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ZDENĚK VOSTRÝ

The spectral factorization of transfer function matrices is an intrinsic problem arising in the optimal control of linear discrete systems with respect to a quadratic performance index.

The presented method is based on the comparison of two approaches — in the time domain and in the frequency domain — to a special control problem.

INTRODUCTION

To develop a numerical method of matrix spectral factorization, we first solve the problem of optimal control for a simple two-input two-output system with respect to a quadratic perfo^r^mace index. Without any loss of generality we have chosen the system transfer function matrix with polynomial entries.

The first part of the paper shows how to transform the solution of the above problem in the frequency domain into the solution of infinitely many simultaneous equations in the time domain. This system of equations enjoys a particular structure so that a solution has been found, as shown in Example 1.

Further, the classical frequency-domain approach leading to the spectral factorization of matrices, is analyzed. The problem having a unique solution, we deduce that both solutions coincide. Thus relating these solutions yields a relationship between the solution of the infinite system of simultaneous equations and the spectral factorization. This method is shown in Example 2, which completes Example 1.

In the last part of the paper the generalization to n-input n-output systems is given.

We stress at this point that full rank of the matrix to be factorized is necessary as well as sufficient for the present method to hold.

This method can be applied equally to the matrices whose entries are rational functions in z^{-1} after having multiplied the matrix by the least common denominator of all its entries. The factorization of the original matrix is obtained by simply multiplying the result by the factorized least common denominator. The spectral factorization of polynomials is described elsewhere [3].

Consider a particular two-input two-output system having the transfer function matrix with polynomial entries.

We seek for those inputs that will bring the outputs of the system to zero for any initial state and, in doing so,

(1) (i) will minimize the squared outputs and

(2) (ii) both inputs will be zero in the steady state.

The above conditions imply the physical realizability as well as stability of the optimal control.

We denote the transfer function matrix of the system as

$$S(z) = z^{-1}\begin{bmatrix} S_{11}(z) & S_{12}(z) \\ S_{21}(z) & S_{22}(z) \end{bmatrix}, \tag{3}$$

the inputs as

$$Y(z) = \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix}, \tag{4}$$

and the outputs as

$$X(z) = \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix}.$$

We find it convenient to further introduce initial conditions in the bilateral Z-transform

$$Y^-(z) = \begin{bmatrix} Y_1^-(z) \\ Y_2^-(z) \end{bmatrix}, \tag{5}$$

where

$$Y_1^-(z) = \sum_{j=1}^{\xi} y^1_{-j} z^j, \quad Y_2^-(z) = \sum_{j=1}^{\eta} y^2_{-j} z^j,$$

$$\xi = \max(n_{11}, n_{21}), \quad \eta = \max(n_{12}, n_{22})$$

and n_{ij} stand for the degrees of the polynomials $S_{ij}(z)$.

The input and the output of the system obey the equation

$$X(z) = S(z)\,(Y(z) + Y^-(z)). \tag{6}$$

Let the performance index be given as

$$I = \sum_{j=0}^{\infty} ((x_j^1)^2 + (x_j^2)^2) \tag{7}$$

where

$$X_1(z) = \sum_{j=0}^{\infty} x_j^1 z^{-j}, \quad X_2(z) = \sum_{j=0}^{\infty} x_j^2 z^{-j}. \tag{8}$$

This is, in fact, the sum of squared outputs x_j^1 and x_j^2.

It follows from the Z-transform theory that

$$I = \frac{1}{2\pi \mathrm{j}} \oint_{\Gamma} (\bar{X}_1 X_1 + \bar{X}_2 X_2) \frac{\mathrm{d}z}{z} \tag{9}$$

where j is the imaginary unit, Γ is the unit circle $|z| = 1$,

$$\bar{X}_1(z) = X_1(z^{-1}), \quad \bar{X}_2(z) = X_2(z^{-1}).$$

In the matrix shortland, performance index (8) can be written as

$$I = \frac{1}{2\pi \mathrm{j}} \oint_{\Gamma} \bar{X}^{\mathrm{T}}(z) \,.\, X(z) \frac{\mathrm{d}z}{z} = \frac{1}{2\pi \mathrm{j}} \oint_{\Gamma} \bar{X}^{\mathrm{T}}(z^{-1}) \, X(z) \frac{\mathrm{d}z}{z} \tag{10}$$

(T denotes the transpose).

The expression (10) is still valid for a general multivariable system.

MINIMIZATION OF (10) IN THE TIME DOMAIN

By (2), for any $\varepsilon > 0$ there exists an N such that

$$|y_i^1| < \varepsilon, \quad |y_i^2| < \varepsilon \quad \text{for any} \quad i > N.$$

Consequently, up to a small error which tends to zero for $N \to \infty$, performance index (10) depends only on a finite sequence of the system inputs. Hence it follows that

$$I = I(y_0^1, y_1^1, \dots, y_N^1, y_0^2, y_1^2, \dots, y_N^2). \tag{11}$$

The minimum of I is achieved if

$$\frac{\partial I}{\partial y_i^1} = 0, \quad i = 0, 1, 2, \dots, N \tag{12}$$

$$\frac{\partial I}{\partial y_i^2} = 0, \quad i = 0, 1, 2, \dots, N. \tag{13}$$

Substituting (6) into (10) gives us

$$I = \frac{1}{2\pi \mathrm{j}} \oint_{\Gamma} (Y^{\mathrm{T}}(z^{-1}) + Y^{-\mathrm{T}}(z^{-1})) \, S^{\mathrm{T}}(z^{-1}) \, S(z) \, (Y(z) + Y^{-}(z)) \frac{\mathrm{d}z}{z}. \tag{14}$$

Now substitute (14) into (12) and (13) and rewrite

$$(15)\qquad \frac{\partial I}{\partial y_i^1} = \frac{1}{2\pi j} \oint_\Gamma [z^i\ 0]\ S^T(z^{-1})\ S(z)\,(Y(z) + Y^-(z))\,\frac{dz}{z} = 0\,,$$

$$i = 0, 1, 2, \ldots, N\,,$$

$$(16)\qquad \frac{\partial I}{\partial y_i^2} = \frac{1}{2\pi j} \oint_\Gamma [0\ z^i]\ S^T(z^{-1})\ S(z)\,(Y(z) + Y^-(z))\,\frac{dz}{z} = 0\,,$$

$$i = 0, 1, 2, \ldots, N\,.$$

On integration, both equations (15) and (16) yield N and N simultaneous linear equations in unknowns y_i^1, y_i^2.

By (3) we carry out

$$(17)\qquad S^T(z^{-1})\,S(z) = z \begin{bmatrix} S_{11}(z^{-1}) & S_{21}(z^{-1}) \\ S_{12}(z^{-1}) & S_{22}(z^{-1}) \end{bmatrix} z^{-1} \begin{bmatrix} S_{11}(z) & S_{12}(z) \\ S_{21}(z) & S_{22}(z) \end{bmatrix} =$$

$$= \begin{bmatrix} A(z) & B(z) \\ B(z^{-1}) & C(z) \end{bmatrix}$$

with the obvious definition of A, B and C.

Write

$$(18)\qquad S_{ij}(z) = \sum_{r=0}^{r_{ij}} s_r^{ij} z^{-r}\,, \quad i = 1, 2\,; \quad j = 1, 2\,,$$

$$(19)\qquad A(z) = \sum_{r=-\xi}^{\xi} a_r z^{-r}\,,$$

$$B(z) = \sum_{r=-\xi}^{\eta} b_r z^{-r}\,,$$

$$C(z) = \sum_{r=-\eta}^{\eta} c_r z^{-r}\,,$$

where

$$\xi = \max(n_{11} n_{21})\,, \quad \eta = \max(n_{12} n_{22})\,;$$

$$(20)\qquad a_r = a_{-r} \sum_{j=0}^{n_{11}-r} s_j^{11} s_{j+r}^{11} + \sum_{j=0}^{n_{21}-r} s_j^{21} s_{j+r}^{21}\,,$$

$$r = 0, 1, \ldots, \xi\,;$$

$$(21)\qquad b_r = \sum_{j=0}^{n_{12}-r} s_j^{11} s_{j+r}^{12} + \sum_{j=0}^{n_{22}-r} s_j^{21} s_{j+r}^{22}\,,$$

$$r = 0, 1, \ldots, \eta\,;$$

$$b_{-r} = \sum_{j=0}^{n_{11}-r} s_j^{12} s_{j+r}^{11} + \sum_{j=0}^{n_{21}-r} s_j^{22} s_{j+r}^{21}\,, \tag{22}$$
$$r = 0, 1, \ldots, \xi\,;$$

$$c_r = c_{-r} = \sum_{j=0}^{n_{12}-r} s_j^{12} s_{j+r}^{12} + \sum_{j=0}^{n_{22}-r} s_j^{22} s_{j+r}^{22}\,, \tag{23}$$
$$r = 0, 1, \ldots, \eta\,.$$

Remark 1. By convention the empty sum is taken zero.

On substituting from (17), equation (15) takes the form

$$I = \frac{1}{2\pi \mathrm{j}} \oint_\Gamma [z^i\ 0] \begin{bmatrix} A(z) & B(z) \\ B(z^{-1}) & C(z) \end{bmatrix} \begin{bmatrix} Y_1(z) \\ Y_2(z) \end{bmatrix} \frac{\mathrm{d}z}{z} = 0\,.$$

Now substitute (20) through (23) into (15) and integrate it for $i = 0$

$$a_0 y_0^1 + a_1 y_1^1 + \ldots + a_\xi y_\xi^1 + b_0 y_0^2 + b_{-1} y_1^2 + \ldots + b_{-\xi} y_\xi^2 +$$
$$+ a_1 y_{-1}^1 + a_2 y_{-2}^1 + \ldots + a_\xi y_{-\xi}^1 + b_1 y_{-1}^2 + b_2 y_{-2}^2 + \ldots + b_\eta y_{-\eta}^2 = 0\,.$$

In a like manner we obtain the remaining equations by setting $i = 0, 1, \ldots, N$ in (15) and (16).

In a matrix form they read

$$\mathbf{M}'\mathbf{Y}' + \mathbf{B}'\mathbf{Y}^- = \mathbf{0}\,, \tag{24}$$

where (for definiteness assume $\xi > \eta$),

(25)

$$\mathbf{M}' = \left[\begin{array}{cccccccccc|cccccccccc}
a_0 & a_1 & \cdot & \cdots & a_\xi & 0 & \cdots & & & 0 & b_0 & b_{-1} & \cdots & b_{-\xi} & 0 & \cdots & & & & 0 \\
a_1 & a_0 & a_1 & \cdots & & a_\xi & 0 & \cdots & & 0 & b_1 & b_0 & b_{-1} & \cdots & b_{-\xi} & 0 & \cdots & & & 0 \\
\cdot & a_1 & \ddots & & & & \ddots & & & \vdots & \cdot & b_1 & \ddots & & & \ddots & & & & \vdots \\
\vdots & \vdots & & \ddots & & & & \ddots & & & \vdots & \vdots & & \ddots & & & \ddots & & & \\
a_\xi & \cdot & & & \ddots & & & & \ddots & & b_\eta & \cdot & & & \ddots & & & \ddots & & \\
0 & a_\xi & & & & \ddots & & & & & 0 & b_\eta & & & & \ddots & & & \ddots & \\
\vdots & 0 & \ddots & & & & \ddots & & & 0 & \vdots & 0 & \ddots & & & & \ddots & & & 0 \\
0 & 0 & \cdots & 0 & a_\xi & \cdots & a_1 & a_0 & a_1 \cdots & a_\xi & 0 & 0 & \cdots & 0 & b_\eta & \cdots & b_1 & b_0 & b_{-1} \cdots & b_{-\xi} \\
\hline
b_0 & b_1 & \cdots & b_\eta & 0 & \cdots & & & & 0 & c_0 & c_1 & \cdots & c_\eta & 0 & \cdots & & & & 0 \\
b_{-1} & b_0 & b_1 & \cdots & b_\eta & 0 & \cdots & & & 0 & c_1 & c_0 & c_1 & \cdots & c_\eta & 0 & \cdots & & & 0 \\
\cdot & b_1 & \ddots & & & \ddots & & & & \vdots & \cdot & c_1 & \ddots & & & \ddots & & & & \vdots \\
\vdots & \vdots & & \ddots & & & \ddots & & & & \vdots & \vdots & & \ddots & & & \ddots & & & \\
b_{-\xi} & \cdot & & & \ddots & & & \ddots & & & c_\eta & \cdot & & & \ddots & & & \ddots & & \\
0 & b_{-\xi} & & & & \ddots & & & \ddots & & 0 & c_\eta & & & & \ddots & & & \ddots & \\
\vdots & 0 & \ddots & & & & \ddots & & & \vdots & \vdots & 0 & \ddots & & & & \ddots & & & \vdots \\
0 & 0 & \cdots & 0 & b_{-\xi} & \cdots & b_{-1} & b_0 & b_1 \cdots & b_\eta & 0 & 0 & \cdots & 0 & c_\eta & \cdots & c_1 & c_0 & c_1 \cdots & c_\eta\ 0
\end{array}\right]$$

dimension $2N \times (2N + 2\xi)$

$$
(26) \qquad \mathbf{Y}' = \begin{bmatrix} y_0^1 \\ y_1^1 \\ \vdots \\ y_N^1 \\ \vdots \\ y_{N+\xi}^1 \\ y_0^2 \\ y_1^2 \\ \vdots \\ y_N^2 \\ \vdots \\ y_{N+\xi} \end{bmatrix} \qquad \mathbf{B} = \left[\begin{array}{cccccc|ccccc} a_1 & a_2 & \cdot & \cdot & \cdot & a_\xi & b_1 & b_2 & \cdot & \cdot & b_\eta \\ b_{-1} & b_{-2} & \cdot & \cdot & \cdot & b_{-\xi} & c_1 & c_2 & \cdot & \cdot & c_\eta \\ a_2 & a_3 & \cdot & \cdot & a_\xi & 0 & b_2 & \cdot & \cdot & b_\eta & 0 \\ b_{-2} & b_{-3} & \cdot & \cdot & b_{-\xi} & 0 & c_2 & \cdot & \cdot & c_\eta & 0 \\ \cdot & & & & 0 & 0 & \cdot & & & & \vdots \\ \cdot & & & & & \vdots & b_\eta & 0 & & & \\ a_{-\xi} & 0 & & & & & c_\eta & 0 & & & \\ b_{-\xi} & 0 & & & & & 0 & \vdots & & & \\ 0 & \vdots & & & & & \vdots & & & & \\ \vdots & & & & & & & & & & \\ 0 & 0 & \cdots & & & 0 & 0 & 0 & \cdots & & 0 \end{array}\right]
$$

dimension $2N \times (\xi + \eta)$

$$
(27) \qquad \mathbf{Y}^- = \begin{bmatrix} y_{-1}^1 \\ y_{-2}^1 \\ \vdots \\ y_{-\xi}^1 \\ y_{-1}^2 \\ y_{-2}^2 \\ \vdots \\ y_{-\eta}^2 \end{bmatrix}.
$$

This equation cannot be solved for $\mathbf{Y}'$ since $\mathbf{M}'$ is rectangular, that is, there are more unknowns than equations.

By virtue of condition (2), y_i^1 and y_i^2 approach zero for increasing i. Now assume N sufficiently large so that

$$
(28) \qquad y_{N+r}^1 \doteq 0
$$

and

$$
y_{N+r}^2 \doteq 0
$$

for

$$
r > 0 .
$$

Then (28) together with (24) constitute a solvable system of simultaneous equations. The $\mathbf{M}'$ and $\mathbf{Y}'$ are modified to

(29)

$$\mathbf{M}'' = \left[\begin{array}{cccccccc|cccccccc}
a_0 & a_1 & \cdots & a_\xi & 0 & \cdots & & 0 & b_0 & b_{-1} & \cdots & b_{-\xi} & 0 & \cdots & & 0 \\
a_1 & a_0 & a_1 & \cdots & a_\xi & 0 & \cdots & 0 & b_1 & b_0 & b_{-1} & \cdots & b_{-\xi} & 0 & \cdots & 0 \\
\vdots & a_1 & \ddots & & & \ddots & \ddots & \vdots & \vdots & b_1 & \ddots & & & \ddots & \ddots & \vdots \\
a_\xi & \vdots & & \ddots & & & \ddots & 0 & b_\eta & \vdots & & \ddots & & & \ddots & 0 \\
0 & a_\xi & & & \ddots & & & a_\xi & 0 & b_\eta & & & \ddots & & & b_{-\xi} \\
\vdots & 0 & \ddots & & & \ddots & & \vdots & \vdots & 0 & \ddots & & & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & & & \ddots & a_1 & \vdots & \vdots & & \ddots & & & \ddots & b_{-1} \\
0 & 0 & \cdots & 0 & a_\xi & \cdots & a_1 & a_0 & 0 & 0 & \cdots & 0 & b_\eta & \cdots & b_1 & b_0 \\
\hline
b_0 & b_1 & \cdots & b_\eta & 0 & \cdots & & 0 & c_0 & c_1 & \cdots & c_\eta & 0 & \cdots & & 0 \\
b_{-1} & b_0 & b_1 & \cdots & b_\eta & 0 & \cdots & 0 & c_1 & c_0 & c_1 & \cdots & c_\eta & 0 & \cdots & 0 \\
\vdots & b_{-1} & \ddots & & & \ddots & \ddots & \vdots & \vdots & c_1 & \ddots & & & \ddots & \ddots & \vdots \\
b_{-\xi} & \vdots & & \ddots & & & \ddots & 0 & c_\eta & \vdots & & \ddots & & & \ddots & 0 \\
0 & b_{-\xi} & & & \ddots & & & b_\eta & 0 & c_\eta & & & \ddots & & & c_\eta \\
\vdots & 0 & \ddots & & & \ddots & & \vdots & \vdots & 0 & \ddots & & & \ddots & & \vdots \\
\vdots & \vdots & & \ddots & & & \ddots & b_1 & \vdots & \vdots & & \ddots & & & \ddots & c_1 \\
0 & 0 & \cdots & 0 & b_{-\xi} & \cdots & b_{-1} & b_0 & 0 & 0 & \cdots & 0 & c_\eta & \cdots & c_1 & c_0
\end{array}\right],$$

(30)
$$\mathbf{Y}'' = \begin{bmatrix} y_0^1 \\ y_1^1 \\ \vdots \\ y_N^1 \\ y_0^2 \\ y_1^2 \\ \vdots \\ y_N^2 \end{bmatrix}$$

while $\mathbf{Y}^-$ and $\mathbf{B}^-$ remain unaffected. Hence (24) reads

(31)
$$\mathbf{M}''\mathbf{Y}'' + \mathbf{B}'\mathbf{Y}^- = \mathbf{0}\,.$$

The solution of equation (31) simplifies greatly if we realize that y_0^1 and y_0^2 depend only upon the initial conditions $\mathbf{Y}^-$. As a matter of fact, the knowledge of these relations yields recurrent equations for y_i^1 and y_i^2. These will qualify as a solution of (10) with increasing order of the $\mathbf{M}''$ matrix.

To solve equation (31) more easily we modify $\mathbf{M}''$ to a special form. The modification is as follows:

(i) We reorder the rows of $\mathbf{M}''$ thus forming a new matrix $\mathbf{M}'''$, namely, the first row of $\mathbf{M}'''$ is the first row of $\mathbf{M}''$, the second row of $\mathbf{M}'''$ is the $(N + 1)$-th row of $\mathbf{M}''$, the third row of $\mathbf{M}'''$ is the second row of $\mathbf{M}''$, the fourth row of $\mathbf{M}'''$ is the $(N + 2)$-th, etc.

(ii) We form another matrix $\mathbf{M}$ by following the same procedure for the columns of $\mathbf{M}'''$.

The resulting matrix $\mathbf{M}$ thus has the form

(32)

$$\mathbf{M} = \begin{bmatrix}
a_0 & b_0 & a_1 & b_{-1} & a_2 & b_{-2} & \cdot & \cdot & a_\eta & b_{-\eta} & \cdot & \cdot & a_\xi & b_{-\xi} & 0 & \cdots & 0 \\
b_0 & c_0 & b_1 & c_1 & b_2 & c_2 & \cdot & \cdot & b_\eta & c_\eta & 0 & \cdot & \cdot & 0 & 0 & 0 \cdots & 0 \\
a_1 & b_1 & a_0 & b_0 & a_1 & b_1 & & & & & & & a_\xi & b_{-\xi} & 0 & \ddots & \vdots \\
b_{-1} & c_1 & b_0 & c_0 & b_1 & c_1 & & & & & & & & & & & \\
a_2 & b_2 & a_1 & b_1 & \ddots & & & & & & & & & & & & \\
b_{-2} & c_2 & b_{-1} & c_1 & \ddots & & & & & & & & & & & & 0 \\
\cdot & & & & & \ddots & & & & & & & & & & & \\
\cdot & & & & & & & & & & & & & & & & \\
a_\eta & b_\eta & & & & & & & & & & & & & & & \\
b_{-\eta} & c_\eta & & & & & & \ddots & & & & & & & & & \\
\cdot & 0 & & & & & & & & & & & & & & & \\
\cdot & \cdot & & & & & & & & & & & & & & & \\
a_\xi & \cdot & & & & & & & & & & & & & & & \\
b_{-\xi} & 0 & & & & & & & & & & \ddots & & & & & \\
0 & 0 & a_\xi & & & & & & & & & & & & & & \\
\vdots & 0 & b_{-\xi} & 0 & & & & & & & & & & & & & \\
\vdots & \vdots & \ddots & 0 & & & & & & & & & & & \ddots & & \\
0 & 0 & \cdots & 0 & & & & & & & & & & & & &
\end{bmatrix}.$$

We can use the following simple method to write down the matrix $\mathbf{M}$. The technique is especially well-adapted for the computer use.

Write the sequence a_i, b_i, c_i on paper tapes P1 and P2 in the following form

(33)
$$\begin{aligned}
\text{P1} &\equiv \{a_0\ b_0\ a_1\ b_{-1}\ a_2\ b_{-2}\ \ldots\ a_\eta\ b_{-\eta}\ \ldots\ a_\xi\ b_{-\xi}\ 0\ \ldots\ 0\}, \\
\text{P2} &\equiv \{c_0\ b_1\ c_1\ b_2\ \ c_2\ \ldots\ b_\eta\ \ c_\eta\ 0\ \ldots\ldots\ldots\ldots\ldots\ 0\}
\end{aligned}$$

and define generally a paper tape as

$$\text{P} \equiv \{p_0\ p_1\ p_2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}.$$

Introduce the corner paper tape $[\![P_i]\!]$ which defines certain entries of a matrix as follows

$$\begin{array}{cc} & i \\ i & \left[\begin{array}{ccc} & [\![P_i]\!] & \\ & & \\ & & \end{array}\right] \end{array} = \begin{array}{cc} & i \\ i & \left[\begin{array}{cccc} p_0 & p_1 & p_2 & \cdots \\ p_1 & & & \\ p_2 & & & \\ \vdots & & & \end{array}\right] \end{array},$$

then the $\mathbf{M}$ matrix is given as

$$\mathbf{M} = \begin{bmatrix} [\![P1]\!] & & & & \\ & [\![P2]\!] & & & \\ & & [\![P1]\!] & & \\ & & & [\![P2]\!] & \\ & & & & \ddots \end{bmatrix} \tag{34}$$

The modification of $\mathbf{M}''$ to $\mathbf{M}$ induces the modifications of $\mathbf{B}'$ to $\mathbf{B}$ and $\mathbf{Y}'$ to $\mathbf{Y}$.

$$\mathbf{B}' = \left[\begin{array}{cccccc|cccccc}
a_1 & a_2 & \cdot & \cdot & \cdot & a_\xi & b_1 & b_2 & \cdot & \cdot & b_\eta & \\
a_2 & \cdot & \cdot & \cdot & a_\xi & 0 & b_\eta & & \cdot & b_\eta & 0 & \\
\cdot & & & & & & \cdot & & & & & \\
\cdot & & & & & & \cdot & & & & & \\
\cdot & & & & & & b_\eta & 0 & & & & \\
a_\xi & 0 & & & & & 0 & & & & & \\
0 & & & & & & & & & & & \\
\vdots & & & & & & \vdots & & & & \vdots & \\
0 & & \cdots & & & & 0 & 0 & & \cdots & 0 & \\
\hline
b_{-1} & b_{-2} & \cdot & \cdot & \cdot & b_{-\xi} & c_1 & c_2 & \cdot & \cdot & c_\eta & \\
b_{-2} & \cdot & \cdot & \cdot & b_{-\xi} & 0 & c_2 & \cdot & \cdot & c_\eta & 0 & \\
\cdot & & & & & & \cdot & & & & & \\
\cdot & & & & & & \cdot & & & & & \\
\cdot & & & & & & c_\eta & & & & & \\
b_{-\xi} & 0 & & & & & 0 & & & & & \\
0 & & & & & & & & & & & \\
\vdots & & & & & & \vdots & & & & \vdots & \\
0 & & \cdots & & & & 0 & & \cdots & & 0 &
\end{array}\right]. \tag{35}$$

$$\mathbf{Y} = \begin{bmatrix} y_0^1 \\ y_0^2 \\ y_1^1 \\ y_1^2 \\ \vdots \\ y_N^1 \\ y_N^2 \end{bmatrix}. \tag{36}$$

To recall, the first rows of **B** and **Y** are the first rows of **B**′ and **Y**′ respectively, the second rows of **B** and **Y** are the $(N + 1)$-th rows of **B**′ and **Y**′ respectively, etc.

The **B** matrix is of dimension $2N \times (\xi + \eta)$.

We are to solve the equation

$$\mathbf{MY} + \mathbf{BY}^- = \mathbf{0}\,. \tag{37}$$

Since **M** is evidently positive definite, it is invertible and hence

$$\mathbf{Y} = -\mathbf{M}^{-1}\mathbf{BY}^-\,.$$

As shown above, it is sufficient to compute just y_0^2 and y_0^1, i.e., we need just the first two rows of the inverse $\mathbf{M}^{-1}$.

Examining closely the matrix **B**, we find that the $(2\xi + k)$-th row is zero for all $k = 1, 2 \ldots$. Therefore we need only the first 2ξ entries in the first two rows of $\mathbf{M}^{-1}$.

Below we shall establish a theorem on the inverse of a multidiagonal positive definite matrix, which will greatly simplify the problem.

Definition 1. A symmetric matrix $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$, is said to be $(2r - 1)$-*diagonal* if there exists a natural number $r < n$ such that $a_{i,j} = 0$ for $|i - j| + 1 > r$, and $a_{i,j} \neq 0$ for at least one i and j for which $|i - j| + 1 = r$.

Theorem 1. *Let A be a $(2r - 1)$-diagonal positive definite matrix of dimension N. Then the $(r - 1)$-dimensional upper-left-corner submatrix of A^{-1} can be computed via operations on an $(r - 1)$-dimensional matrix.*

Proof. Consider the above matrix A and decompose it into the form below:

$$A = \begin{bmatrix} a_n & P_n & & 0 & \cdots & 0 \\ P'_n & \ddots & \ddots & & \ddots & \vdots \\ & \ddots & a_2 & P_2 & & 0 \\ 0 & & \ddots & a_1 & P_1 & \\ \vdots & \ddots & P'_2 & & & \\ 0 & \cdots & 0 & P'_1 & \begin{bmatrix} Q \end{bmatrix} & \end{bmatrix} \tag{38}$$

where Q, P_i, and P'_i are submatrices of dimensions respectively $(r - 1) \times (r - 1)$, $1 \times (r - 1)$ and $(r - 1) \times 1$.

It is well-known [1] that if A is positive definite then all principal submatrices of A are again positive definite.

This allows us to use a block decomposition of A to compute the A^{-1}.

The first step is to compute Q^{-1}. Then we define

$$Q_1 = \begin{bmatrix} a_1 & P_1 \\ P_1' & Q \end{bmatrix} \tag{39}$$

and write

$$Q_1^{-1} = \begin{bmatrix} e_1 & G_1 \\ G_1' & H_1 \end{bmatrix}.$$

Since $Q_1 Q_1^{-1} = I$, we conclude that

$$a_1 e + P_1 G_1' = I\,, \quad a_1 G_1 + P_1 H_1 = 0\,,$$
$$P_1 e + Q G_1' = 0\,, \quad P_1' G_1 + Q H_1 = I\,.$$

Hence

$$e_1 = (a_1 - P_1 Q^{-1} P_1')^{-1}\,, \tag{40}$$

$$G_1 = -e_1 P_1 Q^{-1}\,,$$

$$G_1' = -Q^{-1\prime}{}_1 P e_1\,, \tag{41}$$

$$H_1 = Q^{-1} + G_1' G_1 e^{-1}\,,$$

see [1]. Note that e_1 in (41) is a number. Repeat the above procedure for

$$Q_2 = \begin{bmatrix} a_2 & P_2 & 0 \\ P_2 & & \\ 0 & & Q_1 \end{bmatrix}. \tag{42}$$

Let again

$$Q_2^{-1} = \begin{bmatrix} e_2 & G_2 \\ G_2' & H_2 \end{bmatrix}.$$

Then formulae similar to (41) hold for e_2, G_2 and H_2.

We are going to show that we do not need the last column and row of Q_1^{-1} to compute Q_2^{-1} without its last row and column.

For instance

$$e_2 = (a_2 - P_2 Q_1^{-1} P_2')^{-1} = \left(a_2 - [P_2\ 0]\begin{bmatrix} & & * \\ & & * \\ & Q_1^{-1} & \vdots \\ * & * \cdots & * \end{bmatrix}\begin{bmatrix} P_2 \\ \\ 0 \end{bmatrix}\right)^{-21}, \tag{43}$$

$$G_2 = -e_2 P_2 Q_1^{-1} = -e_2 [P_2\ 0]\begin{bmatrix} & & \\ & Q_1^{-1} & \\ * & \cdots & ** \end{bmatrix}. \tag{44}$$

(The entries denoted as $*$ are multiplied by zero.)

The last entry of G_2 will not evidently be used in the next step and hence need not be computed.

To summarize, the method consists of successive adjoining a row and a column to the matrix Q_{i-1}, of a short computation, and of deleting the last row and column of $\bar{Q}_i^{-1}$:

(45)
$$\bar{Q}_i^{-1} = \begin{bmatrix} a_i & P_i \\ P_i' & \begin{bmatrix} Q_{i-1} \end{bmatrix} \end{bmatrix}^{-1} = \begin{bmatrix} e_i & \bar{G}_i \\ \bar{G}_i' & \begin{bmatrix} \bar{H}_i \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} & Q_i^{-1} & \end{bmatrix} & \begin{matrix} g \\ h_1 \\ h_2 \end{matrix} \\ g \quad h_1 \quad h_2 & h_{r-1} \end{bmatrix}$$

(46)
$$\begin{aligned} e_i &= (a_i - P_i Q_{i-1}^{-1} P_i')^{-1} , \\ \bar{G}_i &= -e_i P_i Q_{i-1}^{-1} , \\ \bar{H}_i &= Q_{i-1}^{-1} + G_i' G_i e_i^{-1} . \end{aligned}$$

Write:

$$\bar{G}_i = [G_i \; g] ,$$

$$\bar{H}_i = \begin{bmatrix} & & h_1 \\ & H_i & h_2 \\ & & \vdots \\ h_1 & h_2 \ldots & h_{r-1} \end{bmatrix} .$$

Delete g and $h_1, h_2, \ldots, h_{r-1}$ to get

(47)
$$Q_i^{-1} = \begin{bmatrix} e_i & G_i \\ G_i' & H_i \end{bmatrix} ,$$

again of dimension $r - 1$.

This completes the proof and also suggests an effective method of computation.

Now we apply Theorem 1 to solve equation (37). Using (46) and (47) we compute the $(r - 1)$-dimensional submatrix of $\mathbf{M}^{-1}$. As stated above, we just take its first two rows; in general, we take as many rows as the number of outputs in the system. It is easy to increase the order of the matrix $\mathbf{M}$ by adjoing additional paper tapes according to (34). This simultaneously increases the number of variables in performance index (11).

The numerical method described is based on successive adjoining the paper tapes to Q_i until $\|Q_i^{-1} - Q_{i+2}^{-1}\| \leqq \varepsilon$, or generally, until $\|Q_i^{-1} - Q_{i+k}^{-1}\| \leqq \varepsilon$, where k is the number of the system outputs. It is important to first adjoin the paper tape Pk, then P$(k - 1)$, …, P2, P1. We shall call each adjoining of Pk, …, P2, P1 an iteration.

 Example 1. (Two-input two-output system)

$$S(z) = \begin{bmatrix} 1 + 0{\cdot}2z^{-1} & 1 - 2z^{-1} \\ 1 + 2z^{-1} & 1 + 0{\cdot}5z^{-1} \end{bmatrix}, \tag{48}$$

$$S^{\mathrm{T}}(z^{-1}) \,.\, S(z) = \begin{bmatrix} 6{\cdot}04 + 2{\cdot}2(z + z^{-1}) & 2{\cdot}6 + 2{\cdot}2z - 1{\cdot}5z^{-1} \\ 2{\cdot}6 + 2{\cdot}2z^{-1} - 1{\cdot}5z & 6{\cdot}25 - 1{\cdot}5(z + z^{-1}) \end{bmatrix}. \tag{49}$$

By (20), (21), (22), (23),

$$\begin{gathered} \xi = 1\,, \quad \eta = 1\,, \\ a_0 = 6{\cdot}04\,, \quad a_1 = a_{-1} = 2{\cdot}2\,, \\ b_0 = 2{\cdot}6\,, \quad b_1 = 1{\cdot}5\,, \quad b_{-1} = 2{\cdot}2\,, \\ c_0 = 6{\cdot}25\,, \quad c_1 = c_{-1} = -1{\cdot}5\,. \end{gathered} \tag{50}$$

By (33),

$$\begin{aligned} \mathrm{P}_1 &\equiv \{a_0\ b_0\ a_1\ b_{-1}\ 0\ 0\ \ldots\} = \{6{\cdot}04\quad 2{\cdot}6\ 2{\cdot}2\ 2{\cdot}2\ 0\ \ldots\}\,, \\ \mathrm{P2} &\equiv \{c_0\ b_1\ c_1\ 0\ \ldots\ldots\ldots\} = \{6{\cdot}25\ -1{\cdot}5\ 0\quad 0\quad \ldots\}\,. \end{aligned} \tag{51}$$

Construct the matrix Q using (34) to compute Q_i^{-1} by (46), (47). As stressed above, P1 must be the first row of Q to get y_0^1 first and then y_0^2.

From the coefficients of P1 and P2 we infer that $r = 4$ (see Def. 1). By (34) we can construct the following matrix, which is of order $r - 1 = 3$ by Theorem 1:

$$Q = \begin{bmatrix} a_0 & b_0 & a_1 \\ b_0 & c_0 & b_1 \\ a_1 & b_1 & a_0 \end{bmatrix} = \begin{bmatrix} 6{\cdot}04 & 2{\cdot}6 & 2{\cdot}2 \\ 2{\cdot}6 & 6{\cdot}25 & -1{\cdot}5 \\ 2{\cdot}2 & -1{\cdot}5 & 6{\cdot}04 \end{bmatrix}. \tag{52}$$

The method was programmed for a digital computer. Taking the form of the B matrix into account, we see that it has 2ξ rows and the

$$B = \begin{bmatrix} a_1 & b_1 \\ b_{-1} & c_1 \end{bmatrix} = \begin{bmatrix} 2{\cdot}2 & -1{\cdot}5 \\ 2{\cdot}2 & -1{\cdot}5 \end{bmatrix}. \tag{53}$$

Denote Q_i as in (46) and write down only the entries needed.

After adjoining successively P2, P1, P2, P1, P2, P1 we obtained

$$Q_6^{-1} = \begin{bmatrix} 0{\cdot}3416918 & -0{\cdot}2011448 \\ -0{\cdot}2011448 & 0{\cdot}2930832 \end{bmatrix},$$

after further adjoining P2, P1, P2, P1, P2, P1, we obtained

$$Q_{12}^{-1} = \begin{bmatrix} 0{\cdot}3406889 & -0{\cdot}2003049 \\ -0{\cdot}2003049 & 0{\cdot}2923799 \end{bmatrix},$$

and finally

$$Q_{20}^{-1} = \begin{bmatrix} 0{\cdot}3406879 & -0{\cdot}2003040 \\ -0{\cdot}2003040 & 0{\cdot}2923793 \end{bmatrix}. \tag{54}$$

Comparing the respective entries we could see that the difference

$$\|Q_{12}^{-1} - Q_{14}^{-1}\| < 10^{-6}$$

and that Q_{20}^{-1} represents the solution with the accuracy 10^{-7}.

Now (37) yields

$$(55) \qquad \begin{bmatrix} y_0^1 \\ y_0^2 \end{bmatrix} = -Q_{20}^{-1} BY^- = \begin{bmatrix} 0{\cdot}30884 & -0{\cdot}21057 \\ 0{\cdot}20256 & -0{\cdot}13811 \end{bmatrix} \begin{bmatrix} y_{-1}^1 \\ y_{-1}^2 \end{bmatrix} = -R \begin{bmatrix} y_{-1}^1 \\ y_{-1}^2 \end{bmatrix}.$$

Equation (55) can be viewed as a control law at time i:

$$(56) \qquad \begin{bmatrix} y_i^1 \\ y_i^2 \end{bmatrix} = -R \begin{bmatrix} y_{i-1}^1 \\ y_{i-1}^2 \end{bmatrix}.$$

This recurrent equation together with the initial conditions y_{-1}^1, y_{-1}^2 yields the optimal control for system (48) with respect to performance index (11).

In terms of the Z-transform equation (56) formally reads

$$(57) \qquad Y(z) = -z^{-1} R\, Y(z) + C\,,$$

where C represents the influence of the initial conditions.

MINIMIZATION OF (10) IN THE FREQUENCY DOMAIN

We take a classical approach [2] to minimize performance index (10), which reads for our problem as

$$(58) \qquad I = \frac{1}{2\pi j} \oint_\Gamma (Y^{\mathrm{T}}(z^{-1}) + Y^{-\mathrm{T}}(z^{-1}))\, S^{\mathrm{T}}(z^{-1})\, S(z)\, (Y(z) + Y^-(z)) \frac{\mathrm{d}z}{z}\,.$$

We set

$$(59) \qquad \frac{\partial I}{\partial J^{\mathrm{T}}} = \frac{1}{2\pi j} \oint_\Gamma S^{\mathrm{T}}(z^{-1})\, S(z)\, (Y(z) + Y^-(z)) \frac{\mathrm{d}z}{z} = 0\,.$$

This is true if the integrand has no pole inside Γ:

$$(60) \qquad S^{\mathrm{T}}(z^{-1})\, S(z)\, (Y(z) + Y^-(z))\, z^{-1} = \Lambda\,.$$

It is well-known [2] that $S^{\mathrm{T}}(z^{-1})\, S(z)$ can be decomposed to the form

$$(61) \qquad S^{\mathrm{T}}(z^{-1})\, S(z) = S^-(z^{-1})\, S^+(z)$$

where $S^-(z^{-1}) = S^{+\mathrm{T}}(z^{-1})$, $S^+(z)$ and $(S^+(z))^{-1}$ have all their poles inside Γ.

Thus

$$(62) \qquad z^{-1}\, S^-(z^{-1})\, S^+(z)\, (Y(z) + Y^-(z)) = \Lambda\,.$$

 We premultiply equation (62) by $(S^-(z^{-1}))^{-1}$,

$$z^{-1}\, S^+(z)\,(Y(z) + Y^-(z)) = (S^-(z^{-1}))^{-1}\, A = A_1$$

or

$$(63) \qquad z^{-1}\, S^+(z)\, Y(z) + \left[\frac{S^+(z)\, Y^-(z)}{z}\right]_+ = A_1 - \left[\frac{S^+(z)\, Y^-(z)}{z}\right]_- .$$

Here $[\,.\,]_+$ denotes the extraction of the poles lying inside Γ.

By Liouville's theorem both sides of (63) are constant, in our case zero:

$$(64) \qquad Y(z) = -z(S^+(z))^{-1} \,.\, \left[\frac{S^+(z)\, Y^-(z)}{z}\right]_+ .$$

The entries of $S^+(z)$ are polynomials in z so that

$$(65) \qquad S^+(z) = A_0 + z^{-1}A_1 + \ldots + z^{-k}A_k$$

where k is the largest degree of all polynomials involved. We shall show that A_0 is nonsingular: S^+ is nonsingular outside Γ by hypothesis and hence it is so also for $z \to \infty$. It follows that A_0 is nonsingular.

We use $A_0^{-1}S^+$ instead of S^+ in (64) and shall show that $Y(z)$ is not affected:

$$(66) \qquad Y(z) = -z(S^+)^{-1}\, A_0 A_0^{-1} \left[\frac{S^+\, Y^-(z)}{z}\right]_+ .$$

We expand $(A_0 S^+)^{-1}$ into a power series in z^{-1} and truncate it after the first term:

$$(67) \qquad (A_0^{-1}S^+)^{-1} = (1 + z^{-1}A_0^{-1} + z^{-2}A_0^{-1}A_2 + \ldots + z^{-k}A_0^{-1}A_k)^{-1} = \\ = 1 + z^{-1}\Theta\,,$$

Θ being the remainder of the series.

Using (67) in (66),

$$(68) \qquad Y(z) = -\,(z\,.\,1 + \Theta)\left[\frac{A_0^{-1}S^+\; Y^-(z)}{z}\right]_+ .$$

By (67) evidently

$$(69) \qquad A_0^{-1}S^+ =$$

$$= \begin{bmatrix} 1 + \varphi_{11}^1 z^{-1} + \varphi_{11}^2 z^{-2} + \ldots + \varphi_{11}^{\xi} z^{-\xi}, & \varphi_{12}^1 z^{-1} + \varphi_{12}^2 z^{-2} + \ldots + \varphi_{12}^{\eta} z^{-\eta} \\ \varphi_{21}^1 z^{-1} + \varphi_{21}^2 z^{-2} + \ldots + \varphi_{21}^{\xi} z^{-\xi}, & 1 + \varphi_{22}^1 z^{-1} + \varphi_{22} z^{-2} + \ldots + \varphi_{22}^{\eta} z^{-\eta} \end{bmatrix} .$$

The degrees of the polynomials result from the finite memory of the system, see (5).

The inverse Z-transform gives us

$$(70)\qquad \begin{bmatrix} y_0^1 \\ y_0^2 \end{bmatrix} = \frac{1}{2\pi j} \oint_\Gamma Y(z) \frac{dz}{z} =$$

$$= \frac{-1}{2\pi j} \oint_\Gamma (z \,.\, 1 + \Theta) \,.\, \left[\frac{A_0^{-1} S^+ \, Y^-(z)}{z} \right]_+ \frac{dz}{z} \,.$$

By definition, Θ contains no term with positive powers of z.

The following column matrix

$$\left[\frac{A_0^{-1} S^+ \, Y^-(z)}{z} \right]_+$$

contains only the negative powers of z. It means that the integrand in (71) will contain the zero coefficient at z^{-1}, i.e.

$$(71)\qquad \frac{1}{2\pi j} \oint_\Gamma \Theta \left[\frac{A_0^{-1} S^+ \, Y^-(z)}{z} \right]_+ \frac{dz}{z} = 0 \,.$$

On modifying,

$$\begin{bmatrix} y_0^1 \\ y_0^2 \end{bmatrix} = \frac{-1}{2\pi j} \oint_\Gamma \left[\frac{A_0^{-1} S^+(z) \, Y^-(z)}{z} \right]_+ dz = - \left[\frac{A_0^{-1} S^+(z) \, Y^-(z)}{z} \right]_+ \,.$$

Otherwise speaking, this represents the extraction of all terms at z^{-1}.

Substitute (69) into the last equation.

Then

$$(72)\qquad \begin{bmatrix} y_0^1 \\ y_0^2 \end{bmatrix} =$$

$$= - \begin{bmatrix} \varphi_{11}^1 y_{-1}^1 + \varphi_{11}^2 y_{-2}^1 + \ldots + \varphi_{11}^\xi y_{-\xi}^1, & \varphi_{12}^1 y_{-1}^2 + \varphi_{12}^2 y_{-2}^2 + \ldots + \varphi_{12}^\eta y_{-\eta}^2 \\ \varphi_{21}^1 y_{-1}^1 + \varphi_{21}^2 y_{-2}^1 + \ldots + \varphi_{21}^\xi y_{-\xi}^1, & \varphi_{22}^1 y_{-1}^2 + \varphi_{22}^2 y_{-2}^2 + \ldots + \varphi_{22}^\eta y_{-\eta}^2 \end{bmatrix}$$

and equivalently

$$(73)\qquad \begin{bmatrix} y_0^1 \\ y_0^2 \end{bmatrix} = - \begin{bmatrix} \varphi_{11}^1 & \varphi_{11}^2 & \ldots & \varphi_{11}^\xi & \varphi_{12}^1 & \varphi_{12}^2 & \ldots & \varphi_{12}^\eta \\ \varphi_{21}^1 & \varphi_{21}^2 & \ldots & \varphi_{21}^\xi & \varphi_{22}^1 & \varphi_{22}^2 & \ldots & \varphi_{22}^\eta \end{bmatrix} Y^-$$

or, in the shorthand notation,

$$Y_0 = -RY^- \,.$$

Rewrite formally (72) in terms of Z-transform

$$(74)\qquad Y = -\Phi Y + C \,;$$

where

$$\Phi(z) = \begin{bmatrix} \sum_{i=1}^{\xi} \varphi_{11}^i z^{-i}, & \sum_{i=1}^{\eta} \varphi_{12}^i z^{-i} \\ \sum_{i=1}^{\xi} \varphi_{21}^i z^{-i}, & \sum_{i=1}^{\eta} \varphi_{22}^i z^{-i} \end{bmatrix}. \tag{75}$$

THE METHOD OF MATRIX SPECTRAL FACTORIZATION

Comparing (75) and (69) yields

$$A_0^{-1} S^+(z) = \Phi(z) + 1 .$$

Hence

$$S^+ = A_0(\Phi + 1) . \tag{76}$$

Let us look for a way of computing the A_0.

We know that $S^+(z)$ is unique up to a unitary matrix U:

$$S^{\mathsf{T}}(z^{-1})\, S(z) = S^-(z^{-1})\, S^+(z) = K^-(z^{-1})\, K^+(z)$$

where

$$S^+(z) = UK^+(z) ,$$
$$S^-(z^{-1}) = K^-(z^{-1})\, U^{\mathsf{T}}$$

and hence

$$S^-(z^{-1})\, S^+(z) = K^-(z^{-1})\, U^{\mathsf{T}} U\, K^+(z) = K^-(z^{-1})\, K^+(z) .$$

Thus the A_0 matrix is not unique and we have the liberty of choosing a way of computation.

It is seen that (76) implies

$$S^-(z^{-1})\, S^+(z) = (1 + \Phi^{\mathsf{T}}(z^{-1}))\, A_0^{\mathsf{T}} A_0 (1 + \Phi(z)) . \tag{77}$$

To simplify the computations we set $z = 1$, and denote this as $S(1) = S_*$:

$$S_*^- S_*^+ = S_*^{\mathsf{T}} S_* = (1 + \Phi_*^{\mathsf{T}})\, A_0^{\mathsf{T}} A_0 (1 + \Phi_*) \tag{78}$$

and on modification

$$(1 + \Phi_*^{\mathsf{T}})^{-1}\, S_*^{\mathsf{T}} S_* (1 + \Phi_*)^{-1} = A_0^{\mathsf{T}} A_0 . \tag{79}$$

The left hand side L of (79) is positive definite and hence $L = A_0^{\mathsf{T}} A_0$, where A_0 can be taken as an upper-triangular matrix. The technique of finding A_0 is described in [1].

The above trick allows us to factorize $S^{\mathsf{T}}(z^{-1})\, S(z)$ without knowing $S(z)$.

Example 2.

To demonstrate the above method, let us go back to Example 1. There we obtained the R matrix (55), which is the same as the R in (73).

Equation (75) gives us

$$\Phi(z) = \begin{bmatrix} 0{\cdot}30884z^{-1} & -0{\cdot}21057z^{-1} \\ 0{\cdot}20256z^{-1} & -0{\cdot}13811z^{-1} \end{bmatrix}$$

and for $z^{-1} = 1$ we have

$$(1 + \Phi_*)^{-1} = \begin{bmatrix} 0{\cdot}7362 & 0{\cdot}17986 \\ -0{\cdot}1732 & 1{\cdot}11797 \end{bmatrix}.$$

Setting $z^{-1} = 1$ in (49),

$$S_*^{\mathrm{T}} S_* = \begin{bmatrix} 10{\cdot}44 & 3{\cdot}3 \\ 3{\cdot}3 & 3{\cdot}25 \end{bmatrix}.$$

Then

$$(1 + \Phi_*^{\mathrm{T}})^{-1}\, S_*^{\mathrm{T}} S_* (1 + \Phi_*)^{-1} = \begin{bmatrix} 4{\cdot}91498 & 3{\cdot}3671 \\ 3{\cdot}3671 & 5{\cdot}7269 \end{bmatrix} = A_0^{\mathrm{T}} A_0$$

where

$$A_0 = \begin{bmatrix} 2{\cdot}21698 & 1{\cdot}51878 \\ 0 & 1{\cdot}8494 \end{bmatrix}$$

by (79).

Finally the result is

$$S^+(z) = A_0(1 + \Phi(z)),$$

$$S^+(z) = \begin{bmatrix} 2{\cdot}21698 + 0{\cdot}99234z^{-1} & 1{\cdot}51878 - 0{\cdot}6766z^{-1} \\ 0{\cdot}37461z^{-1} & 1{\cdot}8494 - 0{\cdot}2554z^{-1} \end{bmatrix}. \tag{80}$$

To check this result, we also computed

$$S_*^- S_*^+ = \begin{bmatrix} 10{\cdot}440067 & 3{\cdot}29969 \\ 3{\cdot}29969 & 3{\cdot}249968 \end{bmatrix}$$

and the determinant of the matrix (80):

$$\det S^+(z) = 4{\cdot}10008 + 0{\cdot}70006z^{-1}.$$

The matrix (80) is indeed the $S^+(z)$ matrix because the poles of the matrix $(S^+(z))^{-1}$ lay inside Γ.

THE GENERALIZATION TO n-INPUT n-OUTPUT SYSTEMS

Consider an n-input n-output system described by the transfer function matrix

$$S = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1n} \\ S_{22} & & & \\ S_{2n} & \dots\dots & & S_{nn} \end{bmatrix}, \tag{1.1}$$

where

$$S_{ij} = \sum_{l=0}^{n_{ij}} s_{ij}^l z^{-l}, \tag{2.1}$$

$$\bar{S}_{ij} = \sum_{l=0}^{n_{ij}} s_{ij}^l z^l. \tag{3.1}$$

Write

$$D = \bar{S}^{\mathrm{T}} S = \begin{bmatrix} D_{11} & D_{12} & \dots & D_{1n} \\ D_{21} & & & \vdots \\ \vdots & & & \vdots \\ D_{n1} & \dots & \dots & D_{nn} \end{bmatrix}. \tag{4.1}$$

Evidently

$$D_{ij} = \sum_{k=1}^{n} \bar{S}_{ki} S_{kj}. \tag{5.1}$$

Denoting

$$\xi_i = \max_{\langle k \in 1, n \rangle} n_{ki}, \tag{6.1}$$

we get

$$D_{ij} = \sum_{r=-\xi_i}^{\xi_j} d_{ij}^r z^{-r} = \bar{D}_{ji}. \tag{7.1}$$

Examine closely expressions (61) and (71). It is easy to see that the highest power of Z in all polynomials of the i-th row of the D matrix does not exceed ξ_i.

Analogously, the highest power of z^{-1} in all polynomials of the j-th column of the D matrix does not exceed ξ_j.

To apply the above algorithm we write down matrices (25) through (29):

$$\mathbf{B}' = \begin{bmatrix} B'_{11} & B'_{12} & \dots & B'_{1n} \\ \vdots & & & \\ B'_{n1} & \dots & \dots & B'_{nn} \end{bmatrix}, \tag{8.1}$$

where B'_{ij} are matrices of dimensions $N \times \xi_j$ having the form

$$B'_{ij} = \begin{bmatrix} d_{ij}^1 & d_{ij}^2 & \dots & d_{ij}^{\xi_j} \\ d_{ij}^2 & d_{ij}^3 & \dots & d_{ij}^{\xi_j} 0 \\ \dots & \dots & \dots & \dots \\ d_{ij}^{\xi_j} & & & \\ 0 & & & \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 \end{bmatrix} \quad \text{for} \quad j \geqq i \tag{9.1}$$

and

$$(10.1)\qquad B'_{ij} = \begin{bmatrix} d_{ij}^{-1} & d_{ij}^{-2} & \dots & d_{ij}^{-\xi_j} & \\ d_{ij}^{-2} & d_{ij}^{-3} & \dots & d_{ij}^{-\xi_j} & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d_{ij}^{-\xi_j} & 0 & & & \\ 0 & & & & \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix} \quad \text{for} \quad i > j\,.$$

Further

$$(11.1)\qquad \mathbf{M}'' = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1n} \\ M_{21} & & & \\ \dots & \dots & \dots & \dots \\ M_{n1} & \dots & \dots & M_{nn} \end{bmatrix},$$

where M_{ij} are $N \times N$ submatrices of the form

$$(12.1)\qquad M_{ij} = \begin{bmatrix} d_{ij}^{0} & d_{ij}^{-1} & d_{ij}^{-2} & \dots & d_{ij}^{-\xi_i} & 0 & \dots & 0 \\ d_{ij}^{1} & d_{ij}^{0} & d_{ij}^{-1} & \dots & & d_{ij}^{-\xi_i} & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & & & & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & d_{ij}^{-\xi_i} \\ d_{ij}^{\xi_j} & & & & & & & \\ 0 & & & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & d_{ij}^{\xi_j} & \dots & \dots & \dots & d_{ij}^{0} \end{bmatrix} \quad \text{for} \quad j \geqq i$$

and

$$M_{ij} = M_{ji}^{\mathrm{T}} \quad \text{for} \quad i > j\,.$$

The **M** matrix can be constructed by successive adjoining paper tapes like in (34). We have to use as many different tapes as is the number of the system outputs.

To demonstrate this procedure we define the following $n \times n$ matrix

$$(13.1)\qquad \Delta_n^r = \begin{bmatrix} d_{11}^{-r} & d_{12}^{-r} & \dots & & d_{1n}^{-r} \\ d_{21}^{r} & d_{22}^{-r} & \dots & & d_{2n}^{-r} \\ \dots & \dots & \dots & \dots & \dots \\ d_{n1}^{r} & & & d_{n,n-1}^{r} & d_{nn}^{-r} \end{bmatrix}$$

Observe that the superscript r is positive below the principal diagonal and negative elsewhere.

Write

$$(14.1)\qquad \Pi_n = [\Delta_n^0, \Delta_n^1, \dots, \Delta_n^{\xi}]\,,$$

where $\xi = \max_{i \in \langle 1, n \rangle} \xi_i$.

Therefore we can define paper tapes P1, P2, ..., Pn as follows:

$$(15.1)\qquad \Pi_n = \left[\begin{bmatrix} \{d^0_{11} & d^0_{12} & \dots & d^0_{1n} \\ d^0_{21} & \{d^0_{22} & \dots & d^0_{2n} \\ \dots & \dots & \dots & \dots \\ d^0_{n1} & \dots & \dots & \{d^0_{nn} \end{bmatrix}, \begin{bmatrix} \\ \Delta^1_n \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \Delta^2_n \\ \\ \end{bmatrix}, \dots, \begin{bmatrix} \\ \Delta^{\xi}_n \\ \\ \end{bmatrix}\right] \begin{matrix} 0 \dots 0\} \equiv \mathrm{P1} \\ 0 \dots 0\} \equiv \mathrm{P2} \\ \dots\dots\dots\dots \\ 0 \dots 0\} \equiv \mathrm{P}n \end{matrix}$$

in words

P1 is given by first row of Π_n,

P2 is given by second row of Π_n starting with d^0_{22},

. . . .

Pn is given by n-th row of Π_n starting with d^0_{nn}.

Now we use the corner paper tapes (see (34)) and construct the **M** matrix as

$$(16.1)\qquad \mathbf{M} = \begin{bmatrix} [\![\mathrm{P1}]\!] & & & & \\ & [\![\mathrm{P2}]\!] & & & \\ \dots & \dots & \dots & \dots & \dots \\ & & [\![\mathrm{Pn}]\!] & & \\ & & & [\![\mathrm{P1}]\!] & \\ & & & & [\![\mathrm{P2}]\!] \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

In words, we choose the tape having a nonzero element furthest in the right. Let the index of this element be r; then **M** is a $(2r - 1)$ – diagonal matrix in accordance with Definition 1.

Construct an $(r - 1) \times (r - 1)$ matrix Q, starting in the upper-left corner as follows: We adjoin the tapes $[\![\mathrm{P1}]\!]$, $[\![\mathrm{P2}]\!]$, ..., $[\![\mathrm{P}n]\!]$, again $[\![\mathrm{P1}]\!]$, $[\![\mathrm{P2}]\!]$, ..., $[\![\mathrm{P}n]\!]$ etc., until all entries of Q are defined. We see that Q is again of the form (38).

Further compute Q^{-1} and repeat this algorithm until the next iteration differs sufficiently little.

It remains to define the matrix B.

Let

$$\xi = \max_{i \in \langle 1,n \rangle} \xi_i .$$

Define

$$B_{1j} = \begin{bmatrix} d^1_{1j} & d^2_{1j} & \dots\dots\dots & d^{\xi_j}_{1j} \\ d^1_{2j} & d^2_{2j} & \dots\dots\dots & d^{\xi_j}_{2j} \\ \dots & \dots & \dots & \dots \\ d^1_{jj} & d^2_{jj} & \dots\dots\dots & d^{\xi_j}_{jj} \\ d^{-1}_{j+1,j} & d^{-2}_{j+1,j} & \dots\dots & d^{-\xi_j}_{j+1,j} \\ \dots & \dots & \dots & \dots \\ d^{-1}_{nj} & d^{-2}_{nj} & & d^{-\xi_j}_{nj} \end{bmatrix}_{n \times \xi_j}$$

and

$$U_j = \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ 0 & 1 & & \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 \end{bmatrix}_{\xi_j \times \xi_j}$$

and

$$B_{i+1,j} = B_{ij} \,.\, U_j \quad \text{for} \quad i = 1, 2, \dots (\xi - 1)$$
$$j = 1, 2, \dots, n \,.$$

Then

$$B = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{1n} \\ B_{21} & \cdots & \cdots & B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{\xi 1} & \cdots & \cdots & B_{\xi n} \end{bmatrix}.$$

Further we proceed in the same way as for a 2-input 2-output system, that is, we compute

$$R = Q_i^{-1} B \,,$$

construct $\Phi(z)$ by (75), and further compute $\Phi(1) = \Phi_*$ and

$$S^{\mathrm{T}}(1) \,.\, S(1) = S_*^{\mathrm{T}} \,.\, S_* \,.$$

The A_0 matrix is given by (80) as

$$(1 + \Phi_*^{\mathrm{T}})^{-1}\, S_*^{\mathrm{T}} S_* \,(1 + \Phi_*) = A_0^{\mathrm{T}} A_0 \,.$$

We choose A to be an upper-triangular matrix and compute it by the formula for the decomposition of a positive definite matrix. Then

$$S^+(z) = A_0(1 + \Phi(z))$$

and we have thus obtained the desired spectral factorization of $S^{\mathrm{T}}(z^{-1})\, S(z)$.

CONCLUSIONS

Examples show that the method described converges quite fast. It has been proved analytically that the method converges monotonically and if there are no zeros of det $S^{\mathrm{T}}(z^{-1})\, S(z)$ on the unite circle, it converges geometrically. If we write λ for that zero of $S^{\mathrm{T}}(z^{-1})\, S(z)$ lying inside Γ and being closest to Γ then the method converges no slowly than $|\lambda|^{2i}$.

The method has been tested by computing $\|S_*^{\mathrm{T}} S_* - S_*^- S_*^+\|$ as well as by the check for stability of $(S^+(z))^{-1}$.

The method allows the accuracy 10^{-6} or better.

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VÝTAH

Numerická metoda spektrální faktorizace matic

ZDENĚK VOSTRÝ

Při syntéze mnohorozměrných diskrétních regulačních obvodů podle kritéria minima kvadrátů je hlavním problémem numerický výpočet spektrální faktorizace matic, jejichž prvky jsou racionální lomené funkce komplexní proměnné z.

Numerická metoda popsaná v tomto článku je odvozena bez újmy na obecnosti pro polynomiální matice, které pro odvození metody chápeme jako matice přenosových funkcí jednoduché mnohorozměrné soustavy. Dále je určeno řízení takovéto soustavy z nenulového počátečního stavu do nulových ustálených hodnot výstupů podle minima součtu kvadratických ploch výstupů.

V první části článku je ukázáno, jak lze danou úlohu v časové oblasti převést na řešení nekonečného systému rovnic o nekonečně neznámých a jak tento systém řešit.

V druhé části článku je výpočet spektrální faktorizace matic pomocí řešení v časové oblasti. Postup je ukázán v příkladech.

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