \colon \mathscr{B}' \to [0, \infty)$ satisfies the following conditions:

(C_1) $p(V) > 0$ whenever $V \notin \mathscr{P}$,

(C_2) for each $\varepsilon > 0$ there exists $V \in \mathscr{B}'$ such that $p(V) \leqslant \varepsilon$,

(C_3) $U \subset V \Rightarrow p(U) \leqslant p(V)$,

(C_4) $p(\text{conv}\, V) = p(V)$,

where $U, V \in \mathscr{B}'$.

Such a function is said to be a p-function.

And now, we can introduce our notion of a $(\mathscr{P},\mathscr{B},p)$-measure of noncompactness ([3]): a function $\mu\colon \mathscr{M}\to[0,\infty)$ is said to be a $(\mathscr{P},\mathscr{B},p)$-measure of noncompactness $[(\mathscr{P},\mathscr{B},p)-mnc]$ iff

$$\mu(W)=\inf\{\varepsilon>0\colon \text{there exist } H\in\mathscr{P} \text{ and } V\in\mathscr{B}' \text{ such that } W\subset H+V,\ p(V)\leqslant\varepsilon\}$$

for each $W\in\mathscr{M}$.

The Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness are, in fact, $(\mathscr{P},\mathscr{B},p)$-measures of noncompactness (see [3]). More information about these measures can be found in [3] and [1].

In the sequel, we will assume that

(i) $\mathscr{P}$ is a family of all relatively compact subsets of E,

(ii) $p(k\cdot V)=k\cdot p(V),\quad k>0, V\in\mathscr{B}'$,

(iii) $U,V\in\mathscr{B}'\Rightarrow U+V\in\mathscr{B}'$.

Under the above assumptions we have the following lemmas:

Lemma 1. ([3]) *Every $(\mathscr{P},\mathscr{B},p)$-mnc μ has the following properties:*

(L_1) $\mu(A+B)\leqslant\mu(A)+\mu(B)$,

(L_2) $A\subset B\Rightarrow\mu(A)\leqslant\mu(B)$,

(L_3) $\mu(k\cdot A)=k\cdot\mu(A)$,

(L_4) $\mu(\overline{\mathrm{conv}}A)=\mu(A)$,

(L_5) $\mu(A\cup\{x_0\})=\mu(A)$,

(L_6) $\mu(A)=0$ iff A is relatively compact in E,

for each $A,B\in\mathscr{M}$, $x_0\in E$ and $k>0$.

Lemma 2. ([3]) *If K is a continuous mapping from a compact interval I of $\mathbb{R}$ to $L(E)$ and W is a bounded subset of E then*

$$\mu\Big(\bigcup_{t\in I}K(t)W\Big)\leqslant\sup_{t\in I}|K(t)|\cdot\mu(W).$$

The assertion of the above lemma is very important. As claimed, in the case of purely axiomatic theory of such measures of noncompactness, it is necessary to assume this fact (see [11])! See also [5] .

Lemma 3. ([5]) *Let W be a bounded, almost equicontinuous subset of $C(\mathbb{R},E)$. For any subset X of W put*

$$\xi(X)=\sup_{t\in\mathbb{R}}\mu(X(t)).$$

Then the index ξ has the properties (L_1)–(L_5) listed in Lemma 1 and if $\xi(x) = 0$ then x is relatively compact in $C(\mathbb{R}, E)$.

3. Main result

We introduce the following assumptions:

(1^0) $A\colon \mathbb{R} \to L(E)$ is strongly measurable and Bochner integrable on every finite subinterval of $\mathbb{R}$. Moreover suppose that the linear equation

$$x'(t) = A(t)x(t) \tag{1}$$

has a trichotomy with constants $\alpha \geqslant 1$ and $\sigma > 0$.

(2^0) Let $f\colon \mathbb{R} \times E \to E$ be a function with the following properties:

(i) for each $t \in \mathbb{R}$ $f(t, \cdot)$ is continuous,

(ii) for each $x \in E$ $f(\cdot, x)$ is measurable,

(iii) there exist real nonnegative functions a and b locally integrable on $\mathbb{R}$ such that

$$\|f(t,x)\| \leqslant a(t) + b(t) \cdot \|x\|$$

for each $t \in \mathbb{R}$ and $x \in E$. Assume in addition that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} a(s)\,\mathrm{d}s \leqslant M_1, \tag{A}$$

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} b(s)\,\mathrm{d}s \leqslant M_2, \tag{B}$$

where $0 < M_1 < \infty$ and $0 < M_2 < \frac{1-e^{-\sigma}}{2^\alpha}$.

(3^0) Let $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and let $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function. Assume

$$\mu(f(I \times X)) \leqslant \sup_{t\in I} g(t) \cdot h(\mu(X)) \tag{C}$$

for each compact subinterval I of $\mathbb{R}$ and each bounded subset X of E.

(4^0) Put

$$L = \sup\left\{ \int_{\mathbb{R}} |G(t,s)| g(s)\,\mathrm{d}s : t \in \mathbb{R} \right\}.$$

Assume that $0 < L < \infty$ and $L \cdot h(t) < t$ for $t > 0$.

Theorem. *Under the above hypotheses there exists at least one bounded solution of*

$$x'(t) = A(t)x(t) + f(t, x(t)) \tag{4}$$

on $\mathbb{R}$.

Proof. Let $\alpha \geqslant 1$ and $\sigma > 0$ be constants from Definition 1 (assumption (1^0)), so $|G(t,s)| \leqslant \alpha \cdot e^{-\sigma|t-s|}$, $t, s \in \mathbb{R}$.

By H we denote the following set:

$$H = \Big\{ x \in C(\mathbb{R}, E) \colon \|x(t)\| \leqslant K,$$
$$\|x(t) - x(\tau)\| \leqslant K \int_\tau^t |A(s)| \, ds + \int_\tau^t a(s) \, ds + K \int_\tau^t b(s) \, ds \,, \ \tau, t \in \mathbb{R} \Big\},$$

where $K = 2\alpha M_1/(1 - e^{-\sigma} - 2\alpha M_2)$.

Note that $K > 0$.

It is clear that H is nonempty, closed, bounded, almost equicontinuous and convex in $C(\mathbb{R}, E)$.

For each $x \in H$ we define

$$F(x)(t) = \int_{\mathbb{R}} G(t,s) f(s, x(s)) \, ds \,.$$

Thus

$$\|F(x)(t)\| = \Big\| \int_{\mathbb{R}} G(t,s) f(s, x(s)) \, ds \Big\| = \Big\| \sum_{m=-\infty}^{\infty} \int_{t+m}^{t+m+1} G(t,s) f(s, x(s)) \, ds \Big\|$$
$$\leqslant \sum_{m=-\infty}^{\infty} \int_{t+m}^{t+m+1} \alpha e^{-\sigma|t-s|} \|f(s, x(s))\| \, ds$$
$$\leqslant \sum_{m=-\infty}^{\infty} \int_{t+m}^{t+m+1} \alpha e^{-\sigma|t-s|} (a(s) + K b(s)) \, ds \,.$$

Similarly as in ([6], Lemma 5.1) we have the estimate

$$\|F(x)(t)\| \leqslant \frac{2\alpha(M_1 + M_2 K)}{1 - e^{-\sigma}} = K.$$

Furthermore, since $F(x)$ is a solution of

$$y'(t) = A(t)y(t) + f(t, x(t))$$

(see [6], Th. 5.2), for $\tau \leqslant t$ $(t, \tau \in \mathbb{R})$ we have

$$\begin{aligned}\|F(x)(t) - F(x)(\tau)\| &\leqslant \int_\tau^t \|A(s)F(x)(s) + f(s, x(s))\| \,\mathrm{d}s \\ &\leqslant K \cdot \int_\tau^t |A(s)| \,\mathrm{d}s + \int_\tau^t (a(s) + K \cdot b(s)) \,\mathrm{d}s .\end{aligned}$$

Finally, $F(x) \in H$ and $F\colon H \to H$.

Now we are in a position to show that F is continuous.

Let (x_n) be a sequence which converges to x in $C(\mathbb{R}, E)$ $(x_n, x \in H,\ n = 1, 2, \ldots)$ and let I be an arbitrary compact subset of $\mathbb{R}$.

Put $\varrho_1 = \inf\{t\colon t \in I\}$ and $\varrho_2 = \sup\{t\colon t \in I\}$. Fix an arbitrary $t \in I$. We have

$$\begin{aligned}\|F(x_n)(t) - F(x)(t)\| &\leqslant \int_{-\infty}^t |G(t,s)| \cdot \|f(s, x_n(s)) - f(s, x(s))\| \,\mathrm{d}s \\ &\quad + \int_t^\infty |G(t,s)| \cdot \|f(s, x_n(s)) - f(s, x(s))\| \,\mathrm{d}s \\ &\leqslant \int_{-\infty}^t \alpha \mathrm{e}^{-\sigma(t-s)} \|f(s, x_n(s)) - f(s, x(s))\| \,\mathrm{d}s \\ &\quad + \int_t^\infty \alpha \mathrm{e}^{-\sigma(s-t)} \|f(s, x_n(s)) - f(s, x(s))\| \,\mathrm{d}s \\ &= \alpha \mathrm{e}^{-\sigma t} \int_{-\infty}^t \mathrm{e}^{\sigma s} \|f(s, x_n(s)) - f(s, x(s))\| \,\mathrm{d}s \\ &\quad + \alpha \mathrm{e}^{\sigma t} \int_t^\infty \mathrm{e}^{-\sigma s} \|f(s, x(s)) - f(s, x(s))\| \,\mathrm{d}s \\ &\leqslant \frac{1}{\sigma} \alpha \mathrm{e}^{-\sigma \varrho_1} \int_{-\infty}^{\varrho_2} \|f(s, x_n(s)) - f(s, x(s))\| d(\mathrm{e}^{\sigma s}) \\ &\quad - \frac{1}{\sigma} \alpha \mathrm{e}^{\sigma \varrho_2} \int_{\varrho_1}^\infty \|f(s, x_n(s)) - f(s, x(s))\| d(\mathrm{e}^{-\sigma s}).\end{aligned}$$

By the Lebesgue dominated convergence theorem we deduce that F is continuous from H into itself.

Set $\xi(Y) = \sup\{\mu(Y(t))\colon t \in \mathbb{R}\}$ for each subset Y of H. By Lemma 3 this index ξ satisfies our conditions (L_1)–(L_5) listed in Lemma 1.

Choose arbitrary $t \in \mathbb{R}, \varepsilon > 0$ and $Y \subset H$.

Note that Y is almost equicontinuous as a subset of H. By the definition of μ for each $\varepsilon > 0$ there exists $V \in \mathscr{B}'$ such that $p(V) \leqslant \varepsilon$. Let us denote by δ a positive constant such that $B(0, \delta) \subset V$. Let $q > 0$ be such that $K \cdot \mathrm{e}^{-\delta q} < 2\delta$.

Thus, if we denote by A_1 the set $\left\{ \int\limits_{-\infty}^{t-q} G(t,s)f(s,y(s))\,\mathrm{d}s : y \in Y \right\}$, then $\mu(A_1) \leqslant p(V) \leqslant \varepsilon$, because

$$\|A_1\| \leqslant \int_{-\infty}^{t-q} \alpha e^{-\delta(t-s)}(a(s) + Kb(s))\,\mathrm{d}s \leqslant \frac{Ke^{-\delta q}}{2} < \delta,$$

$$\text{so } A_1 \subset \{0\} + B(0,\delta) \subset \{0\} + V.$$

Analogously $\mu\Big(\Big\{ \int\limits_{t+q}^{\infty} G(t,s)f(s,y(s))\,\mathrm{d}s : y \in Y \Big\}\Big) \leqslant \varepsilon$ (cf. [11], assumption (e)).

In the sequel, with no loss of generality, we will assume that $0 \notin (t-q, t+q)$.

For an arbitrary $\varepsilon_1 > 0$ there exists a $\delta_1 > 0$ such that $|s_1 - s_2| < \delta_1$ with $s_1, s_2 \in [t-q, t]$ or $s_1, s_2 \in [t, t+q]$ implies $|G(t,s_1) - G(t,s_2)| < \varepsilon_1$ and $|g(s_1) - g(s_2)| < \varepsilon_1$. Let $t_0 = t - q < t_1 < \ldots < t_k = t < \ldots < t_{2k} = t + q$ be a partition of $[t-q, t+q]$ with $t_i - t_{i-1} < \delta_1$ for each $i = 1, 2, \ldots, 2k$.

The interval $[t_{i-1}, t_i]$ will be denoted by I_i.

By continuity of g and $G(t,\cdot)$ (except $G(t,t)$) there exist points $\tau_i, s_i \in I_i$ such that

$$\begin{aligned} |G(t,s_i)| &= \sup\{|G(t,s)| \colon s \in I_i\}, \\ g(\tau_i) &= \sup\{g(s) \colon s \in I_i\}. \end{aligned} \tag{5}$$

Let

$$\begin{aligned} c_1 &= \sup\{|G(t,s)| \colon t-q \leqslant s \leqslant t+q\}, \\ c_2 &= \sup\{g(s) \colon t-q \leqslant s \leqslant t+q\}. \end{aligned}$$

For simplicity, we will denote the set $\{Y(s) \colon t-q \leqslant s \leqslant t+q\}$ by Z. Obviously

$$\mu(Z) = \sup\{\mu(Y(s)) \colon t-q \leqslant s \leqslant t+q\} \leqslant \xi(Y).$$

By the mean value theorem we get

$$\left\{ \int_{t-q}^{t+q} G(t,s)f(s,y(s))\,\mathrm{d}s : y \in Y \right\} \subset \sum_{i=1}^{2k} (t_i - t_{i-1})\, \overline{\mathrm{conv}}\Big(\bigcup_{s \in I_i} G(t,s)f(I_i \times Z) \Big).$$

So, by Lemma 2 (see also [3])

$$\mu\Big(\bigcup_{s \in I_i} G(t,s)f(I_i \times Z) \Big) \leqslant \sup_{s \in I_i} |G(t,s)| \mu(f(I_i \times Z))$$

and consequently, by our assumptions

$$\begin{aligned}
\mu\Big(\Big\{\int_{t-q}^{t+q} G(t,s)f(s,y(s))\,\mathrm{d}s : y\in Y\Big\}\Big)& \\
&\leqslant \sum_{i=1}^{2k}(t_i-t_{i-1})\cdot\sup_{s\in I_i}|G(t,s)|\cdot\mu(f(I_i\times Y))\\
&\leqslant \sum_{i=1}^{2k}(t_i-t_{i-1})\cdot|G(t,s_i)|\cdot\sup_{s\in I_i}g(s)\cdot h(\mu(Y))\\
&= h(\mu(Y))\cdot\sum_{i=1}^{2k}(t_i-t_{i-1})\cdot|G(t,s_i)|\cdot g(\tau_i)\\
&\leqslant h(\mu(Y))\cdot\sum_{i=1}^{2k}\int_{I_i}(|G(t,s_i)-G(t,s)|\cdot g(\tau_i)\\
&\quad+|G(t,s)|\cdot|g(\tau_i)-g(s)|+|G(t,s)|\cdot g(s))\,\mathrm{d}s\\
&\leqslant h(\mu(Z))\cdot[2q(c_1+c_2)\varepsilon_1+\int_{t-q}^{t+q}|G(t,s)|\cdot g(s)\,\mathrm{d}s\,].
\end{aligned}$$

Since ε_1 is arbitrarily small, we get

$$\begin{aligned}
\mu\Big(\Big\{\int_{t-q}^{t+q} G(t,s)f(s,y(s))\,\mathrm{d}s : y\in Y\Big\}\Big)& \\
&\leqslant h(\mu(Z))\cdot\int_{\mathbf{R}}|G(t,s)|\cdot g(s)\,\mathrm{d}s\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\mu(F(Y)(t)) &\leqslant \varepsilon + h(\mu(Z))\cdot\int_{\mathbf{R}}|G(t,s)|\cdot g(s)\,\mathrm{d}s\,+\varepsilon_1\\
&\leqslant 2\varepsilon + L\cdot h(\mu(Z)) \leqslant 2\varepsilon + L\cdot h(\xi(Y)),
\end{aligned}$$

so $\mu(F(Y)(t))\leqslant L\cdot h(\xi(Y))$ and consequently

$$\xi(F(Y))\leqslant L\cdot h(\xi(Y)).$$

By (4^0) and Sadovskii's fixed point theorem ([12]) F has a fixed point in H, which is a bounded solution of (4) (cf. [7], Lemma 7). The proof is complete. □

As claimed in Introduction, this theorem is an extension of previous results: in the case of trichotomy of (1) it generalizes a famous result of Elaydi and Hajek ([6], Th. 5.4) and in the case of exponential dichotomy of (1) other mentioned results ([2], [5], [8], [11], [13], for example).

References

[1] *J. Banaś and K. Goebel*: Measures of Noncompactness in Banach Spaces. Lect. Notes Pure Applied Math., vol. 60, New York, 1980.

[2] *M.A. Boudourides*: On bounded solutions of nonlinear ordinary differential equations. Comm. Math. Univ. Carolinae *22* (1981), 15–26.

[3] *M. Cichoń*: A point of view on measures of noncompactness. Demonstr. Math. *26* (1993). in press.

[4] *J.L. Daleckii and M.G. Krein*: Stability of Solutions of Ordinary Differential Equations in Banach Spaces. Moscow, 1970. (In Russian.)

[5] *M. Dawidowski and B. Rzepecki*: On bounded solutions of nonlinear differential equations in Banach spaces. Demonstratio Math. *18* (1985), 91–102.

[6] *S. Elaydi and O. Hajek*: Exponential trichotomy of differential systems. Jour. Math. Anal. Appl. *129* (1988), 362–374.

[7] *S. Elaydi and O. Hajek*: Exponential dichotomy and trichotomy of nonlinear differential equations. Diff. Integral Equations *3* (1990), 1201–1204.

[8] *D.L. Lovelady*: Bounded solutions of whole-line differential equations. Bull. AMS *79* (1972), 752–753.

[9] *J.L. Massera and J.J. Schäffer*: Linear Differential Equations and Function Spaces. New York-London, 1966.

[10] *P. Preda and M. Megan*: Exponential dichotomy of evolutionary processes in Banach spaces. Czechoslovak Math. Jour. *35* (1985), 312–323.

[11] *B. Przeradzki*: The existence of bounded solutions for differential equations in Hilbert spaces. Ann. Polon. Math. *56* (1992), 103–121.

[12] *B.N. Sadovskii*: A fixed point principle. Functional Analysis and its Applications *1* (1967), 151–153. (In Russian.)

[13] *S. Szufla*: On the existence of bounded solutions of nonlinear differential equations in Banach spaces. Funct. Approx. *15* (1986), 117–123.

Author's address: *Mieczysław Cichoń*, Faculty of Mathematics and Informatics, A. Mickiewicz University, Matejki 48/49 60-769 Poznań, Poland.