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The von Neumann solution is one of the basic solution concepts derived in the classical coalition games theory. Here it is formulated for the general coalition games model and its relations to the set of strongly stable imputations are derived.

0. INTRODUCTION

The model of the general coalition game was suggested in [2]. The solution concept, defined there as an analogy of the core and called the strong stability of imputations, is based on the domination relation. The classical literature on the coalition games theory deals also with another solution based on the same domination relation, namely with the von Neumann solution concept. As it can be easily formulated also in the notions of the general coalition games model, it is useful to compare it with the strong stability and to find some basic relations between both concepts.

As the strong stability is defined in [2] and other related papers by means of so called weak domination relation, we use the same relation even for the definition of the von Neumann solution.

1. GENERAL COALITION GAME

In this section we briefly recall the basic notions of the general coalition game model used in this paper.

The *general coalition game* is a pair (I, V) where I is a finite and non-empty set of *players* and V is a mapping prescribing to aby set $K \subset I$ a subset $V(K)$ of the real space $\mathbb{R}^I$ such that

(1.1) $$V(K) \quad \text{is closed},$$

(1.2) $$\text{if} \quad x \in V(K), \quad y \in \mathbb{R}^I, \quad x_i \geqq y_i \quad \text{for all} \quad i \in K \quad \text{then} \quad y \in V(K),$$

(1.3) $$V(K) \neq \emptyset\,,$$

(1.4) $$V(K) = \mathbb{R}^I \Leftrightarrow K = \emptyset\,.$$

The sets $K \subset I$ are called *coalitions*, any partition of I into non-empty coalitions is a *coalition structure*, and any vector $\boldsymbol{x} \in \mathbb{R}^I$ is an *imputation*. For any coalition structure $\mathscr{K}$ we denote

$$V(\mathscr{K}) = \bigcap_{K \in \mathscr{K}} V(K)\,.$$

If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^I$ are imputations and $K \subset I$ is a coalition then we say that $\boldsymbol{x}$ *dominates* $\boldsymbol{y}$ *via* K and write $\boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}$ iff $x_i \geqq y_i$ for all $i \in K$ and $x_i > y_i$ for at least one $i \in K$. For each $K \subset I$ we denote

(1.5) $$V^*(K) = \{\boldsymbol{y} \in \mathbb{R}^I\colon \textit{there is no } \boldsymbol{x} \in V(K) \textit{ such that } \boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}\}\,.$$

It can be easily verified that for any $K \subset I$ the equality $V(K) \cup V^*(K) = \mathbb{R}^I$ holds, and that $V(\emptyset) = V^*(\emptyset) = \mathbb{R}^I$. The set

(1.6) $$\boldsymbol{P}(K) = V(K) \cap V^*(K) \quad \text{for} \quad K \subset I$$

is called the *Pareto optimum* of the coalition K. We say that $\boldsymbol{P}(K)$, $K \subset I$, is *sufficient* iff for any $\boldsymbol{x} \in V(K) - V^*(K)$ there exists $\boldsymbol{y} \in \boldsymbol{P}(K)$ such that $\boldsymbol{y} \operatorname{dom}_K \boldsymbol{x}$.

We say that an imputation $\boldsymbol{x} \in \mathbb{R}^I$ is *strongly stable* in the game (I, V) iff there exists a coalition structure $\mathscr{K}$ such that

(1.7) $$\boldsymbol{x} \in V(\mathscr{K}) = \bigcap_{K \in \mathscr{K}} V(K)$$

and

$$\boldsymbol{x} \in V^*(L) \quad \text{for all} \quad L \subset I\,.$$

We say that a game (I, V) is *superadditive* iff, for any pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$, the inclusion

$$V(K) \cap V(L) \subset V(K \cup L)$$

holds. We say that (I, V) is *subadditive* iff

$$V^*(K) \cap V^*(L) \subset V^*(K \cup L)$$

for any pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$.

Finally, by $\mathbf{K}$ we denote the class of all coalition structures $\mathscr{K}$ such that there exists a strongly stable imputation $\boldsymbol{x} \in \mathbb{R}^I$ fulfilling $\boldsymbol{x} \in V(\mathscr{K})$.

2. VON NEUMANN SOLUTION

In this section we re-formulate the definition of the von Neumann solution in the terminology of the general coalition game model and of the domination relation introduced above. We present a few results concerning the mutual connections between the von Neumann solution and the strongly stable imputations.

A set of imputations $U \subset \mathbb{R}^I$ is a *von Neumann solution* of the game (I, V) iff

(2.1) for every $x \in U$ there exists a coalition structure $\mathscr{K}$ such that $x \in V(\mathscr{K})$;

(2.2) if $K \subset I$, $K \neq \emptyset$, is a coalition, $x, y \in V(K) \cap U$ are imputations, then the relation $x \operatorname{dom}_K y$ cannot be valid;

(2.3) if $z \in \mathbb{R}^I$, if $\mathscr{K}$ is a coalition structure and $z \in V(\mathscr{K})$, $z \notin U$, then there exist $L \subset I$, $L \neq \emptyset$, and $x \in U \cap V(L)$ such that $x \operatorname{dom}_L z$.

Theorem 1. Any non-empty von Neumann solution in a game (I, V) contains all strongly stable imputations.

Proof. The statement follows from (2.3) and from the fact that a strongly stable imputation belongs to all sets $V^*(K)$, $K \subset I$. □

Lemma 1. If the set of all strongly stable imputations in a game (I, V) is non-empty then it fulfils conditions (2.1) and (2.2).

Proof. The statement follows from the definition of the strong stability immediately. □

The last condition (2.3) need not be generally fulfilled by the set of all strongly stable imputations as follows from the next example.

Example 1. Let us consider a game (I, V) where $I = \{1, 2, 3, 4\}$ and

$$V(\{1,2\}) = \{x \in \mathbb{R}^4 : x_1 \leqq 2, x_2 \leqq 5\},$$
$$V(\{3,4\}) = \{x \in \mathbb{R}^4 : x_3 \leqq 2, x_4 \leqq 2\},$$
$$V(\{1,2,3\}) = \{x \in \mathbb{R}^4 : x_1 \leqq 3, x_2 \leqq 3, x_3 \leqq 3\},$$
$$V(\{2\}) = \{x \in \mathbb{R}^4 : x_2 \leqq 4\}, \quad V(4\}) = \{x \in \mathbb{R}^4 : x_4 \leqq 1\},$$
$$V(K) = \{x \in \mathbb{R}^4 : x_i \leqq 0 \quad \text{for all} \quad i \in K\} \quad \text{for other} \quad K \subset I.$$

Then there exists exactly one strongly stable imputation $x = (2, 5, 2, 2)$ achievable by the coalition structure $\mathscr{K} = (\{1, 2\}, \{3, 4\})$. The imputation $y = (3, 3, 3, 1)$ achievable by the coalition structure $(\{1, 2, 3\}, \{4\})$ is not strongly stable as there exists e.g. $z = 0, 4, 0, 1)$ (in the coalition structure $(\{1\}, \{2\}, \{3\}, \{4\})$) or $z' = (0, 4, 2, 2)$ (in the coalition structure $(\{1\}, \{2), \{3, 4\})$) such that

$$z \operatorname{dom}_{\{2\}} y \quad \text{and} \quad z' \operatorname{dom}_{\{2\}} y.$$

However, there is no coalition $K \subset I$, $K \neq \emptyset$, such that $x \in V(K)$ and $x \operatorname{dom}_K y$.

Lemma 2. Let us suppose that all Pareto optima $P(K)$, $K \subset I$, $K \neq \emptyset$, in the game (I, V) are sufficient and that there exist strongly stable imputations in (I, V). If every coalition structure $\mathscr{L}$ fulfils

$$V(\mathscr{L}) \subset \bigcup_{\mathscr{K} \in \mathbf{K}} V(\mathscr{K}) \tag{2.4}$$

then the set of all strongly stable imputations fulfils condition (2.3) (where $\mathbf{K}$ was introduced in Section 1).

Proof. Let us consider an imputation $x \in \mathbb{R}^I$ such that $x \in V(\mathscr{L})$ for some coalition structure $\mathscr{L}$, and let x be not strongly stable. Then there exists a coalition structure $\mathscr{K} \in \mathbf{K}$ such that $x \in V(\mathscr{K})$, and a coalition $L \subset I$ such that $x \in V(L) - V^*(L)$. Assumption (2.4) implies that there necessarily exists a coalition structure $\mathscr{M} \in \mathbf{K}$ and a coalition $M \in \mathscr{M}$ such that $x \in V(M) - V^*(M)$. As the Pareto optimum $P(M)$ is sufficient, there exists $y \in P(M)$ such that $y \operatorname{dom}_M x$ and, of course, $y_i \geqq x_i$ for all $i \in I$. If y is not strongly stable then the whole procedure may be repeated. There is only finite number of coalition structures in $\mathbf{K}$ and each of them is formed by a finite number of coalitions. Hence, after a finite number of the described steps we construct an imputation $z \in \mathbb{R}^I$ such that z is strongly stable in the game (I, V), $z_i \geqq y_i \geqq x_i$ for all $i \in I$, and $z \operatorname{dom}_M x$. □

Theorem 2. If (I, V) is a game with sufficient Pareto optima $P(K)$ for all $K \subset I$, with a non-empty set of strongly stable imputations and such that (2.4) is fulfilled for any coalition structure $\mathscr{L}$, then there exists exactly one von Neumann solution in (I, V), and it is identical with the set of all strongly stable imputations.

Proof. The set of all strongly stable imputations forms, under the assumptions of this theorem, a von Neumann solution, as follows from Lemma 1 and Lemma 2. On the other hand, if there exist more than one von Neumann solutions then all of them contain the set of strongly stable imputations by Theorem 1. Lemma 2 implies that any imputation $x \in \mathbb{R}^I$ that is not strongly stable is dominated by some strongly stable imputations and, consequently, it cannot be an element of any von Neumann solution by (2.2). It means that the statement is true. □

Theorem 3. If (I, V) is a superadditive game, if all Pareto optima $P(K)$, $K \subset I$, are sufficient and if there exist strongly stable imputations in (I, V) then there exists exactly one von Neumann solution in (I, V), and it is identical with the set of all strongly stable imputations.

Proof. The superadditivity means that for any coalition structure $\mathscr{K}$ the inclusion $V(\mathscr{K}) \subset V(I)$ holds, and any strongly stable imputation $x \in \mathbb{R}^I$ belongs to the set $V(I)$. Hence, condition (2.4) is surely fulfilled, and the statement is an obvious consequence of Theorem 2.

Theorem 4. If the game (I, V) is subadditive then there exists exactly one von Neumann solution containing exactly one imputation $x = (x_i)_{i \in I} \in \mathbb{R}^I$ such that for all $i \in I$

$$V(\{i\}) = \{y \in \mathbb{R}^I : y_i \leqq x_i\} .$$

Proof. It is not difficult to verify (cf. [3]) that the imputation x described by this theorem is the unique strongly stable imputation in the subadditive game (I, V), and that for any coalition K and any $y \in V(K)$ such that $y \neq x$ there exists $i \in K$ such that $x_i > y_i$, i.e. $x \operatorname{dom}_{\{i\}} y$. Moreover, there is no $K \subset I$, $K \neq \emptyset$, and no $y \in V(K)$ such that $y \operatorname{dom}_K x$ as follows from the subadditivity definition, too. It means that the one-element set of imputations $\{x\}$ is the unique set fulfilling (2.1), (2.2) and (2.3). $\square$

Condition (2.4) is not necessary for the equality between the von Neumann solution and the set of strongly stable imputations as follows from the next example.

Example 2. Let us consider a game (I, V) where $I = \{1, 2, 3, 4\}$ and the sets $V(K)$ are the same as in Example 1 except

$$V(\{1, 2\}) = \{x : x_1 \leqq 3, x_2 \leqq 5\} .$$

Then the imputation $x = (3, 5, 2, 2)$ is the unique strongly stable imputation, and for any coalition structure $\mathscr{K}$ and any $V(\mathscr{K})$ there exists $K \subset I$ such that $x \operatorname{dom}_K y$ including the $y = (3, 3, 3, 1)$ where $x \operatorname{dom}_{\{1,2\}} y$. Nevertheless, condition (2.4) is not fulfilled as $\mathbf{K} = \{\mathscr{K}\}$ for $\mathscr{K} = (\{1, 2\}, \{3, 4\})$ and for $\mathscr{L} = (\{1, 2, 3\}, \{4\})$ $V(\mathscr{L})$ is not a subset of $V(\mathscr{K})$.

3. CONCLUSIONS

It was shown in the previous section that the von Neumann solution may be defined for the general coalition games, and that there is no essential difference between both definitions.

Moreover, under not very strong assumptions there exists a strict relation between the von Neumann solution and strong stability concepts, as it was shown in the main results of this paper.

The domination relation used above is sometimes called the weak one in order to distinguish it from the strong domination such that, for $x, y \in \mathbb{R}^I$ and $K \subset I$, $K \neq \emptyset$, *x strongly dominates y via K* iff $x_i > y_i$ for all $i \in K$. It is not difficult to verify that the strong domination relation, also often used in the literature, does not essentially change the results presented above. It can lead to a simplification of some assumptions; e.g. in case of the strong domination all Pareto optima are evidently sufficient (and identical with the boundary sets of $V(K)$). Also the set of strongly stable imputations is generally larger or at least not smaller if the strong domination is considered instead of the weak one. However, the main methods and concepts may be used parallelly for both types of domination and the results would be similar to those presented above.

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