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Suboptimal Control on Finite Time Interval

ZDENĚK VOSTRÝ

In the paper an algebraic approach to the numerical computation of suboptimal open-loop control on finite time interval is developed. The linear time-invariant continuous system with known transfer functions are considered.

INTRODUCTION

Some optimal control problems can be considered as finite time problems. In this paper only linear time-invariant continuous systems are considered and hence their \mathcal{L} transfer functions are rational.

Our problem can be formulated as that of finding the best approximation $g(t)$ of the given impulse response $f(t)$ in the following sense: the integral

$$\int_0^T (f(t) - g(t))^2 dt$$

is minimized subject to the conditions

$$\begin{aligned}\mathcal{L}(f(t)) &= v(s)/p(s), \\ \mathcal{L}(g(t)) &= x(s)/y(s), \\ \partial v < \partial p, \quad \partial x < \partial y,\end{aligned}$$

where v , p and x , y are polynomials and ∂v , ∂p and ∂x , ∂y are their respective degrees. For example an approximation of the high order \mathcal{L} transfer function by a low order one may be needed. Assuming the impulse response $g(t)$ partially predetermined by the system with transfer function b/a , $\mathcal{L}(g(t)) = bx/(ay)$, the problem can be described more generally as that of open-loop control.

Consider the open-loop control system in Fig. 1, where

- \mathcal{P} is a realization of the transfer function $b(s)/a(s)$,
- $u(t)$ – input signal with \mathcal{L} transform $x(s)/y(s)$,
- $f(t)$ – reference signal with \mathcal{L} transform $v(s)/p(s)$,
- $e(t)$ – error signal,

then for any given ∂y we can find an input $u(t)$ such that $\int_0^T e^2(t) dt$ is minimized.

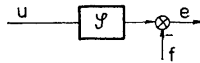


Fig. 1.

Problem formulation:

Let an integer K and polynomials b, a and v, p be given such that $\partial b \leq \partial a, \partial v < \partial p$. Find the polynomials x, y such that $\partial y = K, \partial x < \partial y$ and the following integral

$$\int_0^T (f(t) - g(t))^2 dt$$

is minimized for

$$f(t) = \mathcal{L}^{-1} \left(\frac{v(s)}{p(s)} \right), \quad g(t) = \mathcal{L}^{-1} \left(\frac{b(s)x(s)}{a(s)y(s)} \right).$$

The solution of this problem is complicated due to finite T .

To begin with we introduce some special notation and operations which are more precisely described in [1] and [2]. The mathematical background is the congruence of analytic functions modulo a polynomial.

Consider a polynomial $m(s) = m_0 + m_1s + \dots + m_{\partial m}s^{\partial m}$ with real coefficients and degree $\partial m > 0$. Then the set $\mathcal{M} = \{s : m(s) = 0\}$ is called the spectrum of the polynomial m .

Let functions f, g be analytic on \mathcal{M} . Then f, g are congruent modulo m , written $f = g \pmod m$, if there exists a function h analytic on \mathcal{M} such that $f = g + hm$.

For any function f analytic on \mathcal{M} only one polynomial r exists such that

$$(1) \quad f = r \pmod m, \quad \partial r < \partial m.$$

The polynomial m is called the modulus. The operation which yields such a polynomial r is called the reduction of f modulo m and it is denoted as

$$(2) \quad [f]_m = r.$$

Denote \mathcal{F}_m the set of all functions analytic or having at worst removable singularities on \mathcal{M} .

Let a modulus m , $\partial m > 0$, and function $f, g \in \mathcal{F}_m$ be given. Then for $[f]_m = a$, $[g]_m = b$ and any complex number λ the next equations hold:

(3) $[f + g]_m = [f]_m + [g]_m = a + b$,

(4) $[\lambda f]_m = \lambda [f]_m = \lambda a$,

(5) $[fg]_m = [[f]_m [g]_m]_m = [ab]_m$,

(6) if $f|g \in \mathcal{F}_m$ then

$$\left[\frac{f}{g} \right]_m = \frac{[f]_m}{[g]_m} = \frac{[a]}{[b]}.$$

If the function f is a polynomial then the reduction of f modulo m produces the remainder after dividing f by m . Procedures for the computation of $[f]_m$ for the functions $\ln(s)$, e^{ks} , \sqrt{s} , s^k with k real, and for b/a , $b \cdot a$ with polynomials a, b are described in [1].

Now we introduce a new operation

(7) $\left\langle \frac{s[f]_a}{a} \right\rangle_s = \lim_{s \rightarrow \infty} \frac{s[f]_a}{a} = \frac{c_{k-1}}{a_k}$,

where $k = \partial a$

$$[f]_a = c_0 + c_1 s + \dots + c_{k-1} s^{k-1}$$

and the subscript s denotes the variable with respect to which the operation $\langle \cdot \rangle_s$ is performed.

The next Theorem is proved in [2].

Theorem 1. Let polynomials b, a , $\partial b < \partial a$, with real coefficients be given. Then the inverse Laplace transform of b/a denoted as $\mathcal{L}^{-1}(b/a)$, is the real-valued function $f(t)$ given by

(8) $f(t) = \left\langle \frac{s}{a} [b e^{st}]_a \right\rangle_s$

and furthermore

$$\frac{d^i}{dt^i} f(t) = \left\langle \frac{s}{a} [s^i b e^{st}]_a \right\rangle_s, \quad i = 1, 2, \dots$$

EVALUATION $\int_0^T f(t) g(t) dt$

The basic formula of this paper is in the following Theorem.

Theorem 2. Let polynomials $b, a, v, p, \partial b < \partial a, \partial v < \partial p$ be given. Denote $f(t) = \mathcal{L}^{-1}(b/a), g(t) = \mathcal{L}^{-1}(v/p)$ then

$$(9) \quad I = \int_0^T f(t) g(t) dt = \left\langle \frac{s}{a} \left[b \frac{\bar{v} - e^{sT} [\bar{v} e^{-sT}]_{\bar{p}}}{\bar{p}} \right]_{\bar{a}} \right\rangle_s = \left\langle \frac{s}{p} \left[v \frac{\bar{b} - e^{-sT} [\bar{b} e^{-sT}]_{\bar{a}}}{\bar{a}} \right]_{\bar{p}} \right\rangle_s,$$

where $\bar{v}(s) = v(-s), \bar{b}(s) = b(-s)$.

Proof. Using Theorem 1 and the linearity of the operation $\langle \cdot \rangle_s$, we can write

$$I = \left\langle \frac{s}{a} \left[b \int_0^T e^{st} g(t) dt \right]_{\bar{a}} \right\rangle_s.$$

It is evident that

$$\int_0^T e^{st} g(t) dt = \int_0^{\infty} e^{st} g(t) dt - \int_T^{\infty} e^{st} g(t) dt$$

and $\int_0^{\infty} e^{st} g(t) dt = \bar{v}/\bar{p}$ by definition of \mathcal{L} transform.

By Theorem 1

$$g(t) = \left\langle \frac{s}{p} [v e^{st}]_{\bar{p}} \right\rangle_s$$

and

$$g(T + \tau) = \left\langle \frac{s}{p} [v e^{sT} e^{s\tau}]_{\bar{p}} \right\rangle_s = \left\langle \frac{s}{p} [[v e^{sT}]_{\bar{p}} \cdot e^{s\tau}]_{\bar{p}} \right\rangle_s.$$

Hence

$$\int_T^{\infty} e^{st} g(t) dt = e^{sT} \int_0^{\infty} e^{s\tau} g(T + \tau) d\tau = e^{sT} \frac{[v e^{-sT}]_{\bar{p}}}{\bar{p}}$$

and the proof is complete. Considering $f(t) = \mathcal{L}^{-1}(v/p), g(t) = \mathcal{L}^{-1}(b/a)$ the second formula follows.

Remark. The computation of the formula (9) can be rearranged in the following way. Denote

$$(10) \quad [\bar{v} e^{-sT}]_{\bar{p}a} = x,$$

then $\partial x < \partial p + \partial a$. Use division algorithm for x/\bar{p} , then

$$(11) \quad x = \bar{p}q + r, \quad \partial r < \partial \bar{p}$$

and

$$I = \int_0^T f(t) g(t) dt = \left\langle \frac{s}{a} \left[b e^{sT} \frac{x - [x]_{\bar{p}}}{\bar{p}} \right]_{\bar{a}} \right\rangle_s = \left\langle \frac{s}{a} [b e^{sT} q]_{\bar{a}} \right\rangle_s.$$

Denote

$$g(t) = \mathcal{L}^{-1} \left(\frac{bx}{ay} \right), \quad f(t) = \mathcal{L}^{-1} \left(\frac{t}{p} \right),$$

and

$$I = \int_0^T (f(t) - g(t))^2 dt.$$

Assuming the polynomials a, y, p in normalized form, $a_{2a} = 1, y_{2y} = 1, p_{2p} = 1$, we can write

$$f(t) = \left\langle \frac{s}{s^2 p} [e^{st} t]_p \right\rangle_s,$$

$$g(t) = \left\langle \frac{s}{s^2 a + \epsilon y} [e^{st} bx]_{ay} \right\rangle_s.$$

From the problem formulation the next formula follows.

$$(12) \quad I(x, y) = \int_0^T (f(t) - g(t))^2 dt.$$

Using the variation approach the condition for minimizing (12) is given by

$$(13) \quad \delta I = 2 \int_0^T (f(t) - g(t)) \delta g(t) dt = 0$$

for any variation of $g(t)$ which is given as

$$\delta g(t) = \mathcal{L}^{-1} \left(\frac{b \delta x}{ay} - \frac{bx}{ay^2} \delta y \right).$$

Hence using Theorem 2

$$(14) \quad \delta I = \left\langle \frac{1}{s^2 a + \epsilon y} [b \delta x (\bar{F} - \bar{G})]_{ny} \right\rangle_s - \left\langle \frac{1}{s^2 a + 2 \epsilon y} [bx \delta y (\bar{F} - \bar{G})]_{ny^2} \right\rangle_s = 0,$$

for all $\delta x, \delta y$, where

$$\bar{F} = \frac{\bar{v} - e^{-sT} [e^{-sT} \bar{t}]_p}{\bar{p}},$$

$$\bar{G} = \frac{b \bar{x} - e^{-sT} [e^{-sT} b \bar{x}]_{ay}}{\bar{a} y},$$

$$\bar{a}(s) = a(-s).$$

236 Since

$$\begin{aligned}\delta x &= \delta x_0 + s\delta x_1 + \dots + s^{\partial y - 1}\delta x_{\partial y - 1}, \\ \delta y &= \delta y_0 + s\delta y_1 + \dots + s^{\partial y - 1}\delta y_{\partial y - 1}, \\ y_{\partial y} &= 1\end{aligned}$$

the equation (14) may be written as a set of conditions

$$(15) \quad \begin{aligned}\frac{\partial I}{\partial x_i} &= I_1^{(i)} = \left\langle \frac{s}{s^{\partial a + \partial y}} [s^i b(\bar{F} - \bar{G})]_{ay} \right\rangle_{\mathbf{x}} = 0, \\ \frac{\partial I}{\partial y_i} &= I_2^{(i)} = \left\langle \frac{s}{s^{\partial a + 2\partial y}} [s^i b x(\bar{F} - \bar{G})]_{ay^2} \right\rangle_{\mathbf{x}} = 0, \\ i &= 0, 1, \dots, \partial y - 1.\end{aligned}$$

To solve these nonlinear equations we use Newton's iteration method in the form

$$(16) \quad \begin{aligned}I_1^{(i)} + \delta I_1^{(i)} &= 0, \quad \text{for } i = 0, 1, \dots, \partial y - 1, \\ I_2^{(i)} + \delta I_2^{(i)} &= 0,\end{aligned}$$

where $\delta I_1^{(i)}$, $\delta I_2^{(i)}$ denote variations of $I_1^{(i)}$, $I_2^{(i)}$ with respect to δx , δy .

Note that the application of Theorem 2 to equations (15) gives

$$I_1^{(i)} = \int_0^T (f(t) - g(t)) h^{(i)}(t) dt = 0,$$

where $h^{(i)} = \mathcal{L}^{-1}(s^i H)$, $H = b/ay$

and similarly

$$I_2^{(i)} = \int_0^T (f(t) - g(t)) c^{(i)}(t) dt = 0,$$

where

$$c^{(i)}(t) = \mathcal{L}^{-1}(s^i C), \quad C = \frac{bx}{ay^2}.$$

Hence

$$\begin{aligned}\delta I_1^{(i)} &= \int_0^T ((f(t) - g(t)) \delta h^{(i)}(t) - h^{(i)}(t) \delta g(t)) dt, \\ \delta I_2^{(i)} &= \int_0^T ((f(t) - g(t)) \delta c^{(i)}(t) - c^{(i)}(t) \delta g(t)) dt,\end{aligned}$$

where

237

$$\delta h^{(i)}(t) = \mathcal{L}^{-1} \left(-\frac{s^i b}{ay^2} \delta y \right),$$

$$\delta c^{(i)}(t) = \mathcal{L}^{-1} \left(\frac{s^i b \delta x}{ay^2} - 2 \frac{s^i b x \delta y}{ay^3} \right),$$

$$\delta g(t) = \mathcal{L}^{-1} \left(\frac{b \delta x}{ay} - \frac{b x \delta y}{ay^2} \right).$$

Define

$$\bar{H}^{(i)} = \frac{b(-s)^i - e^{sT} [e^{-sT} (-s)^i b]_{ay}}{\bar{a}y},$$

$$\bar{C}^{(i)} = \frac{\bar{b}\bar{x}(-s)^i - e^{sT} [e^{-sT} \bar{b}\bar{x}(-s)^i]_{\bar{a}y^2}}{\bar{a}y^2}$$

and the elements of the matrices A, B, C, D of the dimension $\partial y \times \partial y$ in the following way.

$$(17) \quad A_{i+1, j+1} = \left\langle \frac{s}{s^{\partial a + \partial y}} [b s^j \bar{H}^{(i)}]_{ay} \right\rangle_s,$$

$$B_{i+1, j+1} = \left\langle \frac{s}{s^{\partial a + 2\partial y}} [b x s^j \bar{H}^{(i)}]_{ay^2} \right\rangle_s - \left\langle \frac{s}{s^{\partial a + 2\partial y}} [b s^{i+j} (F - \bar{G})]_{ay^2} \right\rangle_s,$$

$$C_{i+1, j+1} = B_{j+1, i+1},$$

$$D_{i+1, j+1} = \left\langle \frac{s}{s^{\partial a + 3\partial y}} [2b s^{i+j} (F - G)]_{ay^3} \right\rangle_s - \left\langle \frac{s}{s^{\partial a + 2\partial y}} [b x s^j \bar{C}^{(i)}]_{ay^2} \right\rangle_s,$$

$$i = 0, 1, \dots, \partial y - 1, \quad j = 0, 1, \dots, \partial y - 1.$$

Using Theorem 2 it is simple to prove that A, D are symmetric matrices. Using the matrix notation

$$J_1(X, Y) = \begin{bmatrix} I_1^{(0)} \\ I_1^{(1)} \\ \vdots \\ I_1^{(\partial y - 1)} \end{bmatrix}, \quad J_2(X, Y) = \begin{bmatrix} I_2^{(0)} \\ I_2^{(1)} \\ \vdots \\ I_2^{(\partial y - 1)} \end{bmatrix},$$

$$X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\partial y - 1} \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{\partial y - 1} \end{bmatrix}, \quad y_{\partial y} = 1,$$

238 then

$$(18) \quad \begin{aligned} \delta J_1 &= -A\delta X + B\delta Y, \\ \delta J_2 &= B'\delta X + D\delta Y. \end{aligned}$$

Let us point out that $I_1^{(i)}$ can be computed as

$$\begin{aligned} I_1^{(i)} &= \left\langle \frac{s}{s^{\delta a} + \delta y} [s^i b \bar{F}]_{ay} \right\rangle_s - \left\langle \frac{s}{s^{\delta a} + \delta y} [bx \bar{H}^{(i)}]_{ay} \right\rangle_s, \\ I_1^{(i)} &= \left\langle \frac{s}{s^{\delta a} + \delta y} [s^i b \bar{F}]_{ay} \right\rangle_s - \sum_{j=0}^{\delta y - 1} A_{i+1, j+1} x_j \end{aligned}$$

and $I_2^{(i)}$ corresponds to the second term in the formula for $B_{i+1,1}$. Hence in our iteration process, we can define for given Y the optimal X by

$$(19) \quad X = A^{-1} J_1(0, Y).$$

Substituting (19) into (18) and using Newton's formula

$$\begin{aligned} -A\delta X + B\delta Y &= 0, \\ J_2 + B'\delta X + D\delta Y &= 0 \end{aligned}$$

we obtain

$$\delta Y = -(D + B'A^{-1}B)^{-1} J_2.$$

Now we summarize the iteration algorithm.

- (1) Start from initial condition $y = s^{\delta y}$.
- (2) Compute optimal X given by $X = A^{-1} J_1$.
- (3) Compute the value $Y = Y + \delta Y$, where

$$\delta Y = -(D + B'A^{-1}B)^{-1} J_2.$$

- (4) If the n -th iteration and the $(n-1)$ -th iteration are not sufficiently near go to (2).
- (5) Print results.

This algorithm was programed and tested on IBM 370/135 computer using PL/I language.

Example 1. Consider the reference signal $f(t) = \mathcal{L}^{-1}(v/p)$, where

$$\begin{aligned} v &= 0.03387 - 0.06093s - 1.1634s^2 - 0.43164s^3 + 8.6775s^4 - \\ &\quad - 9.525s^5 + 0.9898s^6 + 1.5379s^7, \end{aligned}$$

$$p = 0.01129 - 0.03329s - 1.1056s^2 + 5.82213s^3 - 1.77933s^4 - 6.36952s^5 + 1.97685s^6 + s^7 + 0.5s^8,$$

and the system transfer function b/a , where

$$b = 1, \\ a = 1 + 0.2s + 0.04s^2.$$

Find polynomials x, y such that $\partial y = 3$ and the integral

$$\int_0^5 (f(t) - g(t))^2 dt, \quad \text{where } g(t) = \mathcal{L}^{-1}\left(\frac{bx}{ay}\right),$$

is minimized.

The result is obtained after five iterations in the form

$$x = 0.8023 - 1.225s + 0.1849s^2, \\ y = -0.01124 - 0.007296s - 0.000608s^2 + s^3.$$

The functions $f(t), g(t)$ are plotted in Fig. 2.

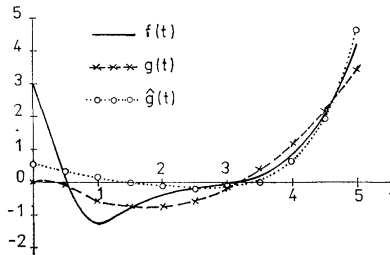


Fig. 2.

Example 2 Consider the transfer function v/p as in Example 1. Find a 3-order approximation of this transfer function such that the integral

$$\int_0^5 (f(t) - \hat{g}(t))^2 dt, \quad \text{where } \hat{g}(t) = \mathcal{L}^{-1}(x/y)$$

is minimized. The solution is similar as in the previous Example, but $b = 1, a = 1$.

After five iterations we obtain the result

$$\begin{aligned}x &= 0.69452 - 1.2376s + 0.5877s^2, \\y &= 0.6626 - 0.6487s - 1.011s^2 + s^3.\end{aligned}$$

The function $\hat{g}(t)$ is plotted in Fig. 2.

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