Milan Medveď

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OSCILLATORY PROPERTIES OF SOME CLASSES OF NONLINEAR DIFFERENTIAL EQUATIONS

MILAN MEDVEĎ, Bratislava

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Summary. A sufficient condition for the nonoscillation of nonlinear systems of differential equations whose left-hand sides are given by n-th order differential operators which are composed of special nonlinear differential operators of the first order is established. Sufficient conditions for the oscillation of systems of two nonlinear second order differential equations are also presented.

Keywords: Oscillation, nonlinear differential equation

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1. INTRODUCTION

M. Švec [7] studied the problem od dependence of oscillatory properties of linear nonhomogeneous differential equations of the form

$$y'' + p(t)\,y = f(t)$$

on the function f. He has used some properties of the wronskian $W(y,x)$ of a solution y of the homogeneous equation and a solution x of the corresponding homogeneous equation. This method has also been used in the papers [3], [4] (another method is used in the proof of [6, Lemma 5]) in a study of nonoscillation of linear nonhomogeneous differential equations of the n-th order and oscillation of linear nonhomogeneous differential equations of the second order, respectively. We use this idea in a study of nonoscillation of some classes of systems of nonlinear differential equations defined by a nonlinear differential operators of the n-th order which are composed of nonlinear differential operators of the first order. We also study oscillatory properties of a system of differential equations consisting of two nonlinear second order differential equations.

Consider a system of differential equations

$$L_m(x,y,t) = f(x,y,t), \tag{1}$$

$$M_n(x,y,t) = 0, \tag{2}$$

where

$$L_m(x,y,t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + a_m\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + a_{m-1}\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}t} + a_1\right) x,$$

$$M_n(x,y,t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} + b_n\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + b_{n-1}\right)\cdots\left(\frac{\mathrm{d}}{\mathrm{d}t} + b_1\right) y,$$

$a_i = a_i(x,y,t)$, $i = 1, 2, \ldots, m$, $b_j = b_j(x,y,t)$, $j = 1, 2, \ldots, n$, $x \in \mathbf{R}$, $y \in \mathbf{R}$. We assume that the following hypotheses are satisfied:

(H1) The functions $a_i \in C^{m-i}(\mathbf{R}^3, \mathbf{R})$, $b_j \in C^{n-j}(\mathbf{R}^3, \mathbf{R})$, $i = 1, 2, \ldots, m$, $j = 1, \ldots, n$ are bounded, $f \in C^0(\mathbf{R}^3, \mathbf{R})$.

(H2) All solutions of the system (1), (2) exist on the interval $(-\infty, \infty)$.

(H3) $f(x,y,t) = 0$ if and only if $y = 0$.

Definition. We call a function $g\colon \mathbf{R} \to \mathbf{R}$ oscillatory on $I = (-\infty, \infty)$ if it has an infinite number of zeros on each of the intervals $(-\infty, -T)$ and (T, ∞) for any $T > 0$. The function g is called nonoscillatory on I if it has only a finite number of zeros on I. The mapping $G\colon \mathbf{R} \to \mathbf{R}^n$ is called oscillatory (nonoscillatory) on I if each of its components is oscillatory (nonoscillatory) on I. We say that a system of differential equations $\dot{u} = F(u,t)$, $u \in \mathbf{R}^n$, is oscillatory (nonoscillatory) on I if all its solutions whose all components are not identically equal to zero are oscillatory (nonoscillatory) on I.

Lemma. *Let $a \in C^0(\mathbf{R}^2, \mathbf{R})$ be a bounded function and $x\colon \mathbf{R} \to \mathbf{R}$ a solution of the equation*

$$\dot{x} + a(x,t)\,x = 0$$

such that $x(t_0) = 0$ for some $t_0 \in \mathbf{R}$. Then $x(t) \equiv 0$.

Proof. Let $|\, a(x,t)\, | \leqslant M$ for all $(x,t) \in R^2$. Then

$$|x(t)| = \left| \int_{t_0}^{t} a\big(x(s), s\big) x(s)\,\mathrm{d}s \right| \leqslant M \int_{t_0}^{t} |x(s)|\,\mathrm{d}s$$

and the Gronwall lemma implies that $x(t) \equiv 0$. □

Theorem 1. *If the hypotheses* (H1)–(H3) *are satisfied then the system* (1), (2) *is nonoscillatory on* $I = (-\infty, \infty)$.

Proof. Let $(x(t), y(t))$ be a solution of the system (1), (2) and $x(t) \not\equiv 0$, $y(t) \not\equiv 0$. First we shall prove that the function $y(t)$ is nonoscillatory on I. Obviously, the function $y(t)$ satisfies the system of differential equations

$$(3_1) \qquad \dot{y} + b_1\big(x(t), y, t\big)\, y = y_1$$

$$(3_2) \qquad \dot{y}_1 + b_2\big(x(t), y, t\big)\, y_1 = y_2$$

$$\dots$$

$$(3_{n-1}) \qquad \dot{y}_{n-2} + b_{n-2}\big(x(t), y, t\big)\, y_{n-2} = y_{n-1}$$

$$(3_n) \qquad \dot{y}_{n-1} + b_n\big(x(t), y, t\big)\, y_{n-1} = 0.$$

Since the function b_n is bounded, Lemma implies that $y_{n-1}(t)$ is either identically equal to zero or $y_{n-1}(t) \neq 0$ for all $t \in I$. If $y_{n-1}(t) \equiv 0$ then the same is valid for $y_{n-2}(t)$. If all functions $y_1(t)$, …, $y_{n-1}(t)$ are identically equal to zero then the equality corresponding to (3_1) has the form

$$\dot{y}(t) + b_1\big(x(t), y(t), t\big)\, y(t) = 0,$$

and since the function b_1 is bounded and $y(t) \not\equiv 0$ we obtain by Lemma that $y(t) \neq 0$ for all t.

Assume that $y_{n-i}(t) \equiv 0$, $i = 1, 2, \dots, k-1$ and $y_{n-k}(t) \neq 0$ for all $t \in I$. The equality corresponding to (3_{n-k}) has the form

$$(4) \qquad \dot{y}_{n-k-1} + b_{n-k}\big(x(t), y(t), t\big)\, y_{n-k-1} = y_{n-k}(t).$$

By Lemma no nontrivial solution of the equation

$$(5) \qquad \dot{z} + b_{n-k}\big(x(t), y(t), t\big)\, z = 0$$

has a zero on I. Let $z(t)$ be a nontrivial solution of (5) and $W\big(y_{n-k-1}(t), z(t)\big) = y_{n-k-1}(t)\dot{z}(t) - \dot{y}_{n-k-1}(t)z(t)$ be the wronskian of the functions $y_{n-k-1}(t)$, $z(t)$. From (4), (5) we have that $W(y_{n-k-1}(t), z(t)) = -y_{n-k}(t)z(t) \neq 0$ for all $t \in I$. Therefore $y_{n-k-1}(t)$ has at most one zero in I. Proceeding by induction with respect to k one can show that $y_1(t)$ has a finite number of zeros in I. Therefore there is a $T > 0$ such that $y_1(t) \neq 0$ for all $t \in (-\infty, -T) \cup (T, \infty)$. By Lemma no nontrivial solution of the equation

$$(6) \qquad \dot{v} + b_1\big(x(t), v, t\big)\, v = 0$$

has a zero in I and therefore $W(y(t), v(t)) = -v(t)\, y_1(t) \neq 0$ for a nontrivial solution $v(t)$ of (6) and all $t \in (\infty, -T) \cup (T, \infty)$. Therefore $y(t)$ has at most one zero in $(-\infty, -T) \cup (T, \infty)$ and thus the function $y(t)$ is nonoscillatory on I. This fact and the hypothesis (H3) imply that the function $f(x(t), y(t), t)$ has no zero in $(-\infty, -T) \cup (T, \infty)$. The function $x(t)$ satisfies the equalities

$$\begin{aligned} \dot{x}(t) + a_1(x(t), y(t), t)\, x(t) &= x_1(t) \\ \dot{x}_1(t) + a_2(x(t), y(t), t)\, x_1(t) &= x_2(t) \\ &\dots \\ \dot{x}_{m-2}(t) + a_{m-1}(x(t), y(t), t)\, x_{m-2}(t) &= x_{m-1}(t) \\ \dot{x}_{m-1}(t) + a_m(x(t), y(t), t)\, x_{m-1}(t) &= f(x(t), y(t), t). \end{aligned} \tag{7}$$

One can show similarly as above that the function $x_{m-1}(t)$ is nonoscillatory on I, and proceeding by induction it is possible to prove that also the function $x(t)$ is nonoscillatory on I. □

3. Oscillation of systems of 2-nd order differential equations

Let us consider the system of differential equations

$$\ddot{x} + p(y, t)\, x = f(x, y, t), \tag{8}$$

$$\ddot{y} + q(y, t)\, y = 0. \tag{9}$$

We assume that the following conditions are satisfied:

(C1) $p, q \in C^0(\mathbf{R}^2, \mathbf{R})$, $f \in C^0(\mathbf{R}^3, \mathbf{R})$.

(C2) All solutions of the system (8), (9) exist on the interval $I = (-\infty, \infty)$.

(C3) $f(x, y, t) = 0$ if and only if $y = 0$.

(C4) $\operatorname{sign} f(x, y, t) = \operatorname{sign} y$ for all $(x, y, t) \in \mathbf{R}^3$.

Theorem 2. *Let the conditions* (C1)–(C4) *be satisfied and let* $(x(t), y(t))$ *be a solution of the system* (8), (9), *where* $x(t) \not\equiv 0$, $y(t) \not\equiv 0$. *Assame that the differential equation*

$$\ddot{u} + q(y(t), t)\, u = 0 \tag{10}$$

is oscillatory on $I = (-\infty, \infty)$ *and*

$$2d_1 < d_2 < \infty, \tag{11}$$

where

$$d_1 = \sup\{|t_1 - t_2| : t_1 < t_2,\ u(t_1) = u(t_2) = 0,\ u(t) \neq 0 \text{ for all } t \in (t_1, t_2),\ u \text{ is a solution of } (10)\},$$

$$d_2 = \inf\{|t_1 - t_2| : t_1 < t_2,\ y(t_1) = y(t_2) = 0,\ y(t) \neq 0 \quad \text{for all} \quad t \in (t_1, t_2)\}.$$

Then the solution $(x(t), y(t))$ *is oscillatory on* I.

Proof. We shall prove that x has an infinite number of zeros in the interval $(0, \infty)$. The same assertion for the interval $(-\infty, 0)$ can be proved analogously.

Assume that there is a $T > 0$ such that $x(t) \neq 0$ for all $t \in (T, \infty)$. The function $y(t)$ satisfies the equation (10) and by the assumption it is oscillatory on I. Let

(I) $x(t) > 0$ for all $t \in (T, \infty)$. From (11) it follows that for any $K > T$ there are $s_1, s_2 \in \mathbf{R}$ such that $K < s_1 < s_2$,

$$y(s_1) = y(s_2) = 0,\ y(t) \neq 0 \quad \text{for all} \quad t \in (s_1, s_2) \tag{12}$$

$$y(t) < 0 \quad \text{for all} \quad t \in (s_1, s_2), \tag{13}$$

and there exist $T_1, T_2 \in (s_1, s_2)$, $T_1 < T_2$ such that

$$u(T_1) = u(T_2) = 0,\ u(t) \neq 0 \quad \text{for all} \quad t \in (T_1, T_2),\ \dot{u}(T_1) > 0. \tag{14}$$

(II) $x(t) < 0$ for all $t \in (T, \infty)$. From (11) it follows that for any $K > T$ there are $r_1, r_2 \in \mathbf{R}$ such that $K < r_1 < r_2$, the condition(12) holds,

$$y(t) > 0 \quad \text{for all} \quad t \in (r_1, r_2) \tag{15}$$

and there exist $T_3, T_4 \in (r_1, r_2)$, $T_3 < T_4$ such that

$$u(T_3) = u(T_4) = 0,\ u(t) > 0 \quad \text{for all} \quad t \in (T_3, T_4),\ \dot{u}(T_3) > 0. \tag{16}$$

Take the wronskian $w(t) = W(u(t), x(t)) = u(t)\,\dot{x}(t) - \dot{u}(t)\,x(t)$. From (8), (9) we obtain that $w(t) = u(t)\,g(t)$, where $g(t) = f(x(t), y(t), t)$ and (14), (16) yield

$$w(t) = -\dot{u}(T_1)\,x(T_1) + \int_{T_1}^{t} u(s)\,g(s)\,\mathrm{d}s, \tag{17}$$

$$w(t) = -\dot{u}(T_3)\,x(T_3) + \int_{T_3}^{t} u(s)\,g(s)\,\mathrm{d}s. \tag{18}$$

The conditions (C4) and (13) imply that in the case (I) we have $g(t) < 0$ for all $t \in (s_1, s_2)$. Therefore (14) and (17) imply that $w(t) < 0$ for all $t \in (T_1, T_2)$ and thus $x(t)$ must have a zero in (T_1, T_2). This contradicts the assumption. In the case (II) we have $g(t) > 0$ for all $t \in (r_1, r_2)$. Therefore (16), (18) imply that $w(t) > 0$ for all $t \in (T_3, T_4)$. Thus $x(t)$ must have a zero in (T_3, T_4) and this again contradicts the assumption. This means that $x(t)$ has a zero in (T, ∞) for any $T > 0$. □

Corollary. *Let* $P(t)$, $Q(t)$ *be continuous functions on* $I = (-\infty, \infty)$, *and let the equations*

$$\ddot{w} + P(t)\,w = 0, \tag{19}$$

$$\ddot{z} + Q(t)\,z = 0 \tag{20}$$

be oscillatory on I, *where*

$$\begin{aligned} &m_1^2 \leqslant P(t) \leqslant M_1^2, \quad m^2 \leqslant Q(t) \leqslant M^2 \quad \text{for all } t \in I, \\ &M \geqslant m, \quad M_1 \geqslant m_1, \quad 2M \leqslant m_1. \end{aligned} \tag{21}$$

Let $f \in C^0(\mathbf{R}^3, \mathbf{R})$ *satisfies the conditions* (C3), (C4) *and let all solutions of the system*

$$\ddot{x} + P(t)\,x = f(x, y, t), \tag{22}$$

$$\ddot{y} + Q(t)\,y = 0 \tag{23}$$

exist on the interval I. *Then the system* (22), (23) *is oscillatory on* I.

P r o o f. Let $(x(t), y(t))$ be any solution of the system (22), (23), $x(t) \not\equiv 0$, $y(t) \not\equiv 0$ and let $u(t) \not\equiv 0$ be any solution of the equation (19). The Sturm comparison theorem and (21) (see e.g. [5]) imply that the following holds: If $u(t_1) = u(t_2) = 0$, $t_1 < t_2$, $u(t) \neq 0$ for all $t \in (t_1, t_2)$, $y(s_1) = y(s_2) = 0$, $s_1 < s_2$, $y(s) \neq 0$ for all $t \in (s_1, s_2)$, then

$$\frac{\pi}{M_1} \leqslant |t_1 - t_2| \leqslant \frac{\pi}{m_1},$$
$$\frac{\pi}{M} \leqslant |s_1 - s_2| \leqslant \frac{\pi}{m}$$

and therefore we have

$$2\,|t_1 - t_2| \leqslant 2\frac{\pi}{m_1} \leqslant \frac{\pi}{M} \leqslant |s_1 - s_2|.$$

We have proved that the condition (11) of Theorem 2 is satisfied. By this theorem $(x(t), y(t))$ is oscillatory on I. □

Now let us consider the system

$$\ddot{x} + R(y, t)\,x = f(x, y, t), \tag{24}$$

$$\ddot{y} + S(y)\,y = 0. \tag{25}$$

We assume that the following conditions are satisfied:

(D1) $R \in C^0(\mathbf{R}^0, \mathbf{R})$, $S \in C^0(\mathbf{R}, \mathbf{R})$, $f \in C^0(\mathbf{R}^3, \mathbf{R})$.

(D2) All solutions of the system (24), (25) exist on the interval $I = (-\infty, \infty)$.

(D3) = (C3),

(D4) = (C4).

Theorem 3. *Let the conditions* (D1)–(D4) *be satisfied. Moreover, assume that the following conditions are satisfied:*

(I) There is an $h > 0$ such that all trajectories of the system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -S(y_1)\, y_1$$

lying in the set

$$L_h = \{(y_1, y_2) \in \mathbf{R}^2 \colon H(y_1, y_2) \leqslant h\}$$

are periodic, where

$$H(y_1, y_2) = \frac{1}{2} y_2^2 + \int_0^{y_1} S(t)\, t \, \mathrm{d}t.$$

(II) There is a constant $L > 0$ such that

$$\|Q(u) - Q(v)\| \leqslant L \|u - v\| \quad \text{for all} \quad u, v \in L_h,$$

where $Q(x) := \big(x_2, -S(x_1)x_1\big)$, $x = (x_1, x_2) \in \mathbf{R}_2$ and $\|.\|$ is the Euclidean norm on R^2.

(III) Let $y(t) \not\equiv 0$ be a solution of the equation (25) *such that $\big(y(t), \dot{y}(t)\big) \in L_h$ for all $t \in I = (-\infty, \infty)$, and let the equation*

$$\ddot{w} + R\big(y(t), t\big)\, w = 0 \tag{27}$$

be oscillatory.

(IV)

$$d_1 < \frac{3}{2L},$$

where

$$d_1 = \max\{|t_1 - t_2| \colon t_1 < t_2,\ w(t_1) = w(t_2) = 0,\ w(t) \neq 0 \ \text{for all} \ t \in (t_1, t_2),\ w \text{ is a solution of } (27)\}.$$

If $x(t) \not\equiv 0$ is such a function that $\big(x(t), y(t)\big)$ is a solution of the system (24), (25), *then it is oscillatory on I.*

Proof. The condition (I) implies that $(y(t), \dot{y}(t))$ is a periodic solution of (26) with a minimal period $T > 0$. Since the condition (II) is satisfied, [1, Theorem 1. 3] yields $T \geqslant \frac{6}{L}$. This implies that

$$\begin{aligned} d_2 = \inf\{|t_1 - t_2| : t_1 < t_2,\ y(t_1) = y(t_2) = 0,\ y(t) \neq 0 \\ \text{for all} \quad t \in (t_1, t_2)\} \geqslant \frac{3}{L}, \end{aligned}$$

and therefore using the condition (IV) we obtain the inequality

$$2\,d_1 \leqslant \frac{3}{L} \leqslant d_2 \quad \text{(see (11))}.$$

□

Example.

$$\ddot{x} + k^2 x = f(x, y, t), \tag{28}$$

$$\ddot{y} - (1 - y)\, y = 0, \tag{29}$$

where $f(x, y, t)$ is a continuous function satisfying the conditions (C3), (C4). The equation (29) can be written in the form

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= (1 - y_1)\, y_1. \end{aligned} \tag{30}$$

This system is Hamiltonian with the Hamiltonian function

$$H(y_1, y_2) = \frac{1}{2}\, y_2^2 - \frac{1}{2}\, y_1^2 + \frac{1}{3}\, y_1^3.$$

There is a homoclinic trajectory Γ of the system (30) corresponding to the level curve $H(y_1, y_2) = 0$, and the compact region K whose boundary is the closed curve $\Gamma \cup \{(0,0)\}$ is filled with periodic trajectories (see [1, pp. 291, Fig. 6.1.2]). The solutions of the equation (29) corresponding to these periodic trajectories are the only oscillatory solutions of this equation.

Denote by $Q(y) = (y_2, (1 - y_1) y_1)$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbf{R}^2$ and let $u = (u_1, u_2)$, $v = (v_1, v_2) \in K$. Since the homoclinic trajectory Γ intersects the y_1-axis at the point $A = (a, 0)$, where $a = \frac{\sqrt{3}}{\sqrt{2}}$, we have that $u_1 \leqslant a$, $v_1 \leqslant a$. Therefore we obtain

$$\begin{aligned} \|Q(u) - Q(v)\|^2 &= (v_2 - u_2)^2 + (v_1 - u_1)\big(1 + (u_1 + v_1)\big)^2 \\ &\leqslant (v_2 - u_2)^2 + (1 + 2a)^2 (v_1 - u_1)^2 \\ &\leqslant (1 + 2a)^2 \|u - v\|^2, \end{aligned}$$

i.e., the mapping Q is Lipschitz with the Lipschitz constant $L = 1+2a$. The distance of any two neighbouring zeros of the equation $\ddot{w} + k^2 w = 0$ is $d_1 = \frac{\pi}{k^2}$ and therefore if

$$k^2 > \frac{2}{3}(1+2a)\pi, \quad a = \frac{\sqrt{3}}{\sqrt{2}},$$

then the condition (IV) of Theorem 3 is satisfied. Since all oscillatory solutions of the system (30) define trajectories lying in K we obtain that if the condition (31) is satisfied and $\bigl(x(t), y(t)\bigr)$ is a solution of the system (28), (29) such that $\bigl(y(t), \dot{y}(t)\bigr) \in K$ for all $t \in \mathbb{R}$ then $\bigl(x(t), y(t)\bigr)$ is oscillatory on $(-\infty, \infty)$.

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Súhrn

OSCILATORICKÉ VLASTNOSTI NIEKTORÝCH TRIED NELINEÁRNYCH DIFERENCIÁLNYCH ROVNÍC

MILAN MEDVEĎ

Dokázaná je postačujúca podmienka pre neoscilatoričnosť systému diferenciálnych rovníc, ktorých ľavé strany sú definované diferenciálnym operátorom n-tého rádu, ktorý je kompozíciou nelineárnych diferenciálnych operátorov prvého rádu. Dokázaná je tiež postačujúca podmienka pre oscilatoričnosť systému dvoch nelineárnych diferenciálnych rovníc druhého rádu.

Author's address: Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava.