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ON A FUNCTIONAL EQUATION CONNECTED TO SUM FORM NONADDITIVE INFORMATION MEASURE ON AN OPEN DOMAIN — I

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Shannon's entropy is additive. However, there are information measures such as entropy of degree β which are nonadditive. The sum form representation of these measures along with additivity and nonadditivity properties yields many interesting functional equations for instance (S) and (4). In this paper, we find the measurable solutions of the functional equation (4) on an open domain.

1. INTRODUCTION

Let $\Gamma_n^0 = \{P = (p_1, p_2, \ldots, p_n) | p_k > 0, \sum_{i=1}^{n} p_i = 1\}$ be the set of all finite complete discrete probability distributions and Γ_n be the closure of Γ_n^0. In analysing the additivity and sum property of Shannon's entropy one comes across the following functional equation

$$\text{(S)} \qquad \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j),$$

where $P \in \Gamma_n$ and $Q \in \Gamma_m$. The entropy of degree β

$$\text{(1)} \qquad H_n^\beta(P) = \frac{\sum_{i=1}^{n} p_i^\beta - 1}{2^{1-\beta} - 1} \quad (\beta \neq 1)$$

proposed by Havrda and Charvát [3] is nonadditive. If we write

$$\text{(2)} \qquad f(p) = \frac{p^\beta - p}{2^{1-\beta} - 1},$$

then the entropy of degree β takes the form

$$\text{(3)} \qquad H_n^\beta(P) = \sum_{i=1}^{n} f(p_i).$$

The function f in (3) is called the generating function and satisfies the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + \lambda \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j)\,, \tag{4}$$

where $P \in \Gamma_n$, $Q \in \Gamma_m$, and $\lambda = (2^{1-\beta} - 1)$. The functional equation (4) was solved in [2] when f is continuous and (4) holds for all $m \geqq 2$, $n \geqq 3$. The continuous solutions of (4) can also be found in [4]. In [5], the equation (4) was solved when the function f is Lebesgue measurable and (4) holds for some (arbitrary but) fixed pair of integers (n, m) with $m \geqq 3$ and $n \geqq 3$. The equation (4) was solved in [7] when f is Lebesgue measurable and (4) holds for some (arbitrary but) fixed pair of integers (n, m) with $n \geqq 2$ and $m \geqq 3$. In [2, 4, 5, 7, 8] the solution of (4) on $[0, 1]$ was found using 0-probability under various regularity conditions. The need for the solution of (4) on the open domain $[0, 1]$ arises from the awkwardness in the definition of $0^\beta = 0$. It is also a priori quite possible that there may exist solutions other than those on $[0, 1]$ restricted to $]0, 1[$. See in this regard [1, 6, 9, 10] for solution of similar equations on open domains. In this paper, we find the measurable solutions of the functional equation (4) on the open domain for m and n greater than 2.

2. SOLUTION OF (4) ON $]0, 1[$

In order to solve (4) we make use of the following results.

Result 1 [10]. Let f_i: $]0, 1[\to \mathbb{R}$ (reals) and satisfy the functional equation

$$\sum_{i=1}^{n} f_i(p_i) = 0 \quad \left(0 < p_i < 1, \sum_{i=1}^{n} p_i = 1\right) \tag{5}$$

for arbitrary (but fixed) $n \geqq 3$ and at least one of the f_i's be measurable. Then the f_i's are given by

$$f_i(p) = ap + b_i\,, \tag{6}$$

where a and b_i's are arbitrary constants satisfying

$$a + \sum_{i=1}^{n} b_i = 0\,. \tag{7}$$

Lemma 2. Let f: $]0, 1[\to \mathbb{R}$ be measurable and satisfy the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j)\,, \tag{8}$$

for a fixed pair of positive integers n, $m (\geqq 3)$ and for all $P \in \Gamma_n^0$ and $Q \in \Gamma_m^0$. Then

$$f(p) = p^\beta, \quad p \in]0, 1[\,, \tag{9}$$

or

(10) $$f(p) = Ap + B\,, \quad p \in]0, 1[\,,$$

where β is an arbitrary constant and A, B are constants satisfying

(11) $$(A + mnB) = (A + nB)(A + mB)\,.$$

Proof. Fix $(q_1, q_2, \ldots, q_m) \in \Gamma_m^0$ temporarily and let

(12) $$F(p) = \sum_{j=1}^{m} f(pq_j) - \sum_{j=1}^{m} f(q_j) f(p),\ p \in]0, 1[\,.$$

Using (12) in (8), we obtain

(13) $$\sum_{i=1}^{n} F(p_i) = 0\,.$$

The measurable solution of (13) is given by (see Result 1)

(14) $$F(p) = A(q_1, q_2, \ldots, q_m)[1 - np]\,.$$

Letting $p = \alpha(\neq 1/n)$, where α is any fixed real number in $]0, 1[$, in (14) and (12), we get

(15) $$\sum_{j=1}^{m} f(pq_j) - \sum_{j=1}^{m} f(p) f(q_j) = [1 - np][1 - n\alpha]^{-1} \Big[\sum_{j=1}^{m} f(\alpha q_j) - \sum_{j=1}^{m} f(\alpha) f(q_j)\Big]\,.$$

Equation (15) holds for all $p \neq 1/n$. But from (12) and (13), we see that (15) holds also for $p = 1/n$. Fixing $p(\neq\alpha)$ we define $\delta(p) = [1 - np][1 - n\alpha]^{-1}$ and

(16) $$g(q) = f(pq) - f(p) f(q) - \delta(p) f(\alpha q) + \delta(p) f(\alpha) f(q)\,, \quad q \in]0, 1[\,.$$

By use of (16), equation (15) can be reduced to (5) and hence

(17) $$g(q) = B(p)[1 - mq]\,.$$

From (17) and (16) we get

(18) $$f(pq) - f(p) f(q) = \delta(p) f(\alpha q) - \delta(p) f(\alpha) f(q) + B(p)[1 - mq]\,, \quad p \neq \alpha\,.$$

Letting $q = \alpha$ in (18), we get

(19) $$f(\alpha p) - f(\alpha) f(p) = [1 - np][1 - n\alpha]^{-1}[f(\alpha^2) - f(\alpha) f(\alpha)] + [1 - m\alpha] B(p)\,.$$

Hence (19) yields

(20) $$B(p) = [1 - m\alpha]^{-1}[f(\alpha p) - f(\alpha) f(p)] - c_1[1 - np]\,,$$

where c_1 is a constant. Putting (20) into (19), we get

(21) $$f(pq) - f(p) f(q) = [1 - n\alpha]^{-1}[1 - np][f(\alpha q) - f(\alpha) f(q)] - c_1[1 - np][1 - mq] + [1 - m\alpha]^{-1}[1 - mq][f(\alpha p) - f(\alpha) f(p)]\,, \quad p \neq \alpha\,.$$

Interchanging p and q in (21), we get

$$(22) \quad [f(\alpha p) - f(\alpha)f(p)]([1 - m\alpha]^{-1}[1 - mq] - [1 - n\alpha]^{-1}[1 - nq]) + [f(\alpha q) - f(\alpha)f(q)]([1 - n\alpha]^{-1}[1 - np] - [1 - m\alpha]^{-1}[1 - mp]) = c_1[1 - np][1 - mq] - c_1[1 - nq][1 - mp].$$

For $m \neq n$, we get from (22)

$$(23) \quad [f(\alpha p) - f(\alpha)f(p) - c_1](q - \alpha) = [f(\alpha q) - f(\alpha)f(q) - c_1](p - \alpha),$$

Hence, from (23), we get

$$(24) \quad f(\alpha p) - f(\alpha)f(p) = a_1 p + b_1$$

and from (24) and (20) we obtain

$$(25) \quad B(p) = a_2 p + b_2 .$$

Using (24) and (25) in (18), we get

$$(26) \quad f(pq) - f(p)f(q) = ap + bp + cpq + d, \quad p, q \neq \alpha,$$

where a, b, c, d are constants, in the case $m \neq n$.

If the right hand side of (18) is not equal to 0, then by using (26), we can write

$$(27) \quad f(pqr) = f(pq)f(r) + apq + br + cpqr + d$$

for $p, q, r(\neq \alpha) \in]0, 1[$. By using (26) in (27), we get

$$(28) \quad f(pqr) = f(p)f(q)f(r) + [ap + bq + cpq + d]f(r) + apq + br + cpqr + d$$

also

$$(29) \quad f(pqr) = f(p)f(q)f(r) + [ap + br + cpr + d]f(q) + apr + bq + cpqr + d .$$

Fixing p and q, i.e. $p = p_0$ and $q = q_0$, in the right hand sides of (28) and (29) so that right hand side of (26) is not 0 (which is possible by the hypothesis), we get

$$(30) \quad f(r) = Ar + B, \quad r \neq \alpha,$$

where A, B are constants. Putting (30) into (15), we get

$$(Ap + mB) - (Ap + B)(A + mB) = [1 - np][1 - n\alpha]^{-1}[(A\alpha + mB) - (A + mB)f(\alpha)] .$$

Equating the coefficients of p and the constant terms from both sides of the above equation, we get

$$(31) \quad A - A^2 - mAB + n[1 - n\alpha]^{-1}[(A\alpha + mB - (A + mB)f(\alpha)] = 0,$$

and

$$(32) \quad mB - AB - mB^2 - [1 - n\alpha]^{-1}[(A\alpha + mB) - (A + mB)f(\alpha)] = 0 .$$

From (31) and (32), first we get the condition (11). Using (11) in (32), we obtain

$$(A + mB) f(\alpha) = (A + mB)(A\alpha + B) . \tag{33}$$

If $A + mB \neq 0$, then

$$f(\alpha) = A\alpha + B . \tag{34}$$

If $A + mB = 0$, then, by (11) $A + mnB = 0$. Hence, $A = 0 = B$ and

$$f(r) = 0 , \quad r \neq \alpha . \tag{35}$$

Letting $p_1 = \alpha = q_1$ in (8) and using (35), we get

$$f(\alpha) = 0 . \tag{36}$$

Hence, by (30), (34), (35) and (36), we get

$$f(p) = Ap + B , \tag{10}$$

for all $p \in]0, 1[$ with (11).

On the other hand, if the right hand side of (18) is zero, we get

$$[1 - np][1 - n\alpha]^{-1} [f(\alpha q) - f(\alpha) f(q)] = -B(p)[1 - mq] , \tag{37}$$

and

$$f(pq) = f(p) f(q) , \quad p \neq \alpha . \tag{38}$$

From (37), we get

$$f(\alpha q) - f(\alpha) f(q) = c_2 [1 - mq] . \tag{39}$$

for all $q \in]0, 1[$ and c_2 is a constant. Using (39), and (38), we have

$$\begin{aligned} c_2(1 - mqt) &= f(\alpha qt) - f(\alpha) f(qt) , \\ &= [f(\alpha q) - f(\alpha) f(q)] f(t) = c_2(1 - mq) f(t) , \end{aligned} \tag{40}$$

from which we can conclude that $c_2 = 0$ so that $f(\alpha q) - f(\alpha) f(q) = 0$, that is,

$$f(pq) = f(p) f(q) , \quad p, q \in]0, 1[. \tag{41}$$

The measurable solutions of (41) are given by (9) and $f(p) = 0$.

Now let us consider *the case* $m = n$. Let $x \in]0, 1[$, $P \in \Gamma_n^0$ and substitute $p = = xp_i$ $(i = 1, 2, \ldots, n)$ in (12) and (14). Using (12), we get

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{n} [f(xp_i q_j) - f(x) f(p_i) f(q_j)] = \\ &= A(p_1, p_2, \ldots, p_n)(1 - nx) \sum_{j=1}^{n} f(q_j) + A(q_1, q_2, \ldots, q_n)(n - nx) . \end{aligned} \tag{42}$$

Interchanging $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ in (42), we get

$$\begin{aligned} &A(p_1, p_2, \ldots, p_n) [(1 - nx) \sum_{j=1}^{n} f(q_j) - (n - nx)] = \\ &= A(q_1, q_2, \ldots, q_n) [(1 - nx) \sum_{i=1}^{n} f(p_i) - (n - nx)] . \end{aligned} \tag{43}$$

If we choose $x = 1/n$ in (43), then we see that A is a constant and using this in (14) we obtain from (12) and (14)

$$\sum_{j=1}^{n} [f(pq_j) - f(p)f(q_j)] - A[1 - np] = 0 . \tag{44}$$

Keeping p fixed and using result 1, we get

$$f(pq) - f(p)f(q) = A[1 - np]\, q + C(p)\,[1 - nq] . \tag{45}$$

Again interchanging p and q in (45) and using the symmetry of the left hand side of (45), we get

$$[C(p) - Ap]\,[1 - nq] = [C(q) - Aq]\,[1 - np] . \tag{46}$$

From (46) (fixing $q(\neq 1/n)$), we get

$$C(p) = a_3 p + b_3 , \quad p \in \,]0, 1[. \tag{47}$$

Equations (45) and (47) yield

$$f(pq) - f(p)f(q) = ap + bq + cpq + d \tag{48}$$

for all $p, q \in \,]0, 1[$ and a, b, c, d are arbitrary constants. As before, it can be shown (cf. (26)) that f is indeed given by (9) and (10).

Now, if $x \neq 1/n$, then writing $\delta_1 = (1 - nx)(n - nx)^{-1}$ in (43), we get

$$A(p_1, p_2, \ldots, p_n)\,[\sum_{j=1}^{n} f(q_j) - \delta_1] = A(q_1, q_2, \ldots, q_n)\,[\sum_{i=1}^{n} f(p_i) - \delta_1] . \tag{49}$$

If $\sum_{j=1}^{n} f(q_j) = \delta_1$ for all $Q \in \Gamma_n^0$, then $f(p)$, using Result 1, is given by (10). If there exists a $Q^* \in \Gamma_n^0$ such that $\sum_{j=1}^{n} f(q_j^*) \neq \delta_1$, then we get from (49)

$$A(p_1, p_2, \ldots, p_n) = \delta_2[\sum_{i=1}^{n} f(p_i) - \delta_1] , \tag{50}$$

where δ_2 is a constant. From (50), (14) and (12) we obtain

$$\sum_{j=1}^{n} [f(pq_j) - (f(p) + [1 - np]\,\delta_2) f(q_j) + \delta_1\delta_2[1 - np]\, q_j] = 0 . \tag{51}$$

As before, by fixing p, the equation (51) can be reduced to (5) and hence

$$f(pq) - (f(p) + [1 - np]\,\delta_2) f(q) + \delta_1\delta_2[1 - np]\, q = D(p)\,[1 - nq] . \tag{52}$$

Interchanging p and q in (52), we get

$$[1 - np]\,[\delta_2 f(q) - \delta_1\delta_2 q - D(q)] = [1 - nq]\,[\delta_2 f(p) - \delta_1\delta_2 p - D(p)] . \tag{53}$$

Keeping q fixed, i.e. letting $q = q_0(\neq 1/n)$ in (53), we get

$$D(p) = \delta_2 f(p) - \delta_1\delta_2 p - K[1 - np] , \tag{54}$$

where K is a constant. From (54) and (52), we get

$$f(pq) = f(p)f(q) + \delta_2[1 - np](f(q) - \delta_1 q) + \\ + \delta_2[1 - nq](f(p) - \delta_1 p) - K[1 - np][1 - nq]. \tag{55}$$

Writing $f(pqr)$ as $f(pq \,.\, r)$ and $f(pr \,.\, q)$ and using (55), we have (after substitutions of $p = 1/n = r$)

$$\delta_2(1 - 1/n)f(q) = c_3 q + c_4 \tag{56}$$

for some constants c_3 and c_4. Note $1 - 1/n \neq 0$. If $\delta_2 \neq 0$, then f is of the form (10). If $\delta_2 = 0$, then, we get from (55)

$$f(pq) - f(p)f(q) = -K[1 - np][1 - nq] \tag{57}$$

which is of the form (26). Thus f is of the form (9) and (10). This completes the proof of the lemma.

Let $f\colon]0, 1[\to \mathbb{R}$ be a measurable function and satisfy the functional equation (4) for $\lambda \neq 0$, $P \in \Gamma_n^0$, $Q \in \Gamma_m^0$ with m, n (fixed) $\geqq 3$. Defining

$$g(p) = p + \lambda f(p), \tag{58}$$

and putting (58) into (4), we obtain a functional equation of the form (8). Hence, using Lemma 2 and (58), we get

$$f(p) = \frac{(\alpha - 1)}{\lambda} p + \frac{b}{\lambda}, \tag{59}$$

$$f(p) = \frac{p^\beta - p}{\lambda}, \tag{60}$$

where β is an arbitrary constant while the constants a, b satisfy the equation

$$(a + mnb) = (a + nb)(a + mb). \tag{61}$$

Thus we have proved the following theorem.

Theorem. Suppose that $f\colon]0, 1[\to \mathbb{R}$ (reals) is measurable and satisfies the functional equation (4) for a fixed pair $m \geqq 3$, $n \geqq 3$ for a constant $\lambda \neq 0$ and for all $(p_1, p_2, \ldots, p_n) \in \Gamma_n^0$, $(q_1, q_2, \ldots, q_m) \in \Gamma_m^0$. Then the function f is given by (59) or (60) with (61).

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