Michael Šebek

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CHARACTERISTIC POLYNOMIAL ASSIGNMENT FOR DELAY-DIFFERENTIAL SYSTEMS VIA 2-D POLYNOMIAL EQUATIONS

MICHAEL ŠEBEK

A new method of characteristic polynomial assignment for delay-differential systems, both retarded and neutral, is presented. The method consists in solving a 2-D polynomial equation. Solvability conditions well suited for practical testing are given and the class of all assignable polynomials is parametrized. The problems of minimality, causality, properness and stabilization are discussed. The method is used to stabilize even not formally stable plants. A finite spectrum assignment is investigated. All the design procedures proposed are based on classical 1-D techniques.

1. INTRODUCTION

Growing presence of algebraic methods can be observed in the control theory in the last decade. So among the methods of synthesis in classical linear systems, the reputation of those based on polynomials in one variable (1-D) stands now very high. In particular, both scalar and matrix linear polynomial equations has already been successfully introduced in control (see, e.g., Kučera [12]). That is why now plenty of researchers interest in developing similar methods for more complex systems. In the case of linear delay-differential systems, good foundations has been laid by Kamen [11] and Morse [16] who have employed polynomials in several variables. This way has further been developed by many authors. In systems with commensurate time delays one encounters two operators — differentiation and delay — so that polynomials in just two variables (2-D) appear to be an appropriate tool.

Whenever one needs to change the dynamics of a plant by a dynamic output feedback, as in a characteristic polynomial assignment (CPA) problem, a simple linear polynomial equation can do the job. To solve such an equation in 2-D polynomials, two different methods has been derived till now by Emre [6] and Šebek [20, 21]. The first method is based on a realization technique while the second one employs knowledge of the minimum degree a solution may have. In the case of a denominator which is monic in one variable, the first method transforms the original scalar 2-D polynomial equation into a 1-D matrix polynomial equation while the second method solves directly the original scalar equation but in the ring of 1-D polynomials with rational coefficients in the second variable. In a general case, the first method employs a matrix equation in 1-D polynomials with rational coefficients in the second variable while the second one uses a matrix equation in real 1-D polynomials. Needless to say that the first method can handle also matrix 2-D polynomials and therefore, when used iteratively, it provides a way to solve general n-D polynomial equations. The same is true for the second method, in principle, but this is still a subject of a further research. In this paper, a refined second type method is employed which is tuned to cope with delay-differential systems.

The CPA problem in delay-differential systems has already been solved by many authors via a current state feedback. Dynamic output of a restricted type has been considered by Paraskevopoulos and Kosmidou [18]. By the general dynamic output feedback has the problem been solved by Emre and Khargonekar [9] and Emre [7] for the class of "split" systems which, however, is not generic within scalar systems as shown by Lee and Olbrot [14]. The above restrictions has been overcome in Šebek [21] with a help of 2-D polynomial equations. This paper is, however, oriented more toward discrete-time 2-D systems and does not profit from the properties of delay-differential ones. Recently, the problem has been solved by Chiasson and Lee [4, 5] and Chiasson [3] using polynomial equations as well. However, they employ the Wolowich structure having let the desired characteristic polynomial divided into two factors. This is a little restrictive as not every 2-D polynomial can be factored such a way. Moreover, to assign the denominator of a transfer function arbitrarily, they force all the fixed poles to lay in the second factor. In addition, only plants having the numerator degree exactly one less then the denominator degree are considered in [4, 5]. On the other hand, the state feedback method has been improved by Manitius and Olbrot [15] and Watanabe, Ito and Kaneko [22] who employed more complex controllers including distributed delays. Such controllers appear to be more powerful but they are difficult to implement.

It is the aim of this paper to present a new method for CPA by a general dynamic output feedback. The method consists in the solution of a linear 2-D polynomial equation. A general linear delay-differential plant with commensurate point delays is considered, both retarded and neutral, and a feedback controller is to be found within the same class. Two types of solvability conditions are given which provide two algorithms. The class of all polynomials, which can be assigned to a plant, is expressed in a parametric form. This is useful for a practical design. Besides, the class of all plants which an arbitrary characreristic polynomial can be assigned to is also parametrized. For retarded plants it is shown that there exists a minimal controller whenever the problem is solvable. Simple conditions are given under which this holds true for neutral plants. Furthermore, questions of causality, properness and stability are discussed. It is shown how this method can be used to stabilize neutral plants which are not formally stable. Finally, a finite spectrum assignment is studied. Namely it is shown that one can always change an originally infinite number of poles to a finite one by means of a point delay controller.

2. BACKGROUND

As to the mathematics and notation, we employ real polynomials in two indeterminates, $\boldsymbol{d}$ and $\boldsymbol{s}$, which will be interpreted as delay and differentiation operators. So $\mathscr{R}[\boldsymbol{d}]$, $\mathscr{R}(\boldsymbol{d})$, $\mathscr{R}[\boldsymbol{d}, \boldsymbol{s}]$ and $\mathscr{R}[\boldsymbol{d}][\boldsymbol{s}]$ $(\mathscr{R}(\boldsymbol{d})[\boldsymbol{s}])$ stand as usual for the ring of (real) polynomials in $\boldsymbol{d}$, the field of rational functions in $\boldsymbol{d}$, the ring of 2-D polynomials in $\boldsymbol{d}$ and $\boldsymbol{s}$, and the ring of polynomials in $\boldsymbol{s}$ with coefficients from $\mathscr{R}[\boldsymbol{d}]$ $(\mathscr{R}(\boldsymbol{d}))$, respectively. The greatest power of $\boldsymbol{s}$ occurring in a polynomial a will be called its $\boldsymbol{s}$-degree and denoted by $\deg_{\boldsymbol{s}} a$.

An ordered couple (d, s) of complex numbers is a *zero* of $a \in \mathscr{R}[\boldsymbol{d}, \boldsymbol{s}]$ iff $a(d, s) = 0$. Polynomials $a, b \in \mathscr{R}[\boldsymbol{d}, \boldsymbol{s}]$ are *zero coprime* iff they have no common zero and *factor coprime* iff they have no nonconstant factor in common. Finally, $a \mid b$ means "a divides b" while (a, b) stands for their greatest common divisor.

A single-input single-output linear delay-differential *plant* with point commensurate delays will be represented by its transfer function

$$b/a \tag{2.1}$$

where

$$a = a_0(\boldsymbol{d}) + a_1(\boldsymbol{d})\,\boldsymbol{s} + \ldots + a_n(\boldsymbol{d})\,\boldsymbol{s}^n$$
$$b = b_0(\boldsymbol{d}) + b_1(\boldsymbol{d})\,\boldsymbol{s} + \ldots + b_m(\boldsymbol{d})\,\boldsymbol{s}^m$$

Here $\boldsymbol{d} = \mathrm{e}^{-hs}$ for a delay duration $h \geqq 0$. $a, b \in \mathscr{R}[\boldsymbol{d}, \boldsymbol{s}]$ so that $a_i, b_j \in \mathscr{R}[\boldsymbol{d}]$, $a_n \neq 0$, $b_m \neq 0$ and $n \geqq m$. In addition, we assume that a and b are factor coprime. On the other hand, they generically have common zeros which are referred to as *fixed poles* of (2.1).

The system (2.1) is said to be ([1]) *causal* or *neutral* iff $a_n(0) \neq 0$, *formally stable* iff $a_n(d) \neq 0$ for $|d| \leqq 1$, and *retarded* iff $a_n \in \mathscr{R}$. Of course, it is *proper* or *strictly proper* iff $n \geqq m$ or $n > m$.

To change the behaviour of a plant we apply a *feedback controller* with transfer function

$$y/x \tag{2.2}$$

where $x, y \in \mathscr{R}[\boldsymbol{d}, \boldsymbol{s}]$. As we assume that both the plant (2.1) and the controller (2.2) are realized without hidden modes, the resultant interconnected system from Figure 1 possesses the characteristic polynomial

$$ax + by = c \tag{2.3}$$

On the other hand, facing a CPA problem, we are given a and b, the plant, and c, a *desired characteristic polynomial*

(2.4) $$c = c_0(d) + c_1(d)\, s + \ldots + c_{2n-1}(d)\, s^{2n-1}$$

where $c_i \in \mathscr{R}[d]$ and $\deg_s c = 2n - 1$. To solve the problem we have to find x and y, a controller, to satisfy (2.3). That is why we need to solve a linear 2-D polynomial equation (2.3).

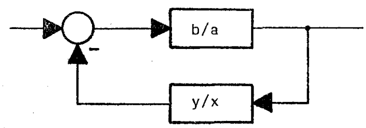

Fig. 1. The resultant system.

3. RETARDED SYSTEMS

A retarded plant is characterized by a denominator which is monic in s, i.e. $a_n = 1$ in (2.1). In addition, to produce a retarded resultant system, we may well assume that also $c_{2n-1} = 1$ in (2.4).

At first, let us glance at (2.3) in $\mathscr{R}(d)[s]$. This ring is Euclidean so that we may apply a standard 1-D theory [12] to get the following lemma (recall that a, b are assumed factor coprime):

Lemma 3.1. There always unique $\tilde{x}, \tilde{y} \in \mathscr{R}(d)[s]$ such that

(3.1) $$a\tilde{x} + b\tilde{y} = c \quad \text{and} \quad \deg_s \tilde{y} < n\,.$$

What we need to find a controller, however, is a solution from $\mathscr{R}[d, s]$ rather than from $\mathscr{R}(d)[s]$. Unfortunately, the ring $\mathscr{R}[d, s]$ is no longer Euclidean so that we seem to be in a more difficult position. Nevertheless, a basic result still resembles the preceding lemma:

Lemma 3.2. Let $a_n = 1$. Then the equation (2.3) possesses a unique solution $x, y \in \mathscr{R}[d, s]$ satisfying

(3.3) $$\deg_s y < n$$

whenever it is solvable.

Proof. The original proof can be found in [20]. Suppose that $x', y' \in \mathscr{R}[d, s]$ is a solution of (2.3). The general solution then reads

3.4) $$x = x' + bt$$
$$y = y' - at$$

for an arbitrary parameter $t \in \mathscr{R}[d, s]$. As $a_n = 1$, one can immediately divide y' by a to get y as a remainder. □

Putting Lemma 3.1 and Lemma 3.2 together and taking into account that $\mathscr{R}[d, s] \subset \mathscr{R}(d)[s]$, we can easy derive the main result of this section – necessary and sufficient solvability conditions:

Theorem 3.3. Given a retarded plant (2.1) and a desired characteristic polynomial (2.4), the CPA problem has a solution if and only if $\tilde{x}$, $\tilde{y}$ given by (3.1) satisfy

$$\tilde{x}, \tilde{y} \in \mathscr{R}[d, s] \tag{3.5}$$

Moreover, then there exists a unique *minimal controller*, for which $\deg_s y < n$, and this controller is given by

$$x = \tilde{x} \quad \text{and} \quad y = \tilde{y} \tag{3.6}$$

Evidently, Theorem 3.3 provides a nice way to solve (2.3): we may simply solve it in $\mathscr{R}(d)[s]$ rather than in $\mathscr{R}[d, s]$. This can be done by any Euclidean ring algorithm e.g. [10]. Once resulting $\tilde{x}$, $\tilde{y}$ from (3.1) are indeed polynomials in both d and s, they immediately yield the desired solution (2.6). On the other hand, whenever they contain a fraction in d then (2.3) has no solution at all.

Example 3.1. As an example, let us consider a plant having $a = d + ds + s^2$ and $b = 1 + s$ along with the desired $c = c_0 + c_1 s + c_2 s^2 + s^3$, $c_0, c_1, c_2 \in \mathscr{R}[d]$. Using any Euclidean ring method in $\mathscr{R}(d)[s]$, we obtain the minimal solution (3.1)

$$x = \tilde{x} = c_0 - c_1 + c_2 + s$$

$$y = \tilde{y} = c_0 - d(c_2 - c_1 + c_0) + (c_1 - c_0 - d)\, s\,.$$

Notice that $\tilde{x}$ and $\tilde{y}$ are polynomials from $\mathscr{R}[d, s]$ for any $c_0, c_1, c_2 \in \mathscr{R}[d]$. It means that we can assign an arbitrary characteristic polynomial to its plant.

Example 3.2. As another example, assign the characteristic polynomial $c = c_0 + s$ to the plant $(b_0 + s)/(a_0 + s)$ where $a_0, b_0, c_0 \in \mathscr{R}[d]$ and $a_0 \neq b_0$. Operating in $\mathscr{R}(d)[s]$ one gets

$$\tilde{x} = 1 - (c_0 - a_0)/(b_0 - a_0)\,, \quad \tilde{y} = (c_0 - a_0)/(b_0 - a_0)\,.$$

It is immediately seen that the problem has a solution iff $(b_0 - a_0) \mid (c_0 - a_0)$. In particular, c_0 can be arbitrary iff $(b_0 - a_0) \in \mathscr{R}$. Such a way, applying any 1-D method (in $\mathscr{R}(d)[s]$) for an undetermined c, can always derive solvability conditions for the given plant. The designer can then vary c, having these conditions satisfied, to meet additional requirements.

It has been shown in the above examples that an arbitrary characteristic polynomial can be assigned to certain plants. In contrast to 1-D, such cases are no longer generic here:

Theorem 3.4. One can assign an arbitrary characteristic polynomial (2.4) to a retarded plant (2.1) if and only if the following equivalent conditions hold:

(i) The matrix $\begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}$ is unimodular, where

(3.7) $$\hat{A} = \begin{bmatrix} a_0, a_1, \ldots, a_n & & \\ & a_0, a_1, \ldots, a_n & \\ & \cdots & \\ & & a_0, a_1, \ldots, a_n \end{bmatrix} \quad \hat{B} = \begin{bmatrix} b_0, b_1, \ldots, b_m & & \\ & b_0, b_1, \ldots, b_m & \\ & \cdots & \\ & & b_0, b_1, \ldots, b_m \end{bmatrix}$$

are, respectively, $m \times (n + m)$ and $n \times (n + m)$.

(ii) The plant has no fixed pole (i.e., a and b are zero coprime)

(iii) There are $u, v \in \mathscr{R}[d, s]$ satisfying the Bezout identity

$$au + bv = 1 .$$

Proof. See [21]. □

In practice, one often looks for a proper controller. Now we show that the minimal controller is generically proper:

Theorem 3.5. The minimal controller solving the CPA problem for a retarded plant (2.1) can be improper only if $n = m$ and $b_m \in \mathscr{R}$. Otherwise it is always proper.

Proof. This is easily seen when inspecting highest power in s coefficients in (2.3). □

Example 3.3. While the Example 3.1 illustrates the situation when $n > m$, for the plant with $a = 1 + ds + s^2$ and $b = 1 + s^2$, the characteristic polynomial $c = 1 - 2d + (1 + d)s + s^2 + s^3$ is assigned by the proper controller $y/x = (-2d - s)/(1 + 2s)$ but the requirement $c = (1 + d)s + s^3$ gives rise to $y/x = (-1 + s)$, the improper one.

4. NEUTRAL SYSTEMS

In a neutral plant (2.1), the leading denominator coefficient $a_n \in \mathscr{R}[d]$ is a general polynomial but $a_n(0) = 1$. As we want to produce a resultant neutral system as well, we may assume that also $c_{2n-1}(0) = 1$. To simplify our notation, we denote by q the difference

(4.1) $$q = n - m$$

Let us start with the generic case of

(4.2) $$(a_n, b_m) = 1$$

Its not generic counterpart, $(a_n, b_m) \notin \mathscr{R}$, will be sketched later on.

At first, the analogue of Lemma 3.2 for neutral systems reads:

Lemma 4.1. Let $(a_n, b_m) = 1$. Then the equation (2.3) possesses (at least one) solution $x, y \in \mathscr{R}[d, s]$ satisfying

(4.3) $$\deg_s y < n + q$$

whenever it is solvable.

Proof. Let $x', y' \in \mathscr{R}[d, s]$ be a solution of (2.3) and let $\deg_s x' = k$ and $\deg_s y' = 1$. Then all other solutions are again described by (3.4). To imitate the proof of Lemma 3.2, however, we must cope with the fact that a is no longer monic in s. Namely, to be able to divide y' by a, we wish to show that a_n divides y'_1, the leading coefficient of y'. To do this, we equate the coefficients at the highest powers of s in (2.3). This yields

$$a_n x'_k + b_m y'_1 = 0 \tag{4.4}$$

whenever $1 \geqq n + q$ so that $a_n \mid b_m y'_1$. As $(a_n, b_m) = 1$ by assumption, $a_n \mid y'_1$ results. Finally, iterating this procedure until $\deg_s y < n + q$, the lemma is proved. □

If $n = m$ ($q = 0$), the inequalities (4.3) and (3.3) are identical: that is why there exists a unique minimal solution satisfying (4.3) in this case.

For $n > m$, however, this is no longer true. All the solutions satisfying (4.3) are called here *low s-degree solutions* and are parametrized via

$$\begin{aligned} x &= \bar{x} + bv \\ y &= \bar{y} - av \end{aligned} \tag{4.5}$$

where $\bar{x}, \bar{y}$ is any of them and $v \in \mathscr{R}[d, s]$ is an arbitrary polynomial parameter up to s-degree $q - 1$. Generically, indeed $\deg_s y = n + q - 1$. Only in particular cases (which will be treated later on) this set has a unique minimum satisfying (4.4) again.

Analogously to the preceding section, Lemma 4.1 will serve us to derive solvability conditions. As now we have the whole set of low s-degree solutions rather than a unique one, we employ a different procedure. Form matrices

$$A = \begin{bmatrix} a_0, a_1, \ldots, a_n & & \\ & a_0, a_1, \ldots, a_n & \\ & \ldots & \\ & & a_0, a_1, \ldots, a_n \end{bmatrix} \quad B = \begin{bmatrix} b_0, b_1, \ldots, b_m & & \\ & b_0, b_1, \ldots, b_m & \\ & \ldots & \\ & & b_0, b_1, \ldots, b_m \end{bmatrix} \tag{4.6}$$

$$C = [c_0, c_1, \ldots, c_{2n-1}]$$

where A is $n \times 2n$, B is $(n + q) \times 2n$, C is $1 \times 2n$ and all their entries are from $\mathscr{R}[d]$. For a low s-degree solution x and y form similarly

$$X = [x_0, x_1, \ldots, x_{n-1}] \quad Y = [y_0, y_1, \ldots, y_{n+q-1}] \tag{4.7}$$

Using the above matrices, the original 2-D *scalar* polynomial equation (2.3) can be transformed into a 1-D *matrix* polynomial equation

$$XA + YB = C \tag{4.8}$$

Now if (4.8) has a solution then clearly (2.3) has a solution as well. On the other hand, the converse is true to Lemma 4.1. Thus, with a little help from 1-D polynomial matrix theory [12], we have derived the desired solvability conditions:

Theorem 4.2. Given a neutral plant (2.1) with $(a_n, b_m) = 1$ and a desired characteristic polynomial (2.4), the CPA problem has a solution if and only if a greatest common right divisor of A and B is a right divisor of C.

Moreover, then there exist low s-degree solutions (4.3) which all can be obtained via the solution of (4.8).

As a consequence of Theorem 4.2, one can solve 1-D polynomial matrix equation (4.8) by any 1-D method (see, e.g., [12]) and thereby gets a low s-degree solution of the CPA problem. In a particular case of $n = m$ possesses (4.2) a unique solution which gives rise directly to the minimal solution of the CPA. In any case, other solutions can be obtained via (3.4).

Example 4.1. As an example, let us consider a plant with $a = d + (1 + d) s^2$ and $b = d$, and a desired $c = c_0 + c_1 s + c_2 s^2 + c_3 s^3$, $c_i \in \mathcal{R}[d]$. For these data (4.8) takes the form

$$[x_0, x_1]\begin{bmatrix} d, & 0, & 1+d, & 0 \\ 0, & d, & 0, & 1+d \end{bmatrix} + [y_0, y_1, y_2, y_3]\begin{bmatrix} d, & 0, & 0, & 0 \\ 0, & d, & 0, & 0 \\ 0, & 0, & d, & 0 \\ 0, & 0, & 0, & d \end{bmatrix} = [c_0, c_1, c_2, c_3]$$

Applying any 1-D method, we can find out that this equation is solvable iff d divides both c_0 and c_1 and then it has got a solution

$$X = [c_2, c_3] + [v_1, v_2]\begin{bmatrix} -d, & 0 \\ 0, & -d \end{bmatrix}$$

$$Y = [-c_2 + c_0/d, -c_3 + c_1/d, -c_2, -c_3] + [v_1, v_2]\begin{bmatrix} d, & 0, & 1+d, & 0 \\ 0, & d, & 0, & 1+d \end{bmatrix}$$

which yields

(4.9)

$$x = c_2 + c_3 s - d(v_1 + v_2 s)$$

$$y = -c_2 + c_0/d + (-c_3 + c_1/d) s - c_2 s^2 - c_3 s^3 + (d + (1 + d) s^2)(v_1 + v_2 s)$$

For an arbitrary $v_1, v_2 \in \mathcal{R}[d]$, this is the parametrization of all the low s-degree solutions.

If we denote by G a greatest common right divisor of A and B, we can easily parametrize all the characteristic polynomials up to s-degree $2n - 1$ which can be assigned to a given plant.

Theorem 4.3. Let be given a plant (2.1) with $(a_n, b_m) = 1$. All the $c \in \mathcal{R}[d, s]$ satisfying $\deg_s c \leqq 2n - 1$ for which the CPA problem is solvable are expressed by means of a matrix C given by

(4.10) $$C = [e_0, e_1, \ldots, e_{2n-1}]\, G$$

with arbitrary parameters $e_i \in \mathcal{R}[d]$.

Consequently, having computed G, the designer is free to choose e_i for the resultant characteristic polynomial to meet additional requirements.

Again, there are plants which an arbitrary characteristic polynomial can be assigned to. Unlike classical 1-D systems, such cases are not generic within delay-differential systems at all.

Theorem 4.4. One can assign an arbitrary characteristic polynomial (2.4) to a neutral plant (2.1) with $(a_n, b_m) = 1$ if and only if the following equivalent conditions hold:

(i) A and B are right coprime (i.e., $\det G \in \mathscr{R}$)

(ii) The plant has no fixed poles

(iii) The matrix $\begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}$ given by (3.7) is unimodular.

Proof. (i) is evident from Theorem 4.3. The equivalence of (i) and (iii) can be seen by inspection and the equivalence of (ii) and (iii) is a standard result [22]. □

The reader might notice in the Example 4.1 that for most of c is the solution (4.3) improper. Only if $(1 + \boldsymbol{d})$ divides both c_2 and c_3, one can take $v_1 = c_2/(1 + \boldsymbol{d})$ and $v_2 = c_3/(1 + \boldsymbol{d})$, $v_1, v_2 \in \mathscr{R}[\boldsymbol{d}]$ so that a proper solution results. We have shown that for retarded plants is the minimal solution generically proper. This holds true for neutral plants as well. There is, however, a basic difference here: minimal controller generically does not exist for neutral plants. Some authors [5, 19, 17] have already noticed that a derivative feedback is necessary to change the leading coefficient. This, however, is exactly true only if $m = n - 1$. We are going to prove here that the improper (derivative) feedback is *not necessary* for $m = n$ and, on the other hand, the first derivative is even not sufficient when $m < n - 1$.

To this analyse, define for $q > 0$ matrices

$$(4.11) \qquad \bar{A}_q = \begin{bmatrix} a_n & & \\ a_{n-1}, a_n & & \\ \cdots & & \\ a_{n-q+1}, & \ldots, & a_n \end{bmatrix} \quad \bar{B}_q = \begin{bmatrix} b_m & & \\ b_{m-1}, b_m & & \\ \cdots & & \\ b_{m-q+1}, & \ldots, & b_m \end{bmatrix}$$

$$\bar{C}_q = [C_{2n-q}, \ldots, C_{2n-1}]$$

Now the properness conditions are as follows:

Theorem 4.5. Let the CPA problem with $(a_n, b_m) = 1$ be solvable.

If $n > m$ then there exists a unique proper solution (the minimal one) if and only if $\bar{A}_q$ is a right divisor of $\bar{C}_q$.

If $n = m$ then the minimal controller is proper whenever b_m does not divide c_{2n-1}.

Proof. The case of $n = m$ can be proved simply by inspecting the highest s-degree coefficients in (2.3).

For $n > m$, denote by x', y' a low s-degree solution. Comparing coefficients at powers of s higher than $n + m - 1$ in (2.3) one gets

$$[x'_m, \ldots, x'_{n-1}]\, \bar{A}_q + [y'_n, \ldots, y'_{n+q-1}]\, \bar{B}_q = \bar{C}_q$$

or, when postmultiplying it by $\bar{A}_q^{-1}$,

$$[x'_m, \ldots, x'_{n-1}] + [y'_n, \ldots, y'_{n+q-1}]\, \bar{B}_q \bar{A}_q^{-1} = \bar{C}_q \bar{A}_q^{-1} \tag{4.12}$$

As the first term is a polynomial, the second is so iff the right hand side is a polynomial. Now $\bar{B}_q \bar{A}_q^{-1} = \bar{A}_q^{-1} \bar{B}_q$ and the both fractions are coprime so that $\bar{A}_q$ is a right divisor of $\bar{C}_q$ iff it is a right divisor of $[y'_n, \ldots, y'_{n+q-1}]$. Just in such a case one can apply (4.5) to reduce the s-degree of y below n which yields the proper (and minimal) solution. □

The reader should notice that, whenever $n > m$ and $a_n \notin \mathscr{R}$, then the requirements of Theorem 4.5 *cannot* be satisfied generically. By means of a proper controller, therefore, we cannot assign the coefficients at high powers of s freely: $\bar{A}_q$ must divide $\bar{C}_q$.

Example 4.2. As an illustration, consider $a = 2 + (2 + \boldsymbol{d})\, s^2$, $b = 1$ and $c = 2 + 2s + 2s^2 + (2 + \boldsymbol{d})\, s^3$. This results in $y/x = (-\boldsymbol{d}s^2 - (2 + (2 + \boldsymbol{d})\, s^2)\, v) : (1 + s + v)$. Although $a_2 = c_3$, the solution is improper for any $v \in \mathscr{R}[\boldsymbol{d}, s]$. The reason is that $\bar{A}_2 = \begin{bmatrix} 2 + \boldsymbol{d} & 0 \\ 0 & 2 + \boldsymbol{d} \end{bmatrix}$ is not a right divisor of $\bar{C}_2 = [\boldsymbol{d}, 2 + \boldsymbol{d}]$.

To make our exposition complete, let us finally discuss the non-generic case of $(a_n, b_m) \notin \mathscr{R}$. The main difference this brings is that an upper bound of low s-degree solutions is now higher than (4.3).

Lemma 4.6. Let $r \geqq 0$ be an integer such that the lower-triangular greatest common right divisor of matrices $\bar{A}_{r+1}$ and $\bar{B}_{r+1}$ defined by (4.11) has its last column-last row entry from $\mathscr{R}$. Then the equation (2.3) possesses (at least one) solution $x, y \in \mathscr{R}[\boldsymbol{d}, s]$ satisfying

$$\deg_s y < n + q + r \tag{4.13}$$

whenever it is solvable.

Proof. Let $x', y' \in \mathscr{R}[\boldsymbol{d}, s]$ be a solution of (2.3) Let $\deg_s x' = k$, $\deg_s y' = 1$ and define $j = 1 - r$. As in the proof of Lemma 4.1 we wish to show that $a_n \mid y'_1$ whenever $1 \geqq n + q + r$. To do this, we compare the coefficients at powers of s higher than $1 - 2r + m$ in (2.3) and, as in the proof of Lemma 4.5, we get

$$[x_{j-q}, \ldots, x_k] + [y'_j, \ldots, y'_1]\, \bar{B}_{r+1} \bar{A}_{r+1}^{-1} = 0 \tag{4.14}$$

Now $\bar{B}_{r+1} \bar{A}_{r+1}^{-1} = \bar{A}_{r+1}^{-1} \bar{B}_{r+1} = F^{-1} H$, where F and H are some left coprime lower triangular matrices. By the definition of r, the last-column last-row entry of F is equal to a_n. As (4.14) can hold only if this F is a right divisor of $[y'_j, \ldots, y'_1]$, $a_n \mid y'_1$ results immediately. □

If we define matrices A_r, B_r and C_r of sizes $(n + r) \times (2n + r)$, $(n + q + r) \times \times (2n + r)$ and $1 \times (2n + r)$, respectively, analogously to (4.6), we can transform the original equation (2.3) into

$$X_r A_r + Y_r B_r = C_r \tag{4.15}$$

(Notice that for $r = 1$ we have $A_1 = A$, $B_1 = B$ and $C_1 = C$ while (4.15) becomes (4.8).) From an analogy, the general solvability conditions are as follows:

Theorem 4.7. Let r be the least integer satisfying the assumption of Lemma 4.6. Then the CPA problem has a solution if and only if a greatest common right divisor of A_r and B_r is a right divisor of C_r.

If $(a_n, b_m) = 1$, then simply $r = 0$ and this theorem is consistent with Theorem 4.2. Let $(a_n, b_m) = g \notin \mathscr{R}$ so that $a_n = g\bar{a}_n$ and $b_m = g\bar{b}_m$. Then $r = 1$ iff $(g, a_{n-1}\bar{b}_m - b_{m-1}\bar{a}_n) = 1$ (see [20]). Otherwise $r > 2$.

5. CAUSALITY AND PROPERNESS

From a practical point of view, it is often desirable to produce a causal (neutral) controller for which the leading coefficient $x_1 \in \mathscr{R}[d]$ satisfies $x_1(0) \neq 0$. Of course, we assume that the plant itself is causal, $a_n(0) = 1$, and also $c_{2n-1}(0) = 1$. When comparing coefficients at highest powers of s in (2.3), we can easy derive the theorem:

Theorem 5.1. If $n > m$ then the minimal controller is always causal. If $n = m$ then it may be non-causal only if $b_m(0) \neq 0$.

In most of practical cases we want to produce a proper resultant system. Naturally, we assume that the plant (2.1) itself is proper. On the other hand, the proper controlers have already been discussed in the last section. However, in the light of recent studies, the properness of a controller is *not* the crucial point. When connecting systems, one should take care of the *internal properness* of the resultant system [13]. This means, roughly speaking, that no puls modes can occur in the system as a response to initial conditions even if the system may include some derivative parts. Adapting 1-D results of [13] to delay-differential systems, one can easily prove the theorem:

Theorem 5.2. The interconnection of a plant (2.1) and the minimal controller results always in an internally proper system. On the other hand, using a higher order (nonminimal) controller (assigning c with $\deg_s c = 2n - 1$), the overall system is never internally proper.

So using a minimal controller we need not take care of the properness. When this minimal controller fails to exist, however, knowledge of an improper low s-degree solution may still be useful: For example, the designer can use it to look for additional measurable signals (derivatives of the output) in the real plant. Feeding them back one eliminates the need of an improper controller. As another way out, one can simply increase the degree of a desired c. Taking $\deg_s c = 2n - 1 + k$, $k > 0$, all the solutions (4.5) with $\deg_s v > k$ give rise to an internally proper resultant system.

6. STABILITY

In most practical cases we want to produce a stable system. That is why we interest whether there exists a stable c (2.4) for which the problem has a solution. For the basic studies of stability of delay-differential systems, the reader is referred to [1]. Emre and Knowles [8] recently developed stabilizability results for retarded and formally stable neutral plants. In our context, their results read as follows:

Theorem 6.1. For a retarded plant (2.1), there exists a stabilizing controller (2.2) if and only if every complex number s such that $a(e^{-hs}, s) = b(e^{-hs}, s) = 0$ satisfies $\operatorname{Re} s < 0$.

For a formally stable neutral plant (2.1) there exists a stabilizing controller (2.2) if (and practically only if) there is a real $\gamma > 0$ such that for every s for which $a(e^{-hs}, s) = b(e^{-hs}, s) = 0$ is $\operatorname{Re} s < -\gamma$.

Proof. See [8]. For the frequency domain interpretation see also [5, 3]. □

Hence a retarded (formally stable neutral) plant can be stabilized if and (practically) only if every its fixed pole (d, s) such that $d = e^{-hs}$ satisfies $\operatorname{Re} s < 0$ $(-\gamma)$. The stabilizability can therefore be checked via $\det \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}$ as it yields the fixed poles. If all of them such that $d = e^{-hs}$ lay in the stability region, one can find in the class of all assignable polynomials (4.10) a stable one. This, however, can still be a difficult task. Moreover, nothing is known about the degree of this polynomial thus far.

For systems which are *not formally stable* no stabilizability results are known up to now. A method to change the leading coefficient by a derivative state feedback was proposed in [17, 19] which is applicable for $n = m + 1$. Using polynomial techniques, however, one can simply change the leading denominator coefficient for general $m \leqq n$. As a result, a formally stable or even retarded system is produced for which the stabilizability conditions are known. We now illustrate the procedure on simple examples.

Example 6.1. For a not formally stable plant with $a = 1 + (1 + \boldsymbol{d})\boldsymbol{s}$ and $b = \boldsymbol{s}$ the controller $y/x = -\boldsymbol{d}$ yields the $c = 1 + \boldsymbol{s}$ and thereby a retarded resultant system, which is directly stable.

Such a way, one can always produce a formally stable system. This can be a proper controller whenever $n = m$. If $m < n$, however, proper controllers cannot help (Theorem 4.5). Still, an improper one will do the job.

Example 6.2. For the plant having $a = 1 + (1 + \boldsymbol{d})\boldsymbol{s}$ but $b = \boldsymbol{d}$ one should use an improper controller to produce a retarded system, for example, $y/x = -\boldsymbol{s}$ yields $c = 1 + \boldsymbol{s}$.

7. FINITE SPECTRUM ASSIGNMENT

In practical applications one may find convenient to produce a closed-loop system having only finite number of poles. In such a case, $c \in \mathscr{R}[s]$ must be taken as in Example 6.1. This problem was first considered by Manitius and Olbrot [15] who employed distributed delay controllers. We will show here that it is always solvable even within the class of point delay controllers.

Theorem 7.1. For any plant (2.1) there is a controller (2.2) such that

$$ax + by = c \in \mathscr{R}[s]$$

Proof. Denote by (d_i, s_i) common zeros of a and b. They are finite in number as far as a and b are factor coprime. The Hilbert Nullstellensatz [21] then implies that there always exist integers l_i such that

$$ax + by = \prod_i (s - s_i)^{l_i} \tag{7.1}$$

has a solution $x, y \in \mathscr{R}[\boldsymbol{d}, \boldsymbol{s}]$ □

To form a suitable c, s_i can be computed as the solutions of equations $a(d_i, s_i) = 0$ where d_i are zeros of $\det \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}$. Alternatively, expressing a and b as polynomials from $\mathscr{R}[s][d]$, one can use matrices $\hat{A}'$ and $\hat{B}'$ analogues to (3.7) and then $c = \det \begin{bmatrix} \hat{A}' \\ \hat{B}' \end{bmatrix} \in \mathscr{R}[s]$ will do the job.

The degree of c depends on the number of fixed poles (counted with their multiplicities in $\boldsymbol{s}$). Moreover, by a controller (2.3), the poles can no longer be shifted at will when they are to be finite in number. This is clear from (7.1). Simply c must vanish for all $\boldsymbol{s} = s_i$. As a consequence, a stable finite spectrum assignment is not possible if Re $s_i \geqq 0$ for some i although the plant can be well stabilizable by assigning an infinite number of stable poles.

8. CONCLUSION

A 2-D polynomial method of characteristic polynomial assignment for linear delay-differential systems has been described in the paper. Although this approach resembles the classical 1-D one for the first sight, and although the design procedures has been transformed into classical (1-D) Euclidean ring algorithms, we have encountered here phenomena witnessing that these systems differ intrinsically from the classical continuous-time systems without delays. First of all, only such characteristic polynomials can be assigned which possess all the fixed poles of the plant (Theorems 3.3 and 4.2). In other words, an arbitrary polynomial assignment is possible only if the plant is lacking of fixed poles (Theorems 3.4 and 4.4). In addition, if one is limited to apply proper controllers, the class of available characteristic polynomials is even more reduced (Theorem 5.5). To assign a polynomial outside this class, the use of an improper controller is a must. As expected, also stabilizability problem is more difficult. Nevertheless, the 2-D polynomial technique can be employed to stabilize even not formally stable plants. Finally, it has been shown that one can always change an originally infinite number of poles to a finite one.

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Ing. Michael Šebek, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.