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ON THE STRUCTURE OF FIXED POINT SETS OF SOME COMPACT MAPS IN THE FRÉCHET SPACE

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Summary. The aim of this note is
1. to show that some results (concerning the structure of the solution set of equations (18) and (21)) obtained by Czarnowski and Pruszko in [6] can be proved in a rather different way making use of a simple generalization of a theorem proved by Vidossich in [8]; and
2. to use a slight modification of the "main theorem" of Aronszajn from [1] applying methods analogous to the above mentioned idea of Vidossich to prove the fact that the solution set of the equation (24), (25) (studied in the paper [7]) is a compact R_δ.

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1. PRELIMINARIES

A non-empty subset F of a metric space X is said to be a *compact R_δ-set in the space X* if F is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts (cf. [5, Section 3]).

(1.1) Lemma ([1, Théorème B], [5, Lemma 5]). *Let X be a metric space, $\{A_n\}$ a sequence of compact absolute retracts in X, F a non-empty subset of X such that*

(i) $\forall n \in \mathbb{N}$: $F \subset A_n$;

(ii) *for each neighbourhood V of F in X there exists an $n_0 \in \mathbb{N}$ such that $A_n \subset V$ for each $n > n_0$.*

Then F is a compact R_δ-set.

(1.2) Theorem (cf. [1, Section 3]). *Let M be a non-empty closed set in a Fréchet space (X,d), $T\colon M \to X$ a compact map (i.e. T is continuous and $T(M)$ is a relatively* *compact set); denote by S the map $I - T$, where I is the identity map on X. Let there exist a sequence $\{U_n\}$ of closed convex sets in X fulfilling*

(iii) $\forall n \in \mathbb{N}: 0 \in U_n$;

(iv) $\lim_{n\to\infty} \operatorname{diam} U_n = 0$

and a sequence $\{T_n\}$ of maps $T_n: M \to X$ fulfilling

(v) $\forall n \in \mathbb{N} \forall x \in M: Tx - T_n x \in U_n$;

(vi) *the map $S_n := I - T_n$ is a homeomorphism of the set $S_n^{-1}(U_n)$ onto U_n.*

Then the set F of all fixed points of the map T is a compact R_δ-set.

Proof. 1. First we shall prove the non-emptiness of the set F. The conditions (vi) and (iii) imply

$$\forall n \in \mathbb{N} \exists x_n \in M: S_n x_n = 0.$$

By (iii) and (v) we have

$$d(Sx_n, 0) = d(Sx_n - S_n x_n, 0) = d(T_n x_n - Tx_n, 0) = d(0, Tx_n - T_n x_n) \leqslant \operatorname{diam} U_n,$$

so by (iv)

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} (x_n - Tx_n) = 0.$$

As T is a compact map and the set M is closed, we must have $Sy = 0$ for some $y \in M$, i.e. the set F is non-empty; by the same argument F is a compact set.

2. Now we shall prove that the sequence $\{A_n\}$ defined by

$$A_n = S_n^{-1}(\overline{\text{co}}\, S_n(F)) \tag{1}$$

and the set F fulfil the conditions (i) and (ii). The assumption (v) implies the inclusion

$$\forall n \in \mathbb{N}: S_n(F) \subset U_n, \tag{2}$$

so by (vi) the set $S_n(F)$ is compact as a continuous image of the compact set F. According to the Mazur theorem the set $\overline{\text{co}}\, S_n(F)$ is convex and compact. As the set U_n is convex and closed, (2) implies

$$\forall n \in \mathbb{N}: \overline{\text{co}}\, S_n(F) \subset U_n. \tag{3}$$

By (vi) and (1) the set A_n is a homeomorphic image of a compact convex set in a locally convex linear space, therefore A_n is a compact absolute retract (see [4, Chapter 4, Theorem 2.1]). As the condition (i) is evidently fulfilled, it suffices to verify (ii). That will be done by a contradiction; let there exist an open neighbourhood V of the set F and a sequence $\{n_k\} \subset \mathbf{N}$ such that

$$\forall k \in \mathbf{N} \exists x_k \in A_{n_k} \setminus V. \tag{4}$$

Then

$$\begin{aligned} d(Sx_k, 0) &= d(Sx_k - S_{n_k}x_k, -S_{n_k}x_k) = d(T_{n_k}x_k - Tx_k, -S_{n_k}x_k) \\ &= d(Tx_k - T_{n_k}x_k, S_{n_k}x_k) \leqslant d(Tx_k - T_{n_k}x_k, 0) + d(S_{n_k}x_k, 0) \\ &\leqslant 2\operatorname{diam} U_{n_k} \end{aligned}$$

(the inequality $d(S_{n_k}x_k, 0) \leqslant \operatorname{diam} U_{n_k}$ is a consequence of (4), (1), (3) and (iii)), so due to (iv)

$$\lim_{k\to\infty} Sx_k = 0. \tag{5}$$

Owing to (5) and to the fact that T is a compact map and the set M is closed, there exist a $y \in F$ and a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ such that

$$\lim_{m\to\infty} x_{k_m} = y. \tag{6}$$

However, (6), (4) and the fact that V is an open set imply $y \notin V$, which contradicts the inclusion $y \in F \subset V$. This completes the proof. □

(1.3) Remarks. 1. From the preceding proof it is easy to see that the assertion of Theorem (1.2) remains in force if the assumption "$T(M)$ is a relatively compact set" is replaced by

(vii) *every sequence $\{x_n\}$ such that $\lim_{n\to\infty} Sx_n = 0$ contains a convergent subsequence (Palais-Smale condition).*

2. In the case that only the proof of non-emptiness, compactness and connectedness of F is needed, it suffices to require that $\{U_n\}$ is a sequence of closed connected sets and (iii), (iv), (v), (vi) and (vii) are fulfilled.

3. The "main theorem" in Aronszajn [1] was modified several times (see, e.g., [6, Lemma (3.1)] or [9, Theorem 2.4]), but all modifications contain the requirement that each "approximating" map S_n is a homeomorphism of $S_n^{-1}(U_n)$ onto U_n, where U_n is a neighbourhood of 0. A "main theorem" of this form cannot be used, e.g., to prove Theorem (2.1) of this paper and therefore in our modification the condition "U_n is a neighbourhood of 0" is replaced by "U_n closed and convex" and conditions (iii) and (iv).

(1.4) Corollary. *Let X, d, T, M, I have the same meaning as in (1.1). If there exists a sequence $\{T_m\}$ of continuous maps $T_m: M \to X$ fulfilling*

(viii) *$I - T_m$ is a homeomorphism of M onto X for each $m \in \mathbb{N}$;*

(ix) *$\{T_m\}$ converges uniformly to T (i.e. $\lim_{m\to\infty} \sup\{d(T_m x, Tx);\ x \in M\} = 0$), then the set F of all fixed points of T is a compact R_δ-set.*

Proof. As X is a Fréchet space, there exists a sequence $\{U_n\}$ of closed convex neighbourhoods of the point $0 \in X$ such that $U_n \subset B(0, 1/n)$ (where $B(0,\varepsilon)$ denotes the closed ball of centre 0 and radius ε), consequently $\operatorname{diam} U_n \leqslant 2/n$. As U_n is a neighbourhood of the point 0,

$$\forall n \in \mathbb{N}\, \exists \varepsilon_n > 0: B(0, \varepsilon_n) \subset U_n$$

and $\lim_{n\to\infty} \varepsilon_n = 0$. In view of (ix) there exists a subsequence $\{T_{m_n}\}$ such that

$$d(T_{m_n} x, Tx) < \varepsilon_n, \quad x \in M,$$

and so $Tx - T_{m_n} x \in U_n$, $x \in M$. The sequences $\{U_n\}$, $\{T_{m_n}\}$ fulfil (iii), (iv), (v), (vi), thus our assertion is a consequence of Theorem (1.2). □

Remark. The statement of the preceding Corollary is known (it can be derived, e.g., from [9, Theorem 2.4]); its simple proof based on Theorem (1.2) is given here only for the sake of completness, as it is an essential part of the proof of Theorem (2.1).

(1.5) Let K be an unbounded convex subset of a normed space $(Z, |.|)$; let $(Y, \|.\|)$ be a Banach space. Let X be the space of all continuous locally bounded maps $f: K \to Y$ equipped with the topology of locally uniform convergence (i.e. X is a Fréchet space whose topology is given by the metric

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(f-g)}{1 + p_n(f-g)}, \tag{7}$$

where

$$p_n(f) = \sup\{\|f(t)\|;\ t \in K,\ |t| \leqslant n\}).$$

Theorem (cf. [8, Theorem 1.1]). *Let $T: X \to X$ be a continuous map, $S = I - T$ (where I denotes the identity map on X). Suppose*

(x) *$\exists t_0 \in K\, \exists y_0 \in Y\, \forall x \in X: Tx(t_0) = y_0$;*

(xi) *$T(X)$ is a set of locally equiuniformly continuous maps, i.e.*

$$\forall \varepsilon > 0\, \forall \eta > 0\, \exists \delta > 0\, \forall x \in X\, \forall t_1, t_2 \in K_\eta:$$

$$|t_1 - t_2| < \delta \Longrightarrow \|Tx(t_1) - Tx(t_2)\| < \varepsilon,$$

where $K_\varepsilon := B(t_0, \varepsilon) \cap K$ *and* $B(t_0, \varepsilon)$ *is the closed ball of center* t_0 *and radius* ε;

(xii) $\forall \varepsilon > 0 \, \forall x, y \in X : x|K_\varepsilon = y|K_\varepsilon \Longrightarrow (Tx)|K_\varepsilon = (Ty)|K_\varepsilon$.

Then there exists a sequence $\{S_n\}$ *of homeomorphisms of* X *onto* X *such that*

$$\lim_{n\to\infty} \sup\{d(S_n x, Sx) ; x \in X\} = 0. \tag{8}$$

Proof. The line of the proof is the same as in Vidossich [8]. Let $n \in \mathbb{N}$; for $t \in K$, $|t - t_0| \geqslant 1/n$, put

$$\alpha_n(t) = t - \frac{1}{n|t - t_0|}(t - t_0) \qquad \left(= \left(1 - \frac{1}{n|t - t_0|}\right)t + \frac{1}{n|t - t_0|}t_0\right),$$

due to the convexity of K, we have $\alpha_n(t) \in K$. Further

$$|\alpha_n(t) - t_0| = |t - t_0| - \frac{1}{n}, \qquad |\alpha_n(t) - t| = \frac{1}{n}. \tag{9}$$

The equality

$$T_n x(t) = \begin{cases} y_0, & \text{if } |t - t_0| \leqslant 1/n \\ (Tx)(\alpha_n(t)), & \text{if } |t - t_0| \geqslant 1/n \end{cases}, \quad x \in X, \tag{10}$$

defines a continuous map $T_n : X \to X$. Denote $S_n := I - T_n$, where I is the identity map on X.

To prove the injectivity of S_n, suppose

$$x(t) - T_n x(t) = y(t) - T_n y(t) \qquad t \in K, \tag{11}$$

and denote $C_i := \{t \in K ; (i-1)/n \leqslant |t - t_0| \leqslant i/n\}$ $(i \in \mathbb{N})$. The equalities (11) and

$$T_n x(t) = T_n y(t) = y_0 \quad \text{for} \quad t \in C_1$$

imply

$$x|C_1 = y|C_1. \tag{12}$$

From (12) and (xii) it follows that $Tx|C_1 = Ty|C_1$. As by (9) we have $\alpha_n(t) \in C_1$ for $t \in C_2$, owing to (10) we have

$$T_n x(t) = (Tx)(\alpha_n(t)) = (Ty)(\alpha_n(t)) = T_n y(t)$$

for $t \in C_2$; this and (11) imply $x|C_2 = y|C_2$. Now we can proceed by induction.

To prove the surjectivity of S_n, let us choose $y \in X$ and look for an $x \in X$ such that $S_n x = y$. From the equalities

$$x(t) - T_n x(t) = y(t) \quad \text{for} \quad t \in K \qquad \text{and} \qquad T_n x(t) = y_0 \quad \text{for} \quad t \in C_1$$

we have for such an x:

$$x(t) = y(t) + y_0 \quad \text{for} \quad t \in C_1.$$

As C_1 is a bounded set and $y \in X$, the set $\{y(t) + y_0\,;\, t \in C_1\}$ is bounded. C_1 is a closed subset of the metric space K, so by the Dugundji extension theorem there exists a bounded continuous map $x_1\colon K \to X$ such that $x_1|C_1 = y|C_1 + y_0$. For $t \in C_2$ we have

$$y(t) = x(t) - T_n x(t) = x(t) - (Tx)(\alpha_n(t)) = x(t) - (Tx_1)(\alpha_n(t))$$

(the last equality is a consequence of (xii)), hence

$$x(t) = y(t) + T_n x_1(t) \quad \text{for} \quad t \in C_2.$$

C_1 and C_2 are closed subsets of the metric space K, the map X_1 is continuous on C_1, the map $y + T_n x_1$ is continuous on C_2 and for $t \in C_1 \cap C_2$ (i.e. $|t - t_0| = 1/n$) we have

$$y(t) + T_n x_1(t) = y(t) + y_0,$$

so the map $\overline{x}_2$ defined by

$$\overline{x}_2(t) = \begin{cases} x_1(t), & \text{if } t \in C_1 \\ y(t) + T_n x_1(t), & \text{if } t \in C_2 \end{cases}$$

is continuous on $C_1 \cup C_2$ and its range is bounded. Again due to the Dugundji extension theorem there exists a bounded continuous map $x_2\colon K \to X$ such that $x_2|C_1 \cup C_2 = \overline{x}_2$. Proceeding by induction we can construct a sequence $\{x_m\}_{m=0}^{\infty}$ of bounded continuous maps such that $x_0 = y_0$ and

(13) $$x_{m+1}|C_m = x_m|C_m, \quad m \in \mathbb{N};$$

(14) $$x_m(t) = y(t) + T_n x_{m-1}(t) \quad \text{for} \quad t \in C_m,\ m \in \mathbb{N}.$$

Due to (13), there exists an $x \in X$, $x = \lim_{m\to\infty} x_m$, and for $t \in C_m$ we have

$$x_m(t) = x_{m+1}(t) = \ldots = x(t),$$

so by (14) and (xii) for $t \in C_m$, $m > 1$,

$$\begin{aligned} x(t) = x_m(t) = y(t) + T_n x_{m-1}(t) = y(t) + (Tx_{m-1})(\alpha_n(t)) \\ = y(t) + (Tx)(\alpha_n(t)) = y(t) + T_n x(t), \end{aligned}$$

i.e.

$$x - T_n x = y,$$

the validity of the last equality for $t \in C_1$ beeing a consequence of the definition of the map x_1.

To check the continuity of S_n^{-1}, we suppose

$$\lim_{m\to\infty} (x_m - T_n x_m) = x - T_n x \tag{15}$$

and prove that $\lim_{m\to\infty} x_m = x$. For $t \in C_1$ we have

$$T_n x_m(t) = y_0 = T_n x(t),$$

therefore (15) implies that $\{x_m|C_1\}$ converges on the bounded set C_1 uniformly to $x|C_1$. Put

$$\varepsilon_m = \sup\{\|x_m(t) - x(t)\|;\, t \in C_1\}.$$

Due to the Dugundji extension theorem, there exists a continuous map $\overline{y}_m : K \to X$ such that $\overline{y}_m|C_1 = (x_m - x)|C_1$, $\sup\{\|\overline{y}_m(t)\|;\, t \in K\} \leqslant \varepsilon_m$. For the map $\overline{x}_m := x + \overline{y}_m$

$$\overline{x}_m|C_1 = x_m|C_1 \tag{16}$$

and $\{\overline{x}_m\}$ converges uniformly on K to x, thus

$$\lim_{m\to\infty} T_n \overline{x}_m = T_n x. \tag{17}$$

By (17) and (xii) for $t \in C_2$ we have

$$x_m(t) = \big(x_m(t) - T_n x_m(t)\big) + T_n x_m(t) = \big(x_m(t) - T_n x_m(t)\big) + T_n \overline{x}_m(t),$$

therefore by (15) and (17) $\{x_m\}$ converges uniformly on C_2 to x. Now we can proceed by induction.

It remains to prove (8). With respect to (7) it suffices to prove the equality

$$\left(\lim_{n\to\infty} \sup\{p_m(S_n x - Sx);\, x \in X\} = \right) \quad \lim_{n\to\infty} \sup\{p_m(T_n x - Tx);\, x \in X\} = 0$$

for each $m \in \mathbf{N}$. We have

$$T_n x(t) - Tx(t) = \begin{cases} Tx(t_0) - Tx(t), & \text{if } |t - t_0| \leqslant 1/n \\ (Tx)(\alpha_n(t)) - Tx(t), & \text{if } |t - t_0| \geqslant 1/n \end{cases},$$

and simultaneously (see (9)) $|\alpha_n(t) - t| = 1/n$. For a given $\varepsilon > 0$ the assumption (xi) (for $\eta = m$) implies

$$\exists n_0 \in \mathbf{N}\, \forall n > n_0\, \forall x \in X\, \forall t \in K_m : |T_n x(t) - Tx(t)| < \varepsilon,$$

which completes the proof. □

(1.6) Remark. Vidossich proved in [8] the same theorem for the case "K convex and bounded". From [9, Theorems 1.1, 2.2 and 2.4] the following statement can be obtained (identically as in [8]):

Let X, T, I, S have the same meaning as in (1.5). If T is a closed map and conditions (x), (xi) and (xii) are fulfilled then the fixed points of T form a nonempty connected set which is a compact R_δ whenever it is compact.

The following Theorem (2.1) is than a simple consequence of this statement.

2. Theorems

(2.1) Theorem. *Let X, Y, Z, K have the same meaning as in* (1.5). *If the compact map $T: X \to X$ satisfies* (x), (xi), (xii), *then the set F of all its fixed points is a compact R_δ.*

Proof. The assertion is a consequence of Corollary (1.4) and Theorem from (1.5). □

(2.2) Theorem (cf. [7, Theorem 1]). *Let X be the Fréchet space of all continuous functions $f: [b, \infty) \to \mathbf{R}^\nu$ equipped with the topology of locally uniform convergence (i.e. the topology on X is given by the metric*

$$d(f, g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(f - g)}{1 + p_m(f - g)},$$

where

$$p_m(x) := \sup\{|x(t)|\,;\, t \in [b, b+m]\}$$

and $|.|$ *denotes the norm in* $\mathbf{R}^\nu$*). Let* φ, $\varphi_n \in C([b,\infty),(0,\infty))$, $n \in \mathbf{N}$, *and let the following condition be satisfied*

(xiii) *for each* $t \in [b,\infty)$ *the sequence* $\{\varphi_n(t)\}$ *is non-increasing and* $\lim_{n\to\infty} \varphi_n(t) = 0$.
Let $r \in \mathbf{R}^\nu$ *and* $M = \{x \in X\,;\, |x(t) - r| \leqslant \varphi(t),\ t \geqslant b,\ x(b) = r\}$. *Suppose that* T: $M \to X$ *is a compact map and there exists a sequence* $\{T_n\}$ *of compact maps* T_n: $M \to X$ *such that*

(xiv)

$$|T_n x(t) - Tx(t)| \leqslant \varphi_n(t),\ \ x \in M,\ t \geqslant b;$$

(xv) *for every* $n \in \mathbf{N}$ *there exists a function* $\varphi_{*n} \in C([b,\infty),[0,\infty))$ *such that*

$$\varphi_{*n} + \varphi_n \leqslant \varphi \quad on \quad [b,\infty)$$

and

$$|T_n x(t) - r| \leqslant \varphi_{*n}(t), \quad x \in M,\ t \geqslant b;$$

(xvi) *the map* $S_n := I - T_n$ *is injective on* M.
Then the set F *of all fixed points of the map* T *is a compact* R_δ.

P r o o f. The set

$$U_n := \{x \in X\,;\, |x(t)| \leqslant \varphi_n(t),\ t \geqslant b\}$$

is convex and closed; we shall show that the sequence $\{U_n\}$ satisfies (iii), (iv), (v), (vi). The condition (iii) is evidently fulfilled. For a given $\varepsilon > 0$ there exists an $m_0 \in \mathbf{N}$ such that $\sum_{m=m_0+1}^{\infty} 1/2^m < \varepsilon/2$. (xiii) and the Dini theorem imply that $\{\varphi_n\}$ converges on $[b,\infty)$ locally uniformly to 0, therefore for ε and m_0 there exists an $n_0 \in \mathbf{N}$ such that $p_m(\varphi_n) \leqslant \varepsilon/4m_0$ for $n \geqslant n_0$ and $m = 1, 2, \ldots, m_0$. Thus for $n \geqslant n_0$ and $f, g \in U_n$ we have

$$\begin{aligned} d(f,g) &= \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(f-g)}{1+p_m(f-g)} \leqslant \sum_{m=1}^{m_0} p_m(f-g) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \\ &\leqslant \sum_{m=1}^{m_0} 2p_m(\varphi_n) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \leqslant 2m_0 \cdot \frac{\varepsilon}{4m_0} + \frac{\varepsilon}{2} \\ &\leqslant \varepsilon, \end{aligned}$$

which implies that the condition (iv) is fulfilled. The assumption (v) is true by (xiv). To fulfil (vi) it suffices to verify the inclusion $U_n \subset S_n(M)$; (xvi) then implies that S_n

is a bijection of $S_n^{-1}(U_n)$ onto U_n and the continuity of $S_n^{-1}|U_n$ is then a consequence of the compactness of T_n. Thus we have to prove

$$\forall y \in U_n \, \exists x_y \in M : x_y - T_n x_y = y,$$

i.e. for every $y \in U_n$ the map $P_n(x) = y + T_n(x)$ has a fixed point. (xv) implies

$$|y(t) + T_n x(t) - r| \leqslant |y(t)| + |T_n x(t) - r| \leqslant \varphi_n(t) + \varphi_{*n}(t) \leqslant \varphi(t), \ t \geqslant b,$$

therefore $P_n(M) \subset M$. As M is a closed convex bounded set and P_n is a compact map, due to the Tichonov fixed point theorem, P_n has a fixed point, which completes the proof. □

Remark. The line of the proof of Theorem (2.2) is the same as in [7, Theorem 1], but in our paper Theorem (1.2) is used instead of [7, Lemma 1] which requires U_n to be a neighbourhood of 0 (cf. Remark 3 in (1.3)) and guarantees only nonemptiness, compactness and connectedness of the set F.

3. Applications

(3.1) Let X be the space from paragraph (1.5), where $Z = \mathbf{R}^2$, $|(x,y)| = \max\{|x|,|y|\}$, $K = [0,\infty) \times [0,\infty)$, $Y = \mathbf{R}^\nu$ with the Euclidean norm $\|.\|$. We say that a map $w \colon [0,\infty) \times [0,\infty) \to \mathbf{R}^\nu$ given by the formula $w(t) = (w_1(t), \dots, w_n(t))$ is ***absolutely continuous,*** if $w_i \colon [0,a] \times [0,a] \to \mathbf{R}$ is absolutely continuous for each $a > 0$ and $i = 1, \dots, n$.

Theorem (see [6, Theorem 2.8]). *Suppose that a map $M \colon [0,\infty) \times [0,\infty) \times \mathbf{R}^\nu \to \mathbf{R}^\nu$ satisfies the following assumptions:*

(xvii) *the map $M(x,y,.)$ is continuous for each $(x,y) \in [0,\infty) \times [0,\infty)$;*

(xviii) *the map $M(.,.,u)$ is Lebesgue measurable for each $u \in \mathbf{R}^\nu$;*

(xix) *there exist locally integrable functions $p, c \colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\|M(x,y,u)\| \leqslant p(x,y)\|u\| + c(x,y)$ for all $(x,y,u) \in [0,\infty) \times [0,\infty) \times \mathbf{R}^\nu$. Let $g, h \colon [0,\infty) \to \mathbf{R}^\nu$ be absolutely continuous functions such that $g(0) = h(0)$. Then the set of all solutions of the problem*

$$(18) \qquad \begin{cases} u_{xy}(x,y) = M(x,y,u(x,y)) & \text{for a.a. } (x,y) \in [0,\infty) \times [0,\infty) \\ u(0,y) = g(y), \quad u(x,0) = h(x) & \text{for } x, y \in [0,\infty) \\ u \colon [0,\infty) \times [0,\infty) \to \mathbf{R}^\nu \text{ is absolutely continuous} \end{cases}$$

is a compact R_δ-set in the space X.

Proof. We use Theorem (2.1), for more detail of the following considerations see [6, paragraph (4.5)].

The set of solutions of (18) coincides with the set of solutions of the equation

$$u(x,y) = h(x) + g(y) - h(0) + \int_0^x \int_0^y M(\xi, \eta, u(\xi, \eta)) d\xi d\eta. \tag{19}$$

The assumption (xix) and the Wendroff inequality (see [2]) implies the existence of a continuous function $\alpha\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that for each continuous solution u of (19)

$$\|u(x,y)\| \leqslant \alpha(x,y), \quad (x,y) \in [0,\infty) \times [0,\infty).$$

Then the set of solutions of (19) coincides with the set of solutions of the equation obtained by replacing the function M in (19) by the function

$$\tilde{M}(x,y,u) := \psi\left(\frac{u}{\alpha(x,y)+1}\right) M(x,y,u),$$

where $\psi\colon \mathbf{R}^\nu \to [0,1]$ is a continuous function, $\psi(u) = 1$ for $\|u\| < 1$, $\psi(u) = 0$ for $\|u\| \geqslant 2$. The map $T\colon X \to X$ given by

$$Tu(x,y) = h(x) + g(y) - h(0) + \int_0^x \int_0^y \tilde{M}(\xi, \eta, u(\xi, \eta)) \mathrm{d}\xi \mathrm{d}\eta \tag{20}$$

is compact (the relative compactness of the set $T(X)$ is a consequence of the inequality

$$\|\tilde{M}(x,y,u(x,y))\| \leqslant p(x,y)\big((2\alpha(x,y)+2) + c(x,y)\big).$$

Its compactness implies the fulfilling of the condition (xi), the assumption (x) is valid for $t_0 = (0,0)$, $y_0 = g(0)$; the fulfilling of (xii) is evident owing to (20). Thus by Theorem (2.1), the set of solutions of (18) is a compact R_δ. □

Remark. The statement of the preceding theorem is identical with [6, Theorem (2.8)] the proof of which in [6] is based on Corollary (1.4), too, but the existence of the sequence $\{T_m\}$ (which in our paper is a consequence of (1.5)) is proved in a different way and the proof of the fact that $I - T_m$ is a homeomorphism is based on the Lasota-Opial condition. Similarly the difference between the proofs of the following Theorem (3.2) and [6, Theorem (2.8)] (whose statements are identical, too) is only in the method of constructing the sequence $\{T_m\}$.

(3.2) Let X be the space from paragraph (1.5), where $Z = \mathbf{R}$, $|.|$ is the Euclidean norm on $\mathbf{R}$, $K = [0, \infty)$, $Y = \mathbf{R}^\nu$, $\|.\|$ is the Euclidean norm on $\mathbf{R}^\nu$.

Theorem (see [6, Theorem (2.9)]. *Suppose that a map $M : [0, \infty) \times [0, \infty) \times \mathbf{R}^\nu \to \mathbf{R}^\nu$ satisfies the following conditions:*

(xx) *the map $M(s, ., .): [0, \infty) \times \mathbf{R}^\nu \to \mathbf{R}^\nu$ is continuous for each $s \in [0, \infty)$;*

(xxi) *the map $M(., t, x): [0, \infty) \to \mathbf{R}^\nu$ is Lebesgue measurable for each $(t, x) \in [0, \infty) \times \mathbf{R}^\nu$;*

(xxii) *there exist locally integrable functions $p, c: [0, \infty) \to [0, \infty)$ such that*

$$\|M(s, t, x)\| \leqslant p(s)\|x\| + c(s) \quad \text{for all} \quad (s, t, x) \in [0, \infty) \times [0, \infty) \times \mathbf{R}^\nu.$$

Then the set of all continuous solutions of the integral equation

$$x(t) = \int_0^t M(s, t, x(s))\,\mathrm{d}s \tag{21}$$

is a compact R_δ in the space X.

Proof. Theorem (2.1) is applied again; more detail can be found in [6, paragraph (4.6).

The assumption (xxii) and the Gronwall inequality imply the existence of a continuous function $\alpha: [0, \infty) \to [0, \infty)$ such that for each continuous solution x of the equation (21)

$$\|x(t)\| \leqslant \alpha(t), \ t \geqslant 0. \tag{22}$$

Put

$$\tilde{M}(s, t, x) = \psi\left(\frac{x}{\alpha(s) + 1}\right) M(s, t, x),$$

where $\psi: \mathbf{R}^\nu \to [0, 1]$ is a continuous function, $\psi(u) = 1$ for $\|u\| \leqslant 1$, $\psi(u) = 0$ for $\|u\| \geqslant 2$. Then the set of continuous solutions of (21) coincides with the set of continuous solutions of the equation

$$x(t) = \int_0^t \tilde{M}(s, t, x(s))\mathrm{d}s.$$

The map $T: X \to X$ defined by

$$Tx(t) = \int_0^t \tilde{M}(s, t, x(s))\mathrm{d}s$$

is compact and fulfils (x), (xi), (xii), so by Theorem (2.1) the set of continuous solutions of (21) is a compact R_δ. □

(3.3) **Remark.** The crucial point of the preceding proof is the existence of the bound for solutions given in (22). A generalization of the Gronwall inequality (which was used to obtain (22)) is the following Bihari inequality.

Lemma (see [3]). *Let* $u\colon [a,b] \to [0,\infty)$ *be a continuous function,* $p\colon [a,b] \to (0,\infty)$ *a locally integrable function,* $k > 0$, $\omega\colon [0,\infty) \to [0,\infty)$ *a non-decreasing function; suppose*

$$\Omega(k) + \int_a^t p(s)\,\mathrm{d}s \leqslant \lim_{s\to\infty} \Omega(s) \quad \textit{for each} \quad t \in [a,b],$$

where

$$\Omega(s) := \int_{u_0}^{s} \frac{\mathrm{d}t}{\omega(t)}, \qquad u_0 > 0,\ s \geqslant 0.$$

Then the inequality

$$u(t) \leqslant k + \int_a^t p(s)\omega(u(s))\,\mathrm{d}s, \quad t \in [a,b],$$

implies the inequality

$$u(t) \leqslant \Omega^{-1}\left(\Omega(k) + \int_0^t p(s)\,\mathrm{d}s\right), \quad t \in [a,b].$$

Replacing in the proof of Theorem from (3.2) the use of the Gronwall inequality by the preceding lemma, we can generalize the assertion of that theorem as follows:

Theorem. *Suppose that a map* $M\colon [0,\infty)\times[0,\infty)\times\mathbf{R}^\nu \to \mathbf{R}^\nu$ *satisfies* (xx), (xxi) *and*

(xxiii) *there exist locally integrable functions* $p, c\colon [0,\infty) \to (0,\infty)$ *and a non-decreasing function* $\omega\colon [0,\infty) \to [0,\infty)$ *such that*

$$\|M(s,t,x)\| \leqslant p(s)\omega(\|x\|) + c(s) \ \textit{for all}\ (s,t,x) \in [0,\infty)\times[0,\infty)\times\mathbf{R}^\nu$$

and

$$\int_0^u p(s)\mathrm{d}s \leqslant \int_{k(u)}^{\infty} \frac{\mathrm{d}s}{\omega(s)} \quad \textit{for each} \quad u > 0,$$

where $k(u) = \int_0^u c(s)\,\mathrm{d}s$.

Then the set of all continuous solutions of the equation (21) *is a compact* R_δ.

(3.4) Let $h > 0$, $b \in \mathbf{R}$, $H = C([-h,0],\mathbf{R}^\nu)$, $\|x\| = \max\{|x(s)|;\ s \in [-h,0]\}$ for $x \in H$ ($|.|$ denotes the Euclidean norm in $\mathbf{R}^\nu$), let X^* be the space $C([b-h,\infty),\mathbf{R}^\nu)$

equipped with the topology of locally uniform convergence. For $x \in X^*$ denote by $x_t \in H$ the function $x_t(s) := x(t+s)$, $s \in [-h, 0]$. Let X have the same meaning as in paragraph (2.2).

Theorem (cf. [7, Theorem 2]). *Let $\psi \in H$, $f \in C([b,\infty) \times H, \mathbf{R}^\nu)$, $\omega \in C([b,\infty),(0,\infty))$, let $g \in C([0,\infty),(0,\infty))$ be a non-decreasing function and*

(xxiv)

$$\int_b^\infty \omega(s)\,\mathrm{d}s \leqslant \int_0^\infty \frac{\mathrm{d}v}{g(v+|\psi(0)|)}.$$

Let

(xxv)

$$|f(t,\chi)| \leqslant \omega(t) g(\|\chi\|) \text{ for each } (t,\chi) \in [b,\infty) \times M^{**},$$

where

$$M^{**} := \{x_t \in H\,;\, x \in X^*,\ |x(t) - \psi(0)| \leqslant \varphi(t) \text{ on } [b,\infty),\ x_b = \psi\}$$

and φ is the solution of the equation

$$y'(t) = \omega(t) g(y + |\psi(0)|),\ y(b) = 0,\ t \in [b,\infty). \tag{23}$$

Then the problem

$$x'(t) = f(t, x_t), \quad t \in [b,\infty), \tag{24}$$

$$x_b = \psi, \tag{25}$$

has a solution satisfying the inequality

$$|x(t) - \psi(0)| \leqslant \varphi(t), \quad t \in [b,\infty),$$

and the set F^* of all such solutions is a compact R_δ in the space X^*.

Remark. (xxiv) is a sufficient condition for the existence of a solution of the equation (23).

Proof of the Theorem. For more detail concerning the following considerations see [7].

The set

$$M = \{x \in X\,;\, |x(t) - \psi(0)| \leqslant \varphi(t) \text{ on } [b,\infty),\ x(b) = \psi(0)\}$$

is a non-empty closed subset of X. Put

$$M^* = \{x \in X^*;\ |x(t) - \psi(0)| \leqslant \varphi(t) \text{ on } [b, \infty),\ x_b = \psi\}.$$

Evidently the map $P\colon X^* \to X$ given by $Px = x|[b, \infty)$ is a homeomorphism of M^* onto M. Let the map $T\colon M \to X$ be defined by

$$Tx(t) = \psi(0) + \int_b^t f(s, (Px)_s)\,\mathrm{d}s, \quad t \in [b, \infty).$$

Then $F^* = P^{-1}(F)$, where F is the set of all fixed points of the map T. As a homeomorphic image of a compact R_δ-set is again a compact R_δ-set, it suffices to prove that F is a compact R_δ-set. That can be done using Theorem (2.2), we put $r = \psi(0)$. The maps $T_n\colon M \to X$ defined by

$$T_n x(t) = \begin{cases} \psi(0), & \text{if } t \in [b, b + 1/n] \\ \psi(0) + \int_b^{t-1/n} f(s, (Px)_s)\,\mathrm{d}s, & \text{if } t \in [b + 1/n, \infty) \end{cases}$$

are compact (it is a consequence of (xxv)) and again by (xxv) we have

$$|T_n x(t) - Tx(t)| \leqslant \varphi_n(t),$$

where

$$\varphi_n(t) := \begin{cases} \int_b^t \omega(s) g(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, & \text{if } t \in [b, b + 1/n] \\ \int_{t-1/n}^t \omega(s) g(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, & \text{if } t \in [b + 1/n, \infty) \end{cases}.$$

The sequence $\{\varphi_n\}$ evidently satisfies the condition (xiii); the last inequality implies the condition (xiv).

For the functions $\varphi_{*n}\colon [b, \infty) \to [0, \infty)$ defined by

$$\varphi_{*n}(t) = \begin{cases} 0, & \text{if } t \in [b, b + 1/n] \\ \int_b^{t-1/n} \omega(s) g(\varphi(s) + |\psi(0)|)\,\mathrm{d}s, & \text{if } t \in [b + 1/n, \infty) \end{cases}$$

we have $|T_n x(t) - \psi(0)| \leqslant \varphi_{*n}(t)$ on $[b, \infty)$ for $x \in M$, and as φ is a solution of (23), we obtain

$$\varphi_{*n}(t) + \varphi_n(t) = \int_b^t \omega(s) g(\varphi(s) + |\psi(0)|)\,\mathrm{d}s = \varphi(t), \quad t \in [b, \infty),$$

thus the condition (xv) is fulfilled.

It remains to check the validity of (xvi): if $x, y \in M$, $x \neq y$, then there exists a $t_0 \in [b, \infty)$ such that $x(t_0) \neq y(t_0)$. Two cases may occur:

a) If $t_0 \in [b, b + 1/n]$, then $x(t_0) - T_n x(t_0) = x(t_0) - \psi(0) \neq y(t_0) - \psi(0) = y(t_0) - T_n y(t_0)$.

b) There exists a $t_1 \geqslant b + 1/n$ such that $t_1 = \sup\{\tau > b;\ x(t) = y(t) \text{ for } t \in [b, \tau)\}$. Then there exists a $t_0 \in (t_1, t_1 + 1/n)$ such that $x(t_0) \neq y(t_0)$. This implies $T_n x(t_0) = \psi(0) + \int_b^{t_0 - 1/n} f(s, (Px)_s)\,\mathrm{d}s = \psi(0) + \int_b^{t_0 - 1/n} f(s, (Py)_s)\,\mathrm{d}s = T_n y(t_0)$, therefore $x(t_0) - T_n x(t_0) \neq y(t_0) - T_n y(t_0)$.

As all assumptions of Theorem (2.2) are fulfilled, our assertion is a consequence of this theorem. □

(3.5) **Remark.** The statement of Theorem in (3.4) is rather stronger as that of [7, Theorem 2] (which guarantees only the fact that F^* is a continuum), though the ideas of the proofs are the same. The reason for this difference is the replacing of [7, Lemma 1] by Theorem (2.2) (cf. Remark in (2.2)).

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