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Kybernetika, Vol. 23 (1987), No. 4, 319--328

Persistent URL: <http://dml.cz/dmlcz/125651>

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MODEL MATCHING OF 2-D MULTI-INPUT MULTI-OUTPUT SYSTEMS¹

MICHAEL ŠEBEK

An analysis of the exact model matching problems of 2-D linear multi-input multi-output systems is presented. The approach is based on input-output 2-D polynomial models and all solutions to the problem are expressed in a parametric form. Compact necessary and sufficient conditions for the existence of causal and stable solutions are given. A design procedure is proposed which provides a stable causal solution. It consists essentially of solving a linear equation in 2-D polynomial matrices.

1. INTRODUCTION

The problem of exact model matching consists in compensating a given system (plant) so that the resultant system has a prespecified (model) transfer matrix. This is of great practical importance since it makes it possible to adapt an unsatisfactory performance of the given system to that of the desired model.

This problem is now well understood in 1-D systems. Some partial results in 2-D have already been reported by several authors using different approaches. As far as multi-input multi-output 2-D systems are concerned, Paraskevopoulos [6] and Yasuda [12] employed the state feedback of restricted types (proportional and dynamical but using only one of two states, respectively). The more natural output feedback was used by Paraskevopoulos [7] and Paraskevopoulos, Kosmidou [8] but again only in its restricted form (proportional and PID, respectively). Unfortunately, all these restricted controllers offer a limited number of free parameters which may be used to satisfy the design requirements. Very often, this is insufficient.

A different approach was developed by Emre [1] when solving this problem for linear systems over rings. Specializing his result for 2-D systems one obtains a solution which is applicable to our problem only if the numerator and the denominator (matrices) of the given transfer matrix have no zero in common.

¹ The original version of this paper was presented at the IFAC'84 Congress in Budapest, Hungary, July 1984.

Recently, a new approach to the exact model matching in single-input single-output 2-D systems was described in Šebek [9] which is based on 2-D polynomial equations whereby it allows to avoid all the restrictions mentioned above.

The first attempt to use this approach for multi-input multi-output 2-D systems appeared in Gajowniczek [2]. However, an unnecessary complicated structure of the controller is considered and some achievements of polynomial approach are not employed. Besides, solvability conditions are not discussed and the stability of the resulting system is not treated.

The purpose of this paper is to generalize the 2-D polynomial approach for 2-D multi-input multi-output systems by use of 2-D polynomial matrix equations but holding all the nice features of the scalar case.

First an (unconstrained) exact model matching problem is solved and all solutions are described in a parametric form. Then the solvability conditions under constraints of causality and/or stability are given, and, finally, a design procedure is proposed which yields a causal stable solution. This procedure consists essentially in solving a polynomial matrix equation in 2-D.

The approach followed here is essentially an extension of a technique previously reported for 1-D systems by Kučera and Šebek [4].

2. PRELIMINARIES

As to the mathematics and notation, we employ real polynomial matrices in two indeterminates denoted by v and w . These indeterminates are to be interpreted in dependence on the problem at hand, for example, as integration (differentiation) and/or delay (forward shift) operators, as parameters, etc., since the 2-D system may describe, for example, delay-differential system, 2-D digital filter, 1-D system depending on a parameter, etc. Since the notation on this field has almost been stabilized during recent years, we just recall that by \mathcal{C} , $\mathcal{R}[v]$, $\mathcal{R}(v)$, $\mathcal{R}[v, w]$ and $\mathcal{R}(v)[w]$ are denoted the field of complex numbers, the ring of real polynomials in v , the field of rational functions in v , the ring of (2-D) polynomials in v and w and the ring of polynomials in w with coefficients in $\mathcal{R}(v)$, respectively. For brevity, the same notation will be used for matrices. Various definitions of *stability* in 2-D can be found in the literature since various problems can be approached by 2-D systems. So whenever reading a word “stable” the reader should simply substitute the definition he needs. (Only some pathological concepts of stability depending on the both denominator and numerator of a transfer function are excluded). Then a polynomial matrix $P(v, w) \in \mathcal{R}[v, w]$ having full rank is said to be *stable* iff all its *zeros* (i.e. all $(z_1, z_2) \in \mathcal{C} \times \mathcal{C}$ for which $\text{rank } P(z_1, z_2) < \text{rank } P(v, w)$) fall inside the “stable region”.

As far as *causality* is concerned, it turned out convenient (Šebek [9]) to employ such system descriptions in which only the denominator of a transfer function plays

a role when checking causality. The most of 2-D systems in which the causality has a sense is then believed to fall within the following two classes: A polynomial matrix $P(v, w) \in \mathcal{R}[v, w]$ having full rank is said to be *causal in one indeterminate* (in w) iff $\text{rank } P(v, 0) = \text{rank } P(v, w)$ and it is said to be *causal in the both indeterminates* iff $\text{rank } P(0, 0) = \text{rank } P(v, w)$.

Besides, we shall need two more concepts defined below. Polynomial matrices $P, Q \in \mathcal{R}[v, w]$ with the same number of columns have a *common right zero* $(z_1, z_2) \in \mathcal{C} \times \mathcal{C}$ iff

$$\text{rank} \begin{bmatrix} P(z_1, z_2) \\ Q(z_1, z_2) \end{bmatrix} < \text{rank} \begin{bmatrix} P(v, w) \\ Q(v, w) \end{bmatrix}$$

Finally, a polynomial matrix will be called *unimodular* iff it has a polynomial inverse and *right unimodular* iff it has a polynomial right inverse.

3. FORMULATION

Consider a 2-D linear system (*plant*) described by the equations

$$(1) \quad u = Ax_p, \quad z = Bx_p, \quad y = Cx_p$$

where u is the m -vector input, z is the n -vector measured output, y is the p -vector output to be controlled, and x_p is an internal variable (a 2-D partial state). A, B and $C \in \mathcal{R}[v, w]$ are respectively $m \times m$, $n \times m$ and $p \times m$. The A and B are assumed right factor coprime (having only unimodular right common divisors), A is causal while $B(v, 0) = 0$ ($B(0, 0) = 0$) so that the plant is *strictly causal*. Finally, we assume the plant is realized such a way that its characteristic polynomial equals to $\det A$ (i.e., that there are no hidden modes in the plant).

It is desired to find a compensator such that the resultant system is governed by the *model equations*

$$(2) \quad u_N = Fx_M, \quad y = Gx_M$$

where u_N is a new q -vector input and x_M is an internal variable. F and $G \in \mathcal{R}[v, w]$ are respectively $q \times q$ and $p \times q$. Again F and G are assumed right factor coprime with F causal.

This basic formulation of the exact model matching problem makes no specific assumptions on the resulting system. From the practical point of view, however, it is desirable to produce a causal and internally stable system. The effect of these requirements will be analysed in the sections to follow.

4. ANALYSIS

A general linear 2-D compensator which processes the available signals z and u_N to produce u can be described by the equation

$$(3) \quad Pu = -Qz + Ru_N$$

where P , Q and $R \in \mathcal{R}[v, w]$ are $m \times m$, $m \times n$ and $m \times q$, respectively. The P must be invertible. This compensator can be thought of as a combination of a feedback and a feedforward. However, it is to be realised as a single system with characteristic polynomial $\det P$.

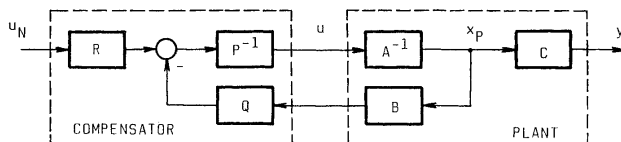


Fig. 1. Overall system.

The overall compensated system defined by (1) and (3) is shown in Figure 1. It obeys the equations

$$(4) \quad y = Cx_p, \quad (PA + QB)x_p = Ru_N$$

Comparing (2) and (4), the problem of exact model matching is equivalent to the algebraic problem of finding polynomial matrices P , Q and R such that the two matrices coincide:

$$(5) \quad C(PA + QB)^{-1}R = GF^{-1}$$

Now the straightforward analysis can be done which follows strictly that of 1-D case reported in Kučera and Šebek [4].

First of all, the relation (5) can be satisfied if and only if

$$\text{rank } [C \ G] = \text{rank } C$$

Denoting this rank by r , then clearly $r \leq m$. Let D be a greatest common left divisor of C and G and write

$$(6) \quad C = D\bar{C}, \quad G = D\bar{G}$$

To avoid trivia, suppose that D is chosen to have full column rank. Then it is $p \times r$ while \bar{C} is $r \times m$ of rank r and \bar{G} is $r \times q$. Further let \hat{F} and $\hat{G} \in \mathcal{R}[v, w]$ be two left coprime matrices such that

$$(7) \quad \hat{F}^{-1}\hat{G} = \bar{C}\bar{F}^{-1}$$

Thus \hat{F} is $r \times r$ and \hat{G} is $r \times q$.

Using (6) and (7), equation (5) can be written in the form

$$\hat{F}\bar{C}(PA + QB)^{-1}R = \hat{G}$$

Let T denote a greatest common left divisor of $PA + QB$ and R and write

$$(8) \quad PA + QB = TX$$

$$(9) \quad R = TY$$

Then $\hat{F}\bar{C}X^{-1}Y = \hat{G}$, a polynomial matrix. Since X and Y are left coprime, X must be a right divisor of $\hat{F}\bar{C}$. Therefore

$$(10) \quad \hat{F}\bar{C} = UX, \quad \hat{G} = UY$$

for a polynomial matrix $U \in \mathcal{R}[v, w]$. \hat{F} and \hat{G} are left coprime by definition and \bar{C} and \hat{G} are so. $\hat{F}\bar{C}$ and \hat{G} are left coprime and U must be right unimodular (in $\mathcal{R}[v, w]$).

We have thus expressed all matrices P , Q and R satisfying (5) in a convenient parametric form from (8), (9) and (10), where T and $X \in \mathcal{R}[v, w]$ are invertible $m \times m$ polynomial matrices, $U \in \mathcal{R}[v, w]$ is a right unimodular $r \times m$ matrix and $Y \in \mathcal{R}[v, w]$ is an $m \times q$ matrix. The X , Y and U are related via (10) while T is arbitrary invertible but such that the equation (8) has a solution. For example, taking $U = [I_r, 0]$ we obtain

$$(11) \quad PA + QB = T \begin{bmatrix} \hat{F}\bar{C} \\ \bar{T} \end{bmatrix}$$

$$(12) \quad R = T \begin{bmatrix} \hat{G} \\ \hat{T} \end{bmatrix}$$

where $\bar{T} \in \mathcal{R}[v, w]$ complements \bar{C} to an invertible matrix and $\hat{T} \in \mathcal{R}[v, w]$ is arbitrary.

5. CAUSAL SOLUTION

From the practical point of view, it is often desirable to find a causal solution to the problem. This means that both the compensator and the overall system must be causal. Naturally, the model itself is causal by assumption. It turns out, however, that this is not enough. The complete answer is given below.

Theorem 1. The exact model matching problem has a causal solution iff \bar{C} is causal.

Proof. In view of (5) and the way the compensator is realised, the overall system is causal iff the matrix $PA + QB$ is causal. From (8) this is equivalent to the causality of T and X , but T can always be chosen causal. From (10) this is further equivalent to the causality of \hat{F} and \bar{C} because U is right unimodular and hence causal. \hat{F} is causal iff F is causal (this is assumed) and the claim follows. Now, since the plant is assumed strictly causal, every solution to (8) yields a causal compensator. To observe this, write, in the case of causality in the equation (8) $w = 0$

$$P(v, 0)A(v, 0) = T(v, 0)X(v, 0)$$

Clearly $P(v, 0) = T(v, 0)X(v, 0)A^{-1}(v, 0) \in \mathcal{R}[v]$ is an invertible matrix so that

the compensator is causal (in w). In the case of causality in the both indeterminates substitute analogously $v = w = 0$ into (8) to get $P(0, 0) = T(0, 0) X(0, 0) A^{-1}(0, 0) \in \mathcal{R}$ invertible and hence the causal compensator.

6. STABLE SOLUTION

In most cases we want to produce a system which is internally stable. Needless to say, the model itself must be stable. This is not sufficient, as expected, and the complete answer is given below.

Theorem 2. The exact model matching problem has a stable solution iff

- 1) F is stable
- 2) \bar{C} is stable
- 3) A and B has no unstable right zero in common.

Proof. Using the same reasoning as in the proof of the last theorem, the overall system is stable iff the matrix $PA + QB$ is stable and hence iff all the matrices \hat{F} , \bar{C} and T are stable. Now \hat{F} is stable iff F is stable (condition 1) while stability of \bar{C} is expressed directly by the condition 2. If both these conditions are satisfied, clearly X is stable so that the stability of $PA + QB$ then hinges on T . T can be chosen arbitrarily but so that the equation (8) has a solution. Clearly, every unstable common right zero of A and B must be a right zero of T . Therefore, the common zeros must be unstable. On the other hand, if A and B have no unstable right zero in common, a stable T assuring solvability of (8) can always be found. This can be proved either using a matrix analogy of the Hilbert's Nullstellensatz (van der Waerden [11]) or directly by the construction described in the next but one section. \square

7. DESIGN PROCEDURE

A procedure will now be described which provides a stable causal solution to the exact model matching problem in 2-D systems whenever one exists. Given A , B , C and F G in the following steps P , Q and R are calculated.

Step 1: Calculate any greatest common left divisor, D , of C and G which has full column rank and factor it out:

$$C = D\bar{C}, \quad G = D\bar{G}$$

If \bar{C} is not causal or not stable, stop. There is no causal or no stable solution.

Step 2: Determine left coprime polynomial matrices \hat{F} and \hat{G} such that

$$\hat{F}^{-1}\hat{G} = G\bar{F}^{-1}$$

Step 3: Choose a stable polynomial matrix $\bar{T} \in \mathcal{R}[v, w]$ which complete \bar{C} to an invertible matrix and an arbitrary $(m - r) \times q$ matrix $\hat{T} \in \mathcal{R}[v, w]$.

The choice

$$\bar{T} = \text{a constant matrix, } \hat{T} = 0$$

is recommended.

Step 4: Choose a causal stable polynomial matrix $T \in \mathcal{R}[v, w]$ such that the equation

$$PA + QB = T \begin{bmatrix} \hat{F}\bar{C} \\ \bar{T} \end{bmatrix}$$

is solvable and solve it for unknown matrices P and Q . If there is no such causal stable T (this happens if A and B have some unstable right zero in common), stop. There is no stable solution.

Step 5: Set

$$R = T \begin{bmatrix} \hat{G} \\ \hat{T} \end{bmatrix}$$

This design procedure is immediate from the analysis given in Section 4. Algorithms to implement all steps of the procedure are described in the next section.

8. ALGORITHMS

To implement *Step 1* of the above procedure, a greatest common left divisor of 2-D polynomial matrices can be found by the method of Morf, Lévy and Kung [5].

In *Step 2*, a left coprime polynomial fraction can be calculated as follows. Using elementary row operations in $\mathcal{R}(v)[w]$ perform the reduction

$$(13) \quad \begin{bmatrix} F & I & 0 \\ \bar{G} & 0 & I \end{bmatrix} \rightarrow \begin{bmatrix} I & \bar{X}_1 & \bar{X}_2 \\ 0 & \bar{X}_3 & \bar{X}_4 \end{bmatrix}$$

where $\bar{X}_1, \bar{X}_2, \bar{X}_3$ and $\bar{X}_4 \in \mathcal{R}(v)[w]$. Denoting by $T \in \mathcal{R}[v]$ a least common left denominator of the matrix $[\bar{X}_3 \ \bar{X}_4]$ we have

$$(14) \quad [T\bar{X}_3 \ T\bar{X}_4] = T[\bar{X}_3 \ \bar{X}_4]$$

for polynomial matrices $\bar{X}_3, \bar{X}_4 \in \mathcal{R}[v, w]$. Unfortunately, they may still have a nonunimodular common left factor (with determinant from $\mathcal{R}[v]$). If this is the case, we must employ the primitive factorization algorithm [5] to get $[\bar{G} \ \bar{F}]$ as the primitive part of $[\bar{X}_3 \ \bar{X}_4]$.

To implement *Step 4*, one of the following two methods is recommended.

a) If A and B have no common right zero which is "unstable in one indeterminate", say in v (which means whenever $z_1 \in \mathcal{C}$ is from the unstable region then

$$\text{rank} [A(z_1, z_2), B(z_1, z_2)]^T = \text{rank} [A(v, w), B(v, w)]^T$$

for all $z_2 \in \mathcal{C}$) we can simply solve the equation

$$\bar{P}A + \bar{Q}B = \begin{bmatrix} \hat{F}\bar{C} \\ \bar{T} \end{bmatrix}$$

for unknown matrices $\bar{P}, \bar{Q} \in \mathcal{R}(v)[w]$ applying standard Euclidean ring algorithms [3]. This is justified since the ring $\mathcal{R}(v)[w]$ is Euclidean. Taking then T as a least common left denominator of $[\bar{P} \ \bar{Q}]$ we have immediately a solution to (11)

$$P = T\bar{P}, \quad Q = T\bar{Q}$$

b) The second algorithm is as follows. Writing $A \in \mathcal{R}[v, w]$ as

$$A(v, w) = A_0(v) + A_1(v)w + \dots + A_d(v)w^d$$

denoting by ${}_i A$ the i th column of A so that

$${}_i A(v, w) = {}_i A_0(v) + {}_i A_1(v)w + \dots + {}_i A_{d_i}(v)w^{d_i}$$

then using the same notation for B and denoting by d_i the highest power of w occurring

in the i th column a composite matrix $\begin{bmatrix} A \\ B \end{bmatrix}$ form matrices (with entries from $\mathcal{R}[v]$) for $k = 1, 2, \dots$

$$\begin{bmatrix} A_k \\ B_k \end{bmatrix} = \begin{matrix} km \\ kn \end{matrix} \left\{ \begin{array}{cc} \overbrace{\begin{matrix} {}_1 A_0, {}_1 A_1, \dots, {}_1 A_{d_1} \\ {}_1 A_0, {}_1 A_1, \dots, {}_1 A_{d_1} \\ \dots \\ {}_1 A_0, {}_1 A_1, \dots, {}_1 A_{d_1} \end{matrix}}^{d_1 + k} & \dots & \overbrace{\begin{matrix} {}_m A_0, \dots, {}_m A_{d_m} \\ {}_m A_0, \dots, {}_m A_{d_m} \\ \dots \\ {}_m A_0, \dots, {}_m A_{d_m} \end{matrix}}^{d_m + k} \\ \overbrace{\begin{matrix} {}_1 B_0, {}_1 B_1, \dots, {}_1 B_{d_1} \\ \dots \\ {}_1 B_0, {}_1 B_1, \dots, {}_1 B_{d_1} \end{matrix}}^{d_1 + k} & \dots & \overbrace{\begin{matrix} {}_m B_0, \dots, {}_m B_{d_m} \\ \dots \\ {}_m B_0, \dots, {}_m B_{d_m} \end{matrix}}^{d_m + k} \end{array} \right. \\ l = km + \sum d_i$$

Find j which is the least integer k such that the matrix $\begin{bmatrix} A_k \\ B_k \end{bmatrix}$ has a full column rank and, at the same time, such that $d_i + k \geq h_i$ for all $i = 1, 2, \dots, m$ (the h_i will be defined below). Further define a matrix $H \in \mathcal{R}[v, w]$

$$H = \begin{bmatrix} \hat{P}\hat{C} \\ T \end{bmatrix}$$

denote by h_i the highest power of w occurring in the i th column of H , define

$$j = \min_i \{d_i + j - h_i\} + 1$$

and form the matrix $\#H_j \in \mathcal{R}[v]$

$$\#H_j = \begin{matrix} jm \end{matrix} \left\{ \begin{array}{cc} \overbrace{\begin{matrix} {}_1 H_0, {}_1 H_1, \dots, {}_1 H_{h_1} \\ \dots \\ {}_1 H_0, {}_1 H_1, \dots, {}_1 H_{h_1}, 0, \dots \end{matrix}}^{d_1 + j} & \dots & \overbrace{\begin{matrix} {}_m H_0, {}_m H_1, \dots, {}_m H_{h_m} \\ \dots \\ {}_m H_0, {}_m H_1, \dots, {}_m H_{h_m}, 0, \dots \end{matrix}}^{d_m + j} \end{array} \right. \\ l$$

Now we can start the calculation. First, using elementary row operations in $\mathcal{R}[v]$ performed the reduction

$$\begin{array}{l} mj \\ nj \end{array} \left\{ \begin{array}{l} [A_j \ I \ 0] \\ [B_j \ 0 \ I] \end{array} \right\} \rightarrow \begin{array}{l} l \\ l \end{array} \left\{ \begin{array}{l} [D_j \ \overbrace{X_1 \ X_2}^{mj \ nj}] \\ [0 \ \overbrace{X_3 \ X_4}^{jm}] \end{array} \right\} l$$

Then, using again elementary row operations in $\mathcal{R}[v]$ perform the reduction

$$\begin{array}{l} l \\ jm \end{array} \left\{ \begin{array}{l} [D_j \ I \ 0] \\ [H_j \ 0 \ I] \end{array} \right\} \rightarrow \begin{array}{l} l \\ l \end{array} \left\{ \begin{array}{l} [E \ \overbrace{Y_1 \ Y_2}^{mj \ nj}] \\ [0 \ \overbrace{Y_3 \ Y_4}^{jm}] \end{array} \right\} l$$

where all the above matrices have entries from $\mathcal{R}[v]$. Denoting now

$$(15), (16) \quad \begin{aligned} T &= [T_1, T_2, \dots, T_j] \\ P &= [P_1, P_2, \dots, P_j] \\ Q &= [Q_1, Q_2, \dots, Q_j] \end{aligned}$$

choose an arbitrary $m \times j$ polynomial matrix $V \in \mathcal{R}[v]$ such that the T given by (15) and the relation

$$(17) \quad T = -VY_4$$

is a stable matrix. Then the desired solution of (11) with T given by (15) and (17) is read from

$$(18) \quad [P, Q] = [VY_3X_1, VY_3X_2]$$

with a help of (16).

(Received January 7, 1987.)

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