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CONVEX ISOMORPHIC ORDERED SETS

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Summary. V. I. Marmazejev introduced in [5] the following concept: two lattices are convex isomorphic if their lattices of all convex sublattices are isomorphic. He also gave a necessary and sufficient condition under which lattices are convex isomorphic, in particular for modular, distributive and complemented lattices.

The aim of this paper is to generalize this concept to ordered sets and to characterize convex isomorphic ordered sets in the general case of modular, distributive or complemented ordered sets. These concepts were defined in [1], [2], [4].

Keywords: convex ordered set, convex isomorphism

AMS classification: 06A10, 06B15

1. The lattice of convex subsets of an ordered set

Let $A = (A, \leqslant)$ be an ordered set. We say that its subset $S = (S, \leqslant)$ is convex if for each $x \in A$ the following implication holds: if $a, b \in S$, $a \leqslant x \leqslant b$ then $x \in S$. The set of all convex subsets of the ordered set A will be denoted by $CS(A)$; evidently $\emptyset$, $A \in CS(A)$ and any one-element subset of A is convex.

Let $\{X_i\,;\, i \in I\}$ be an arbitrary system of subsets of the ordered set A. The set of all $Z \in CS(A)$ such that $X_i \subset Z$ for each $i \in I$ will be denoted by $CS_A(X_i, i \in I)$. If $\{X_i\,;\, i \in I\} = \{X, Y\}$, we will denote $CS_A(X_i, i \in I)$ briefly by $CS_A(X, Y)$, and for $X = \{a\}$, $Y = \{b\}$ we will use the brief notation $CS_A(a, b)$.

Definition 1.1. Let A be an ordered set and let $M \subseteq A$. Let $CS_A\,\langle M\rangle = \bigcap\{K_i\,;\, i \in I\}$, where K_i run over all convex subsets of A which contain the subset M. As usual, we write $CS_A\,\langle a, b\rangle$ instead of $CS_A\,\langle\{a, b\}\rangle$, etc. The set $CS_A\,\langle a, b\rangle$ is called the convex subset of A generated by the elements a, b.

Evidently, $CS_A\langle a,b\rangle = [a,b]$ for $a \leqslant b$ and $CS_A\langle a,b\rangle = \{a,b\}$ for $a \parallel b$. It is also evident that $CS_A\langle M\rangle = \bigcap CS_A(M)$, in particular $CS_A\langle a,b\rangle = \bigcap CS_A(a,b)$.

Lemma 1.1. *Let A be an ordered set and let $\{X_i\,;\,i \in I\}$ be an arbitrary system of convex subsets of A. Then $(CS(A), \subseteq)$ is a complete lattice, where $\bigwedge\{X_i\,;\,i \in I\} = \bigcap\{X_i\,;\,i \in I\}$ and $\bigvee\{X_i\,;\,i \in I\} = \bigcap CS_A(X_i, i \in I)$ are the infimum and supremum, respectively.*

P r o o f. Clearly $\emptyset$ is the least element in $(CS(A), \subseteq)$ and $\bigcap\{X_i\,;\,i \in I\} \in CS(A)$, thus $\bigwedge\{X_i\,;\,i \in I\} = \bigcap\{X_i\,;\,i \in I\}$. Evidently $\bigcap CS_A(X_i, i \in I) \in CS(A)$. Hence $\bigvee\{X_i\,;\,i \in I\} = \bigcap CS_A(X_i, i \in I)$ and $(CS(A), \subseteq)$ is a complete lattice according to Theorem 17 in [6]. □

Let $L = (L, \leqslant)$ be a lattice with the least element 0. We say that an element $a \in L$ is an atom in L if $a \neq 0$ and for each $b \in L$ we have: $b \leqslant a \implies (b = a \text{ or } b = 0)$. The lattice $L = (L, \leqslant)$ with the least element 0 is called an atomic lattice if for each $a \in L$, $a \neq 0$ there exists an atom p such that $p \leqslant a$. Let $A = (A, \leqslant)$ be an ordered set, $A \neq \emptyset$. Then $(CS(A), \subseteq)$ is obviously an atomic lattice, where $\emptyset$ is the least element and all one-element subsets of A form the set of all its atoms.

2. Convex isomorphic ordered sets

Definition 2.1. We say that ordered sets A, A' are convex isomorphic if and only if the lattices $(CS(A), \subseteq)$ and $(CS(A'), \subseteq)$ are isomorphic.

Let A be an ordered set and let a, $b \in A$, $a \leqslant b$. Denote by $[a,b]$ the interval generated by a, b, i.e. $[a,b] = \{x \in A\,;\, a \leqslant x \leqslant b\}$.

Let F be a mapping of A into B and $\emptyset \neq C \subseteq A$. Denote by F/C the restriction of F onto the subset C, i.e. $F/C = F \cap (C \times B)$.

As it was mentioned above all one-element subsets of A are atoms in the lattice $(CS(A), \subseteq)$. Let F be an isomorphism of the lattices $(CS(A), \subseteq)$, $(CS(A'), \subseteq)$. Since every isomorphism of atomic lattices maps atoms onto atoms we have $F(\{a\}) = \{a'\} \in CS(A')$, where $a' \in A'$. Therefore we can define the following concept:

Definition 2.2. Let F be an isomorphism of the lattices $(CS(A), \subseteq)$ and $(CS(A'), \subseteq)$. Let f be a mapping of A into A' such that $\{f(a)\} = F(\{a\})$ for each $a \in A$. We say that the mapping f is associated with the isomorphism F.

Let us denote $f(S) = \{f(x)\,;\, x \in S\}$ for a subset $S \subseteq A$. We can prove:

Lemma 2.1. *$F(S) = f(S)$ for any $S \in CS(A)$.*

P r o o f. If $a \in S$ then $\{a\} \subseteq S$ and also $F(\{a\}) = \{f(a)\} \subseteq F(S)$ because F is an isomorphism. Hence $f(a) \in F(S)$ and thus $f(S) \subseteq F(S)$. Conversely, if $a' \in F(S)$ then $\{a'\} \subseteq F(S)$ and $F^{-1}(\{a'\}) = \{f^{-1}(a')\} \subseteq S$ because F^{-1} is an isomorphism as well. Then we have $f^{-1}(a') \in S$ and $a' \in f(S)$, i.e. $F(S) \subseteq f(S)$. Thus we get $F(S) = f(S)$. □

Proposition 2.1. *If f is associated with an isomorphism F of the lattices $(CS(A), \subseteq)$ and $(CS(A'), \subseteq)$, then $f(CS_A\langle M\rangle) = CS_{A'}\langle f(M)\rangle$ for any $M \subseteq A$.*

P r o o f. Since $M \subseteq \bigcap CS_A(M)$, we have $f(M) \subseteq f(CS_A(M))$. Now, $f(\bigcap CS_A(M)) = F(\bigcap CS_A(M)) \in CS(A')$ by Lemma 2.1 and so $CS_{A'}\langle f(M)\rangle \subseteq f(CS_A\langle M\rangle)$. On the other hand, let $Z \in CS(A')$ be such that $f(M) \subseteq Z$. Since F is surjective, there exists $W \in CS(A)$ with $F(W) = f(W) = Z$. It follows that $M \subseteq W$ and, therefore, $\bigcap CS_A(M) \subseteq W$. Consequently, $f(\bigcap CS_A(M)) \subseteq Z$ and we can see that $f(CS_A\langle M\rangle) \subseteq CS_{A'}\langle f(M)\rangle$. □

Theorem 2.1. *Let A and A' be ordered sets. Then the following three conditions are equivalent:*

(i) *The ordered sets A and A' are convex isomorphic.*

(ii) *There exists a bijection $f\colon A \to A'$ such that $f(CS_A\langle M\rangle) = CS_{A'}\langle f(M)\rangle$ for any $M \subseteq A$.*

(iii) *There exists a bijection $f\colon A \to A'$ such that $f(CS_A\langle a,b\rangle) = CS_{A'}\langle f(a), f(b)\rangle$ for each a, $b \in A$.*

P r o o f. The condition (ii) follows from (i) by Proposition 2.1. Clearly, the third condition is a consequence of the second one.

Now, let f be a bijection satisfying (iii). Denote by $P(A)$ the set of all subsets of A and define a mapping $F\colon P(A) \to P(A')$ such that $F(S) = f(S)$ for each $S \in P(A)$. We are going to prove that for any convex set S, its image $F(S)$ is also convex. Clearly, $f(a)$, $f(b) \in f(S) = F(S)$ for each a, $b \in S$. If $S \in CS(A)$ then $CS_A\langle a,b\rangle \subseteq S$ for arbitrary a, $b \in S$ and according to (iii) we have $[f(a), f(b)] = CS_{A'}\langle f(a), f(b)\rangle = f(CS_A\langle a,b\rangle) \subseteq f(S) = F(S)$. This means that the mapping F maps convex subsets of A onto convex subsets of A', and F is a bijection because f is a bijection. Evidently, the restriction of the mapping $F/CS(A)\colon CS(A) \to CS(A')$ is also a bijection. Since $S \subseteq T \iff F(S) \subseteq F(T)$ for each S, $T \in CS(A)$, the mapping $F/CS(A)$ is an isomorphism of the lattices $(CS(A), \subseteq)$ and $(CS(A'), \subseteq)$. □

The bijection f is called a convex isomorphism.

E x a m p l e 2.1. The ordered sets A, A' in Figure 1 are convex isomorphic because the mapping f satisfies the condition (iii) of Theorem 2.1.

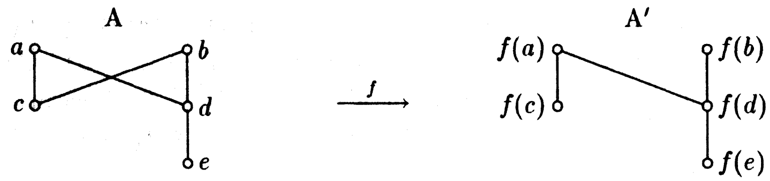

Fig. 1

3. Convex isomorphism of modular, distributive, complemented, uniquely complemented and boolean ordered sets

Let $A = (A, \leqslant)$ be an ordered set. For $S \subseteq A$ denote by $L(S)$ the set $\{x \in A;\ x \leqslant a;\ \forall\, a \in S\}$ and let $U(S) = \{y \in A; a \leqslant y \ \forall\, a \in S\}$. We will also write

$$L(S) = L(\ldots, a_i, \ldots), \quad U(S) = U(\ldots, a_i, \ldots) \quad \text{for} \quad S = \{\ldots, a_i, \ldots\} \subseteq A.$$

Modular distributive, complemented, uniquely complemented and boolean ordered sets are defined in [1], [2], [4] as follows:

Definition 3.1. We say that an ordered set $A = (A, \leqslant)$ is

a) modular if

$$\forall\, a, b, c \in A; \quad a \leqslant c \implies L\big(U(a,b), c\big) = L\big(U\big(a, L(b,c)\big)\big);$$

b) distributive if

$$\forall\, a, b, c, \in A; \quad L\big(U(a,b), c\big) = L\big(U\big(L(a,c), L(b,c)\big)\big);$$

c) complemented if

$$\forall\, a \in A\ \exists\, b \in A; \quad L\big(U(a,b)\big) = A \quad \text{and} \quad U\big(L(a,b)\big) = A$$

(in this case we say that the elements a, b are complemented, or that a is a complement of b or b is a complement of a);

d) uniquely complemented if for each $a \in A$ there exists just one complement $a' \in A$;

e) boolean if A is distributive and complemented.

V. I. Marmazejev proved in [5] that if L, L' are convex isomorphic lattices then L is modular (distributive, complemented or boolean), if and only if L' is modular (distributive, complemented of boolean, respectively). The following examples show that a similar theorem does not hold for modular, distributive, complemented and boolean ordered sets.

E x a m p l e 3.1. The ordered sets A, A' in Figure 2 are convex isomorphic (an arbitrary bijection of A onto A' satisfies (iii)), but A is modular while A' is not modular.

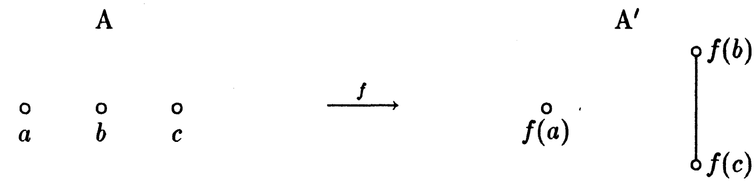

Fig. 2

E x a m p l e 3.2. The ordered sets A, A' in Figure 3 are convex isomorphic (an arbitrary bijection of A onto A' satisfies (iii)), but A is distributive while A' is not distributive.

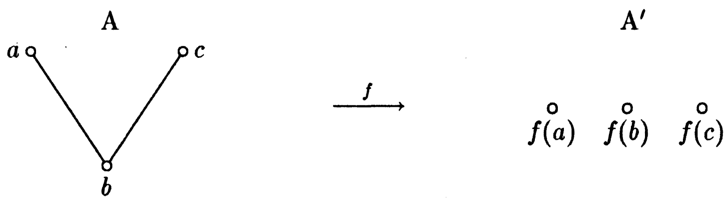

Fig. 3

E x a m p l e 3.3. The ordered sets A, A' in Figure 4 are convex isomorphic (the mapping f satisfies (iii)) but A is complemented while A' is not complemented (a complement of element $f(b)$ does not exists in A').

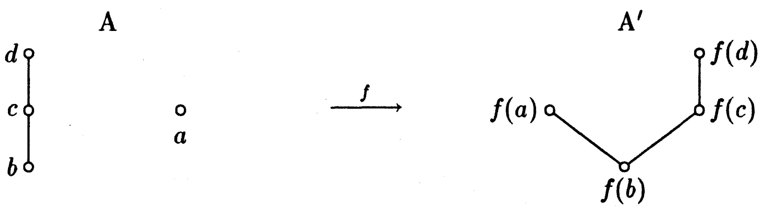

Fig. 4

E x a m p l e 3.4. The ordered sets A, A' in Figure 5 are convex isomorphic, but A is uniquely complemented, even boolean while A' is not complemented because the element $f(b)$ has not a complement in A'. Moreover, A is a boolean ordered set but A' is not boolean.

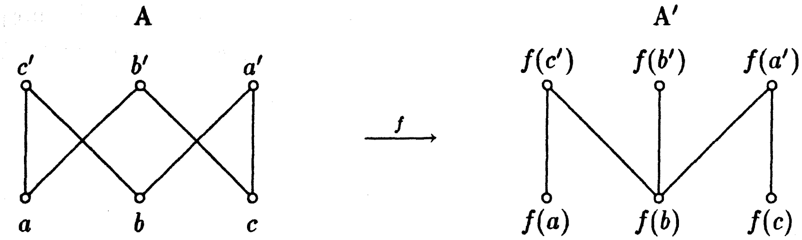

Fig. 5

4. Other properties of convex isomorphic ordered sets

An ordered set $A = (A, \leqslant)$ is called directed if for each a, $b \in A$ there exist c, $d \in A$ such that $c \leqslant a$, $c \leqslant b$, $a \leqslant d$ and $b \leqslant d$. If A is a two-element chain and A' is a two-element antichain (Fig. 6), then A, A' evidently are convex isomorphic, but A is directed and A' is not directed.

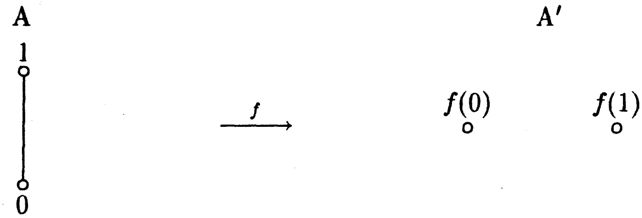

Fig. 6

Taking into account the foregoing example, we will investigate ordered sets with at least two incomparable elements. Then we obtain the following theorem:

Theorem 4.1. *let A, A' be convex isomorphic ordered sets such that both of them contain at least two incomparable elements. Then A is directed if and only if A' is directed.*

P r o o f. Let A, A' be convex isomorphic ordered sets. Suppose A is directed and A has at least two incomparable elements. Then clearly for each a, $b \in A$, $a \neq b$ there exist c, $d \in A$ such that $c \leqslant a$, $c \leqslant b$, $a \leqslant d$, $b \leqslant d$ where at least three of the elements a, b, c, d are different. Hence $a, b \in [c, d] = CS_A \langle c, d\rangle$, where $CS_A \langle c, d\rangle$ is at least a three-element set. If f is a convex isomorphism of A onto A', then $f(a)$, $f(b) \in f(CS_A \langle c, d\rangle) = CS_{A'} \langle f(c), f(d)\rangle = [f(c), f(d)]$ (or $[f(d), f(c)]$) because the set $f(CS_A \langle c, d\rangle)$ also has at least three elements (f is a bijection). Thus $f(c) \leqslant f(a)$, $f(b) \leqslant f(d)$, (or $f(d) \leqslant f(a)$, $f(b) \leqslant f(c)$) and A' is directed. It is clear that the converse implication is also true. □

The convex isomorphic ordered sets A, A' in Figure 6 are such that A contains the least element 0 and the greatest element 1, but A' contains neither the least nor the greatest element. However, if we take ordered sets with at least three elements we get the following theorem:

Theorem 4.2. *Let A, A' be convex isomorphic ordered sets such that both of them contain at least three elements. Then A contains the least and the greatest element if and only if A' contains the least and the greatest element.*

Proof. Let 0 be the least and 1 the greatest element of the ordered set A. Then $A = [0,1] = CS_A\langle 0,1\rangle$ whence $CS_A\langle 0,1\rangle$ has at least three elements. If f is a convex isomorphism of A onto A', then $f(CS_A\langle 0,1\rangle) = CS_{A'}\langle f(0), f(1)\rangle = A'$. Since f is a bijection, the set $f(CS_A\langle 0,1\rangle)$ also has at least three elements. Hence $A' = [f(0), f(1)]$, i.e. $f(0)$ is the least and $f(1)$ is the greatest element in A' (or $A' = [f(1), f(0)]$, i.e. $f(1)$ is the least and $f(0)$ is the greatest element in A'). □

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