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Juraj Hromkovič

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ON THE NUMBER OF MONOTONIC FUNCTIONS FROM TWO-VALUED LOGIC TO k-VALUED LOGIC

JURAJ HROMKOVIČ

We deal with the generalized Dedekind's problem, i.e. with the determination of the number — $\varphi(n)$ of monotonic functions of n variables from two-valued logic to k-valued logic in this paper. Improving the lower and upper bounds of $\varphi(n)$ we obtain an asymptotic estimate of $\log_2 \varphi(n)$.

0. INTRODUCTION

The problem of number determination of monotonic functions of n variables-$\psi(n)$ was formulated and solved for $n = 4$ by Dedekind [3] in 1897. For $n = 5$ and $n = 6$, this problem was solved in Church [2] and in Ward [12] respectively. The further authors bringing the essential improvement of the estimates of $\psi(n)$ were Gilbert [4], Korobkov [8, 9, 10], Hansel [5], and Kleitman [7] who gave an asymptotic estimate of $\log_2 \psi(n)$. The best known result obtained is Korshunov's asymptotic estimate (*) of $\psi(n)$ (of [11])

$$(*)\qquad \psi(n) \simeq 2^{\binom{n}{n/2}} \exp\left\{\binom{n}{n/2}\left(\frac{1}{2^{n/2}} + \frac{n^2}{2^{n+5}} - \frac{n}{2^{n+4}}\right)\right\} \quad \text{for } n \text{ even},$$

$$\psi(n) \simeq 2 . 2^{\binom{n}{(n-1)/2}} \exp\left\{\binom{n}{(n-3)/2}\left(\frac{1}{2^{(n+3)/2}} - \frac{n^2}{2^{n+6}} - \frac{n}{2^{n+3}}\right) + \binom{n}{(n-1)/2}\left(\frac{1}{2^{(n+1)/2}} + \frac{n^2}{2^{n+4}}\right)\right\} \quad \text{for } n \text{ odd}.$$

Besides the classical Dedekind's problem a more general problem — the problem of the determination of the number of n variables monotonic functions from m-valued logic to k-valued logic has been formulated. The best known results concerning the solution of this generalized task can be found in Alexejev [1]. Since we shall deal with a special cases of the task introduced, with the number determination of n-variables monotonic functions from two-valued logic to k-valued logic (denoted by $\varphi(n)$), we state Alexejev's results (1′) and (2′) for $\varphi(n)$ only

$$(1') \qquad 2^{\frac{k-1}{\sqrt{(2\pi n)}} 2^n(1+\varepsilon_1'(n))} \leqq \varphi(n) \leqq 2^{\frac{k-1}{\sqrt{(2\pi n)}} 2^n(1+\varepsilon_2'(n))},$$

where

$$\varepsilon_2'(n) = \frac{c \cdot \log_2 (2n+1)}{\sqrt[4]{n}}, \quad \lim_{n\to\infty} \varepsilon_1'(n) = 0,$$

$$(2') \qquad \log_2 \varphi(n) = \frac{k-1}{\sqrt{(2\pi n)}} 2^n(1 + \varepsilon'(n)), \quad \text{where} \quad \varepsilon'(n) = \frac{c \cdot \log_2 (2n+1)}{n^{1/4}}.$$

The results of this paper are lower bound (Theorem 2) and an upper bound (Theorem 5) on $\varphi(n)$ which does not contain the additional member in the exponent of 2. Using these bounds we obtain in Section 4 the underlying asymptotic estimate of $\varphi(n)$ which is more precise than (2′):

$$\log_2 \varphi(n) = (k-1)\binom{n}{\lfloor n/2 \rfloor}(1 + \varepsilon(n)), \quad \text{where} \quad |\varepsilon(n)| \leqq \frac{k^2}{n}.$$

The paper consists of four sections. In Section 1 the basic definitions and notations used are given. The lower bound and the upper bound of $\varphi(n)$ are obtained in Section 2 and 3 respectively. The above stated estimate of $\log_2 \varphi(n)$ is given in Section 4.

1. DEFINITIONS AND NOTATIONS

In this section we define some basic notions which we shall use in this paper.

The set $B^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in \{0, 1\}, \ i = 1, 2, \ldots, n\}$ is called *n-dimensional cube*. The vectors $\tilde{\alpha}^n = (\alpha_1, \ldots, \alpha_n)$ [or simply $\tilde{\alpha}$] in B^n are called the *vertices* of the n-dimensional cube B^n.

The *norm* of a vertex $\tilde{\alpha}^n$ is defined as the number of coordinates which are equal to one, i.e.

$$\|\tilde{\alpha}^n\| = \sum_{i=1}^{n} \alpha_i.$$

The set of all vertices of B^n having the norm k is called the *k-th sphere* of B^n, and denoted by B_k^n.

The *distance* between $\tilde{\alpha}$ and $\tilde{\beta}$ in B^n is the number

$$\varrho(\tilde{\alpha}, \tilde{\beta}) = \sum_{i=1}^{n} |\alpha_i - \beta_i|,$$

where $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \ldots, \beta_n)$. The vertices $\tilde{\alpha}$ and $\tilde{\beta}$ of B^n are called *adjacent* iff $\varrho(\tilde{\alpha}, \tilde{\beta}) = 1$. An unordered pair of adjacent vertices is called the *edge* of B^n.

We say that the vertex $\tilde{\alpha}^n$ *precedes* the vertex $\tilde{\beta}^n$ (we denote $\tilde{\alpha}^n \leqq \tilde{\beta}^n$) iff $\alpha_i \leqq \beta_i$ for all $i = 1, \ldots, n$. If $\tilde{\alpha}^n \leqq \tilde{\beta}^n$ or $\tilde{\beta}^n \leqq \tilde{\alpha}^n$ holds then $\tilde{\alpha}^n$ and $\tilde{\beta}^n$ are called *comparable*. In the opposite case $\tilde{\alpha}^n$ and $\tilde{\beta}^n$ are called *incomparable*.

The set $A \subseteq B^n$ is called *independent* iff, for all $\tilde{\alpha}$, $\tilde{\beta}$ in A, $\tilde{\alpha}$ and $\tilde{\beta}$ are incomparable. We shall denote the class of all independent sets of B^n by $\mathbb{A}^n$. Clearly $\mathbb{A}^n \subseteq 2^{B^n}$.

The function $f(x_1, \ldots, x_n)$ $[f : B^n \to \{0, 1\}]$ defined on B^n and gaining the values from $\{0, 1\}$ is called the *Boolean function of n variables*. The *monotonic* Boolean function f is each Boolean function f satisfying condition $f(\tilde{\alpha}) \leqq f(\tilde{\beta})$ for all $\tilde{\alpha}, \tilde{\beta}$ in B^n such that $\tilde{\alpha} \leqq \tilde{\beta}$. $\psi(n)$ will denote the number of all monotonic Boolean functions of n variables.

The function $f(x_1, \ldots, x_n)$ $[f : B^n \to \{0, 1, \ldots, k - 1\}$ is called the *function from two-valued logic to k-valued logic*, or simply the *(2, k) function*. The $(2, k)$ function f is called *monotonic* $(2, k)$ function if, for all $\tilde{\alpha}$, $\tilde{\beta}$ in B^n such that $\tilde{\alpha} \leqq \tilde{\beta}$, $f(\tilde{\alpha}) \leqq f(\tilde{\beta})$ holds. The number of all monotonic $(2, k)$ functions of n variables is denoted by $\varphi(n)$. Obviously, the notions $(2, 2)$ function and Boolean function are equivalent.

A set $A_{\{\tilde{\varepsilon}\}} = \{\tilde{\alpha} \text{ in } B^n \mid \tilde{\alpha} \geqq \tilde{\varepsilon}\}$, for $\tilde{\varepsilon} \in B^n$, is said to be the *interval* of B^n. Using the notation of interval we introduce the following notation. Let $C \subseteq B^n$. Then $A_C = \bigcup_{\tilde{\varepsilon} \in C} A_{\{\tilde{\varepsilon}\}}$.

For each $(2, k)$ function f, we shall consider the set system $N_f = \{N_f^0, N_f^1, \ldots \ldots, N_f^{k-1}\}$, where $N_f^i = \{\tilde{\alpha} \text{ in } B^n \mid f(\tilde{\alpha}) = i\}$. Clearly, $B^n = \bigcup_{i=0}^{k-1} N_f^i$ and $N_f^j \cap N_f^k = \emptyset$ for $j \neq k$.

Concluding this section we give some notations. Let L be a set. Then $|L|$ denotes the number of elements in L. Let m be a real number $\lfloor m \rfloor$ ($\lceil m \rceil$) is the *floor* (*ceiling*) of m.

2. THE LOWER BOUND OF $\varphi(n)$

We shall obtain the lower bound (1) of the number of all monotonic functions from two-valued logic to k-valued logic in this section. We shall use a similar idea as in Alexejev [1] but our proof technique utilizing the nice properties of the n-dimensional cube helps to obtain the finer estimate of $\varphi(n)$ than (1').

Theorem 1. Let $\{S_1, \ldots, S_{k-1}\} \subseteq 2^{B^n}$ be a set system, where S_i are independent for all $i = 1, \ldots, k - 1$, and $S_i \cap S_j = \emptyset$ for $i \neq j$. Let $1 \leqq r < s \leqq k - 1$. and for no two $\tilde{\alpha}$ in S_r and $\tilde{\beta}$ in S_s $\tilde{\alpha} \geqq \tilde{\beta}$ holds. Then $\varphi(n) \geqq 2^d$, where $d = \sum_{i=1}^{k-1} |S_i|$.

Proof. We show using the set system $\{S_1, S_2, \ldots, S_{k-1}\}$ that 2^d different, monotonic functions from 2-valued logic to k-valued logic can be constructed. Let $\{S'_1, S'_2, \ldots, S'_{k-1}\}$ be a set system, where $S'_i \subseteq S_i$ for all $i = 1, \ldots, k - 1$. Clearly, for $1 \leqq i < j \leqq k - 1$ and all $\tilde{\alpha}$ in S_i, all $\tilde{\beta}$ in S_j the negation of $\tilde{\beta} \leqq \tilde{\alpha}$ holds (i.e. $\tilde{\alpha} < \tilde{\beta}$ or $\tilde{\alpha}$ and $\tilde{\beta}$ are incomparable). It can be easy seen that there exists exactly

$$\prod_{i=1}^{k-1} 2^{|S_i|} = 2^{\sum_{i=1}^{k-1} |S_i|} = 2^d$$

different set systems $\{S'_1, S'_2, \ldots, S'_{k-1}\}$ chosen from the basic set system $\{S_1, S_2, \ldots \ldots, S_{k-1}\}$.

In what follows we shall show that a monotonic $(2, k)$ function can be assigned to each set system $\{S'_1, \ldots, S'_{k-1}\}$ in such a way that two different, monotonic $(2, k)$ functions are assigned to different set systems $\mathscr{S}_1, \mathscr{S}_2$. Obviously, this will prove our asssertion.

We define a decomposition of B^n to k disjoint sets $D_0, D_1, \ldots, D_{k-1}$ according to a set system $\{S'_1, \ldots, S'_{k-1}\}$ in the following way.

$$D_{k-1} = A_{S'_{k-1}}, \quad D_{k-2} = A_{S'_{k-2}} - D_{k-1}, \ldots, D_i = A_{S_i'} - \bigcup_{j=i+1}^{k-1} D_j, \ldots$$
$$\ldots, D_1 = A_{S_1'} - \bigcup_{j=2}^{k-1} D_j, \quad D_0 = B^n - \bigcup_{j=1}^{k-1} D_j.$$

Then putting $N_f^i = D_i$, for all $i = 0, \ldots, k-1$, the set system N_f determines unambiguously a $(2, k)$ function.

Let us shown that the $(2, k)$ function defined in the way introduced above is monotonic. We prove it by contradiction. Let there exist $\tilde{\alpha}$ and $\tilde{\beta}$ in B^n, such that $\tilde{\alpha} > \tilde{\beta}$ and $i = f(\tilde{\alpha}) < f(\tilde{\beta}) = j$ for some i, j in $\{0, 1, \ldots, k-1\}$. Then

$$\tilde{\alpha} \in D_i = A_{S_i'} - \bigcup_{c=i+1}^{k-i} D_c \quad \text{and} \quad \tilde{\beta} \in D_j = A_{S_j'} - \bigcup_{c=j+1}^{k-1} D_c.$$

But, considering the properties of $\{S'_1, S'_2, \ldots, S'_{k-1}\}$ and the construction of D_i and D_j we see that for all $\tilde{\gamma}$ in D_i and all $\tilde{\varepsilon}$ in D_j the negation of $\tilde{\gamma} \leqq \tilde{\varepsilon}$ holds. It means that either $\tilde{\alpha} < \tilde{\beta}$ or $\tilde{\alpha}$ and $\tilde{\beta}$ are incomparable, what is the contradiction with the assumption $\tilde{\alpha} > \tilde{\beta}$.

Now we shall show that different $(2, k)$ functions f' and f'' are assigned to different set systems $\mathscr{S}' = \{S'_1, S'_2, \ldots, S'_{k-1}\}$, and $\mathscr{S}'' = \{S''_1, S''_2, \ldots, S''_{k-1}\}$. Since $\mathscr{S}' \neq \mathscr{S}''$ there exists c in $\{0, 1, \ldots, k-1\}$ such that $S'_c \neq S''_c$. Without loss of generality we can assume that there exists $\tilde{\alpha}$ in S'_c such that $\tilde{\alpha}$ does not belong to S''_c. Clearly, $f'(\tilde{\alpha}) = c$. Let us assume $f' \equiv f''$ what implies $f''(\tilde{\alpha}) = c$. It follows that there exists $\tilde{\beta}$ in S''_c such that $\tilde{\beta} < \tilde{\alpha}$. So, $f''(\tilde{\beta}) = c$ implies $f'(\tilde{\beta}) = c$, what can hold iff there exists $\tilde{\gamma}$ in S'_c such that $\tilde{\gamma} \leqq \tilde{\beta}$. But this is a contradiction with the independence of the set S'_c because $\tilde{\gamma} \leqq \tilde{\beta} < \tilde{\alpha}$ and $\tilde{\gamma}, \tilde{\alpha}$ belong to S'_c. □

Theorem 2. Let n, k be natural numbers, $n \geqq k \geqq 2$. Then

1. $\varphi(n) \geqq 2^{\binom{n}{n/2} + 2\sum_{i=1}^{(k-2)/2} \binom{n}{(n/2)-i}}$ for n, k even,

2. $\varphi(n) \geqq 2^{2\sum_{i=0}^{(k-3)/2} \binom{n}{\lfloor n/2 \rfloor - i}}$ for n, k odd,

3. $\varphi(n) \geqq 2^{\binom{n}{\lfloor n/2 \rfloor + k/2} + 2\sum_{i=0}^{(k-1)/2} \binom{n}{\lfloor n/2 \rfloor - i}}$ for n odd and k even,

4. $\varphi(n) \geqq 2^{\binom{n}{n/2} + \binom{n}{n/2 + (k-1)/2} + 2\sum_{i=1}^{(k-3)/2} \binom{n}{n/2 - i}}$ for n even and k odd.

Proof. Considering the result of theorem 1 it is sufficient to show that there exists a set system $\mathcal{S} \subseteq 2^{B^n}$ fulfilling the assumptions of Theorem 1 such that the cardinality sum of sets in $\mathcal{S}$ is equal to binary logarithm of the lower bound of $\varphi(n)$. Clearly, the spheres B_i^n are independent sets and the set system $\mathcal{S} = \{B_{a_1}^n, B_{a_2}^n, \ldots, B_{a_{k-1}}^n\}$, where $a_i < a_m$ for $i < m$, fulfils the assumptions of Theorem 1. So, choosing the most powerful (according to the cardinality) spheres to $\mathcal{S}$ we obtain the assertion of Theorem 2. □

3. THE UPPER BOUND OF $\varphi(n)$

To obtain the upper bound of $\varphi(n)$ we use a new method based on the following two theorems. We shall not prove the assertion formulated in Theorem 3 because it is well-known [6, 7].

Theorem 3. The number of monotonic Boolean functions of n variables is equal to the number of all independent sets in 2^{B^n}.

Theorem 4. Let $\mathbb{M}_{k-1}^n$ be the set of all $(k-1)$-tuples $(S_1, S_2, \ldots, S_{k-1})$, where S_i is an independent set of B^n for $i = 1, 2, \ldots, k-1$. Then $\varphi(n) \leqq |\mathbb{M}_{k-1}^n|$.

Proof. We shall prove the assertion introduced showing that a $(k-1)$-tuple of independent sets of B^n can be unambiguously assigned to each monotonic $(2, k)$ function in such a way that different $(k-1)$-tuples are assigned to different, monotonic $(2, k)$ functions f_1, f_2.

Let f be a monotonic $(2, k)$ function of n variables. Let $\mathbb{N}_f = \{N_f^0, N_f^1, \ldots, N_f^{k-1}\}$. Let $S_i \subseteq N_f^i$ be the set of minimal vectors of N_f^i for $i = 1, 2, \ldots, k-1$. Then we have a $(k-1)$-tuple $(S_1, S_2, \ldots, S_{k-1})$ for each monotonic $(2, k)$ function. Clearly, the sets S_i are independent.

Now, we shall show that two different $(k-1)$-tuples $(S_1, S_2, \ldots, S_{k-1})$ and $(S'_1, S'_2, \ldots, S'_{k-1})$ are assigned to different monotonic $(2, k)$ functions f and f'. Let us consider two set systems $\mathbb{N}_f$ and $\mathbb{N}_{f'}$ for two different monotonic $(2, k)$ functions f and f' respectively. Then there exists i in $\{1, 2, \ldots, k-1\}$ such that $N_f^i \neq N_{f'}^i$. We can assume without the loss of generality that there exists $\tilde{\alpha}$ in N_f^i such that $\tilde{\alpha} \notin N_{f'}^i$. Let $\tilde{\beta}$ be such vector in S_i that $\tilde{\beta} \leqq \tilde{\alpha}$ (obviously, such a vector must exist). If $\tilde{\beta}$ does not belong to S'_i the proof is completed. Let us consider the possibility that $\tilde{\beta} \in S'_i$. Realizing that $\tilde{\alpha} \notin N_{f'}^i$ and f' is monotonic we obtain $\tilde{\alpha} \in N_{f'}^j$ for $j > i$. So, there exists $\tilde{\gamma}$ in $N_{f'}^j$ such that $\tilde{\gamma} \leqq \tilde{\alpha}$ and $\tilde{\gamma} \in S'_j$. Obviously $\tilde{\gamma}$ cannot belong to S_j because $\tilde{\gamma} \in S_j$ implies $j = f(\tilde{\gamma}) > f(\tilde{\alpha}) = i$, what is a contradiction with the fact $\tilde{\gamma} \leqq \tilde{\alpha}$. □

Before formulating the upper bound of $\varphi(n)$ in the following Theorem 5 we note that the equality between $\varphi(n)$ and $|\mathbb{M}_{k-1}^n|$ does not hold.

Theorem 5.

$$\varphi(n) \leqq 2^{(k-1)\binom{n}{n/2}} \exp\left\{(k-1)\binom{n}{n/2-1}\left(\frac{1}{2^{n/2}} + \frac{n^2}{2^{n+5}} - \frac{n}{2^{n-4}}\right)\right\}$$

$$(1 + \gamma_1(n)) \text{ for } n \text{ even, where } \lim_{n\to\infty} \gamma_1(n) = 0,$$

$$\varphi(n) \leqq 2^{k-1}\, 2^{(k-1)\binom{n}{(n-1)/2}} \exp\left\{(k-1)\left[\binom{n}{(n-3)/2}\left(\frac{1}{2^{(n+3)/2}} - \frac{n^2}{2^{n+6}} - \frac{n}{2^{n+3}}\right) + \right.\right.$$

$$\left.\left. + \binom{n}{(n-1)/2}\left(\frac{1}{2^{(n+1)/2}} + \frac{n^2}{2^{n+4}}\right)\right]\right\}(1 + \gamma_2(n))$$

$$\text{for } n \text{ odd, where } \lim_{n\to\infty} \gamma_2(n) = 0.$$

Proof. Considering the result of Theorems 3 and 4 we obtain

$$\varphi(n) \leqq |\mathbb{M}^n_{k-1}| = [\psi(n)]^{k-1}.$$

Using Korshunov's estimate of $\psi(n)$, and a simple arrangement we obtain the assertion of Theorem 5. □

4. THE ASYMPTOTIC ESTIMATE OF BINARY LOGARITHM OF $\varphi(n)$

In this section we give an asymptotic estimate of binary logarithm of the number of monotonic $(2, k)$ functions which is more precise than the estimate of Alexejev [1]. We obtain it in the following two lemmas.

Lemma 1. $\log_2 \varphi(n) \geqq (k-1)\binom{n}{\lfloor n/2\rfloor}(1 - k^2/n)$.

Proof. It is no hard technical problem to show that

$$\binom{n}{\lfloor n/2\rfloor - i} \geqq \binom{n}{\lfloor n/2\rfloor}(1 - i/n),$$

where i as a constant. Using this fact and the assertion of Theorem 2 we have

$$\varphi(n) \geqq 2^{(k-1)\binom{n}{\lfloor n/2\rfloor}(1-k^2/n)}.$$ □

Lemma 2. $\log_2 \varphi(n) \leqq (k-1)\binom{n}{\lfloor n/2\rfloor}\left(1 + \frac{c}{\binom{n}{\lfloor n/2\rfloor}}\right)$, for a constant c.

Proof. Taking logarithms of the upper bound of $\varphi(n)$ and doing some simple arrangements the result of Lemma 2 can be obtained. □

Theorem 6. $\varphi(n) = (k - 1)\binom{n}{\lfloor n/2 \rfloor}(1 + O(1/n))$.

Proof. It is the direct consequence of Lemmas 1 and 2. □

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RNDr. Juraj Hromkovič, Katedra teoretickej kybernetiky MFF Univerzity Komenského (Department of Theoretical Cybernetics — Comenius University), 842 15 Bratislava. Czechoslovakia.