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NONDIFFERENTIABLE AND QUASIDIFFERENTIABLE DUALITY IN VECTOR OPTIMIZATION THEORY

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In the paper two concepts of duality, namely nondifferentiable and quasidifferentiable are introduced for a class of vector optimization programs. Weak and partially strong duality are established. The obtained results are then applied to define dual programs for vector fractional programs.

0. INTRODUCTION

Duality theory may be regarded as the most delicate subject in optimization theory and its theoretical importance cannot be questioned (e.g. in the theory of prices and markets in economics). In vector optimization duality theory has been established mostly for linear and convex minimization programs (see [1]–[7]).

In [8], [9], [10] a unified duality theory has been introduced for a considerably wider class of optimization programs. Nevertheless, that theory is, in some concrete cases, too abstract to give a satisfying form of dual programs. This paper is concerned with a smaller class of optimization programs than those in [8], [9] and [10], but dual programs of which have more concrete and analytical form.

In Section 2 resp. 3 a nondifferentiable resp. quasidifferentiable duality is proposed. Weak and partially strong duality are established. Results of Sections 2 and 3 are then applied to define dual programs for vector fractional programs in Section 4.

1. NOTATION AND PRELIMINARIES

1.1. Throughout this paper X, Y, Z and W denote locally convex spaces,

Let $V \subset X$ then int $V, \overline{V}$, co V and $\overline{\text{co}}\ V$ denote the *interior*, *closure*, *convex hull* and *closed convex hull* of V respectively.

Note that X' denotes the *dual* of X equipped with the weak* topology.

$(x_\alpha)_{\alpha \in A} \subset X$ is called a *net* in X if A is a *directed set* (see [13,], p. 21).

For $V \subset X$ we denote the following

$$V^0 = \{v \in X' \mid v(x) \geqq -1 \ \forall x \in V\} \quad \text{the } \textit{polar set} \text{ of } V$$

$$V^* = \{v \in X' \mid v(x) \geqq \ 0 \ \forall x \in V\} \quad \text{the } \textit{dual cone} \text{ of } V$$

$$\operatorname{cone}(V) = \{\lambda x \mid \lambda \geqq 0 \,\&\, x \in V\} \quad \text{the } \textit{cone generated by } V.$$

For $a \in X$ let

$$K(a, V) = \{x \in X \mid \exists \lambda > 0 \quad \forall 0 < \varepsilon < \lambda : a + \varepsilon x \in V\} \quad \text{be the } \textit{tangent cone} \text{ of } V \text{ at } a.$$

1.2. Let X_0 be a nonempty subset of X. A function $f: X_0 \to W$ is called (*weakly*) *directionally differentiable at* $a \in X_0$ if the limit

$$f'(a, x) = \lim_{\lambda \downarrow 0} (f(a + \lambda x) - f(a))/\lambda$$

exists for each $x \in K(a, X_0)$ in the weak topology of W.

1.3. Let T be a nonempty closed convex cone of Z. A function $h: X_0 \to Z$ is said to be T^*-*quasidifferentiable* at $a \in X_0$ if h is directionally differentiable at a, and if for each $t \in T^*$ there exists a nonempty, weak* closed convex set $\partial^{\sim}(th)(a) \subset X'$ such that

$$th'(a, x) = \inf_{v \in \partial^{\sim}(th)(a)} v(x) \quad \forall x \in X$$

If $\partial^{\sim}(th)(a)$ is weak* compact for each $t \in T^*$ we shall say h is *continuously* T^*-*quasidifferentiable* at a since in this case $th'(a, x)$ is continuous.

1.4. A function $g: X_0 \to Y$ is said to be *arc-wise directionally differentiable* at $a \in X_0$ if (in the weak topology of Y)

$$g'(a, x) = \lim_{\lambda \downarrow 0} (g(a + w(\lambda)) - g(a))/\lambda$$

for each continuous arc $w: [0, 1] \to X$ such that $w(0) = 0$ and $w'(0) = x$.

This strengthing of directional differentiability is possible if the limit defining $g'(a, x)$ exists uniformly in x.

1.5. A function $k: X_0 \to R$, where R is the set of all reals, is called *directionally pseudoconcave* at $a \in X_0$ if k is directionally differentiable and

$$k(x) > k(a) \Rightarrow k'(a, x - a) > 0 \quad \forall x \in X$$

1.6. Let X_0 be a nonempty convex subset of X and S a nonempty closed convex cone of W. A function $f: X_0 \to W$ is S-*concave at* $a \in X_0$ if

$$\forall x \in X_0 \quad \forall \lambda \in [0, 1]: f(\lambda x + (1 - \lambda) a) - \lambda f(x) - (1 - \lambda) f(a) \in S$$

f is S-*concave* on X_0 if it is S-concave at a for all $a \in X_0$.

If $W = R$ and $S = R_+$ we say f is *concave* at $a \in X_0$ or on X_0.

A function f is called *S-convex at a*, *S-convex*, *convex at a* or *convex* if $-f$ is S-concave at a, S-concave, concave at a or concave respectively.

1.7. Let $f: X \to [-\infty, +\infty)$ be a concave function, not identically $-\infty$, and let $a \in X$ $f(a) > -\infty$ then the *superdifferential* of f at a is the set

$$\partial^- f(a) = \{v \in X' \mid f(x) - f(a) \leqq v(x - a) \ \forall x \in X\}$$

If f is a convex function, not identically $+\infty$, then the *subdifferential* of f at a is the set

$$\partial_- f(a) = -\partial^-(-f)(a)$$

1.8. Proposition. Let $f: X \to [-\infty, +\infty)$ $((-\infty, +\infty])$ be a concave (convex) function, finite and continuous at $a \in X$. Then $\partial^- f(a)$ $(\partial_- f(a))$ is nonempty, weak* compact and convex.

Proof. See [11], Proposition 5.2, p. 22.

1.9. It is easy shown that every continuous concave function and every linearly Gâteaux differentiable function is continuously (R_+-) quasidifferentiable. Zalinescu [14] has shown that every continuous concave or convex function is arc-wise directionally differentiable.

1.10. Let $V \subset X_0 \subset X$, $f: X_0 \to W$ a function and $W_+ \subset W$ a closed convex cone with $\text{int}\, W_+ \neq \emptyset$. Consider the program

$$f \to \sup \qquad (\mathscr{P}_1)$$
$$x \in V$$

Every $x \in V$ is called a *feasible solution* of program $(\mathscr{P}_1)$. A point $w \in W$ is said to be a (*weak*) *supremum* of program $(\mathscr{P}_1)$ if there exists a net $(x_\alpha) \subset V$ such that $w = \lim_\alpha f(x_\alpha)$ and

$$\forall x \in V: f(x) - w \notin \text{int}\, W_+ .$$

The set of all suprema of program $(\mathscr{P}_1)$ is denoted by $\mathscr{S}(\mathscr{P}_1)$. A point $x \in V$ is called and *optimal solution* of program $(\mathscr{P}_1)$ if $f(x) \in \mathscr{S}(\mathscr{P}_1)$. A net $(x_\alpha) \subset V$ is called an *asymptotic optimal solution* of program $(\mathscr{P}_1)$ if $\lim f(x_\alpha)$ exists and $\lim_\alpha f(x_\alpha) \in \mathscr{S}(\mathscr{P}_1)$.

1.11. Analogously are defined *feasible, optimal, asymptotic optimal solutions* and *infimum* of program

$$f(x) \to \inf \qquad (\mathscr{P}_2)$$
$$x \in G$$

The set of all infima of program $(\mathscr{P}_2)$ is denoted by $\mathscr{I}(\mathscr{P}_2)$.

2. NONDIFFERENTIABLE DUALITY

2.1. In this section we suppose that $Y_+ \subset Y$ and $W_+ \subset W$ are closed convex cones with int $Y_+ \neq \emptyset$ and int $W_+ \neq \emptyset$. Let us have the functions $f : X_0 \to W$ and $g : X_0 \to Y$, where $X_0 \subset X$ is a nonempty set. We shall considere the following program

$$\left.\begin{array}{c} f(x) \to \sup \\ g(x) \in Y_+ \\ x \in X_0 \end{array}\right\} \qquad (P)$$

2.2. In order to establish a dual program to (P) we assume that there exist a nonempty set $W_0 \subset W$, a locally convex space $\tilde{W}$ with a closed convex cone $\tilde{W}_+$ such that int $\tilde{W}_+ \neq \emptyset$ and a function $\varphi : X_0 \times W_0 \to \tilde{W}$ such that

(2.2.1) $$\forall (x, w) \in X_0 \times W_0 : f(x) - w \in \text{int } W_+ \Leftrightarrow \varphi(x, w) \in \text{int } \tilde{W}_+$$

2.3. The following program

(2.3.1) $$\left.\begin{array}{c} w \to \inf \\ \sup_{x \in X_0} (\mu(\varphi(x, w)) + \eta(g(x)) \leqq 0 \\ w \in W_0 \;\&\; \mu \in \tilde{W}_+^* \setminus \{0\} \;\&\; \eta \in Y_+^* \end{array}\right\} \qquad (D)$$

is called a *nondifferentiable dual* of program (P).

2.4. Theorem (Weak Duality). Let x and (w, μ, η) be feasible solutions of programs (P) and (D) respectively. Then

$$f(x) - w \notin \text{int } W_+$$

Proof. Let x and (w, μ, η) be feasible solutions of programs (P) and (D) respectively. If $f(x) - w \in \text{int } W_+$ then, by (2.2.1), $\varphi(x, w) \in \text{int } \tilde{W}_+$ and

$$\mu(\varphi(x, w)) + \eta(g(x)) \geqq \mu(\varphi(x, w)) > 0$$

which contradicts (2.3.1).

2.5. Theorem (Partially Strong Duality). Suppose X_0 is convex $\varphi(\cdot, w)$ is $\tilde{W}_+$-concave on X_0 for all $w \in W_0$, $g(x)$ is Y_+-concave on X_0 and the constraint $g(x) \in Y_+$ satisfies the Slater constraint qualification

(2.5.1) $$\exists x_0 \in X_0 : g(x_0) \in \text{int } Y_+$$

Then

$$\mathcal{S}(P) \cap W_0 \subset \mathcal{I}(D) .$$

Proof. Let $w^* \in \varphi(P) \cap W_0$. Obviously

$$f(x) - w^* \notin \text{int } W_+ \quad \forall x \in V$$

which implies, by (2.2.1),

(2.5.2) $$\varphi(x, w^*) \notin \operatorname{int} \tilde{W}_+ \quad \forall x \in V$$

where

$$V = \{x \in X_0 \mid g(x) \in Y_+\}.$$

Put

$$U = \{(w, y) \in \tilde{W} \times Y \mid \exists x \in X_0 : \varphi(x, w^*) - w \in \tilde{W}_+ \ \& \ g(x) - y \in Y_+\}.$$

Obviously U is convex and from (2.5.2) it follows

$$U \cap \operatorname{int} \tilde{W}_+ \times \operatorname{int} Y_+ = \emptyset$$

So, by a separation theorem (see Holmes [12] or Ekeland, Temam [11], p. 5 Corollary 1.1), there exist $\mu \in W'$, $\eta \in Y'$, $(\mu, \eta) \neq (0, 0)$ such that

(2.5.3) $$\mu(w) + \eta(y) \leqq \mu(w') + \eta(y') \quad \forall(w, y) \in U \ \& \quad \forall(w', y') \in \tilde{W}_+ \times Y_+.$$

Obviously $\mu \in \tilde{W}_+^*$ and $\eta \in Y_+^*$. If $\mu = 0$, then $\eta \neq 0$ and by (2.5.1)

$$\mu(\varphi(x_0, w^*)) + \eta(g(x_0)) = \eta(g(x_0)) > 0$$

which contradicts (2.5.3) for $(\varphi(x_0, w^*), g(x_0)) \in U$ and $(0, 0) \in \tilde{W}_+ \times Y_+$. Hence $\mu \neq 0$. In view of (2.5.3) the constraint (2.3.1) is fulfilled. So (w^*, μ, η) is a feasible solution of program (D) and by the weak duality $w^* \in \mathscr{I}(D)$.

2.6. Remark.

2.6.1. The Slater constraint qualification can be replaced by a weakened condition the generalized Slater constraint qualification (see Golstein [13], p. 89).

2.6.2. From the proof we see that (w^*, μ, η) is actually an optimal solution of dual (D), so that the direct duality holds.

2.6.3. Let (x_α^*) be an asymptotic optimal solution of program (P) with $w^* = \lim_\alpha f(x_\alpha^*)$. Then, by Theorem 2.5, there exists an optimal solution (w^*, μ^*, η^*) of dual (D). It is easy to verify that for this pair of optimal solutions the asymptotic complementary condition

$$\lim_\alpha \eta^*(g(x_\alpha^*)) = 0$$

holds.

3. QUASIDIFFERENTIABLE DUALITY

3.1. In this section $W_+ \subset W$, $Y_+ \subset Y$ and $T \subset Z$ are nonempty closed convex cones with $\operatorname{int} W_+ \neq \emptyset$ and $\operatorname{int} Y_+ \neq \emptyset$, X_0 is a nonempty subset of X and $f : X_0 \to W$, $g : X_0 \to Y$ and $h : X_0 \to Z$ are functions mapping X_0 to W, Y and Z respectively.

We shall be concerned with the following program

$$\left.\begin{array}{r} f(x) \to \sup \\ g(x) \in Y_+ \\ h(x) \in T \\ x \in X_0 \end{array}\right\} \qquad (\mathscr{P})$$

3.2. Similarly as in Section 2 we suppose that there exist a nonempty subset W_0 of W and a function $\varphi : X_0 \times W_0 \to W$ such that

(3.2.1) $$f(x) - w \in \text{int } W_+ \Leftrightarrow \varphi(x, w) \in \text{int } W_+ \quad \forall(x, w) \in X_0 \times W_0$$

and

(3.2.2) $$f(x) = w \Leftrightarrow \varphi(x, w) = 0$$

3.3. Supposing $\varphi(\cdot, w)$ $(w \in W_0)$, $g(x)$ and $h(x)$ are W_+^*-, Y_+^*- and T^*-quasidifferentiable respectively, the following program

$$\left.\begin{array}{lc} & w \to \inf \\ (3.3.1) & \mu\, \varphi(x, w) + \eta\, g(x) + \tau\, h(x) \leqq 0 \\ (3.3.2) & 0 \in \partial^{\sim}(\mu\varphi)(x, w) + \partial^{\sim}(\eta g)(x) + \partial^{\sim}(\tau h)(x) \\ & x \in X_0 \;\&\; w \in W_0 \\ & \mu \in W_+^* \setminus \{0\} \;\&\; \eta \in Y_+^* \;\&\; \tau \in T^* \end{array}\right\} \qquad (\mathscr{D})$$

is called a *quasidifferentiable dual* of program $(\mathscr{P})$.

3.4. Theorem (Weak Duality). Let (x, w, μ, η, τ) be a feasible solution of program $(\mathscr{D})$. If function $k(x') = \mu\, \varphi(x', w) + \eta\, g(x') + \tau\, h(x')$ is directionally pseudoconcave at x then for any feasible solution x' of program $(\mathscr{P})$

$$f(x') - w \notin \text{int } W_+ .$$

Proof. Let, on the contrary, $f(x') - w \in \text{int } W_+$ for feasible solution x' of program $(\mathscr{P})$. Then, by (3.2.1), $\varphi(x', w) \in \text{int } W_+$ which implies

$$k(x') = \mu\, \varphi(x', w) + \eta\, g(x') + \tau\, h(x') \geqq \mu\, \varphi(x', w) > 0 \geqq$$
$$\geqq \mu\, \varphi(x, w) + \eta\, g(x) + \tau\, h(x) = k(x) .$$

The inequality $k(x') > k(x)$ implies, by directional pseudoconcavity of function $k(x')$ at x, $k'(x, x' - x) > 0$ which contradicts constraint (3.3.2).

3.5. The constraint $h(x)$ (or simply h) is *locally solvable* at $a \in X_0$ if $h(a) \in T$ and whenever $d \in X$ satisfies $h(a) + h'(a, d) \in T$ there exists a solution $x = a + \lambda d + 0(\lambda)$ to $h(x) \in T$ valid for all sufficiently small $\lambda > 0$ (note $0(\lambda)/\lambda \to 0$ as $\lambda \downarrow 0$).

3.6. Theorem (Strict Duality). Suppose X is a Banach space, X_0 is a nonempty convex and open set and x^* is an optimal solution of program $(\mathscr{P})$. Let $\varphi(\cdot, w^*)$ $(w^* = f(x^*))$, $g(x)$ and $h(x)$ be continuously W_+^*-, Y_+^*- and T^*-quasidifferentiable at x^* respectively. Let $k(x) = \mu\, \varphi(x, w) + \eta\, g(x) + \tau\, h(x)$ be directionally pseudoconcave on X_0 for any $w \in W_0$, $\mu \in W_+^* \setminus \{0\}$, $\eta \in Y_+^*$ and $\tau \in T^*$. If h is nonlinear let $\varphi(\cdot, w^*)$ and g be arc-wise directionally differentiable at x^*. Let h be locally solvable at x^* with

$$h'(x^*, X) + \operatorname{cone}(h(x^*)) + T = Z \tag{3.6.1}$$

and the constraint $g(x) \in Y_+$ satisfy the Slater constraint qualification

$$\exists x_0 \in X_0 : g(x_0) \in \operatorname{int} Y_+ \ \&\ h(x_0) \in T\,. \tag{3.6.2}$$

Then there exist $\mu^* \in W_+^* \setminus \{0\}$, $\eta^* \in Y_+^*$ and $\tau^* \in T^*$ such that $(x^*, w^*, \mu^*, \eta^*, \tau^*)$ is an optimal solution of program $(\mathscr{D})$.

Proof. From (3.2.1) it is easy seen that $\varphi(\cdot, w^*)$ reaches (weak) maximum on $V = \{x \in X_0 \mid g(x) \in Y_+ \ \&\ h(x) \in T\}$ at x^*. Hence there exist, by [15] Theorem 4 and Corollary 2, $(\mu^*, \eta^*) \in W_+^* \times Y_+^*$ $(\mu^*, \eta^*) \neq (0, 0)$ and $\tau^* \in T^*$ such that

$$0 \in \partial^{\sim}(\mu^*\varphi)(x^*, w^*) + \partial^{\sim}(\eta^* g)(x^*) + \partial^{\sim}(\tau^* h)(x^*) \tag{3.6.3}$$

and

$$0 = \eta^*\, g(x^*) + \tau^*\, h(x^*) \tag{3.6.4}$$

In view of assumption (3.2.2) we have $\varphi(x^*, w^*) = 0$ which, together with equality (3.6.4), gives

$$\mu^*\, \varphi(x^*, w^*) + \eta^*\, g(x^*) + \tau^*\, h(x^*) = 0 \tag{3.6.5}$$

If $\mu^* = 0$ then $\eta^* \neq 0$ for $(\mu^*, \eta^*) \neq (0, 0)$. So for $x_0 \in V$ with $g(x_0) \in \operatorname{int} Y_+$ (existence of such an x_0 is guaranteed by assumption (3.6.2)). We have

$$k(x_0) = \mu^*\, \varphi(x_0, w^*) + \eta^*\, g(x_0) + \tau^*\, h(x_0) \geqq \eta^*\, g(x_0) > 0 =$$
$$\mu^*\, \varphi(x^*, w^*) + \eta^*\, g(x^*) + \tau^*\, h(x^*) = k(x^*)$$

which implies, by directional pseudoconcavity of function $k(x)$ at x^*, $k' \,.\, (x^*, x_0 - x^*) > 0$, a contradiction with (3.6.3). Hence $\mu^* \neq 0$. We have thus proved, by (3.6.3), (3.6.5) and $\mu^* \neq 0$, that $(x^*, w^*, \mu^*, \eta^*, \tau^*)$ is a feasible solution of program $(\mathscr{D})$. Optimality of $(x^*, w^*, \mu^*, \eta^*, \tau^*)$ is then derived from the weak duality 3.4. The proof is complete.

3.7. Remark. In case $T = R_+^p \times \{0\}$ and $h(x)$ is Gâteaux differentiable at x^* local solvability of function $h(x)$ at x^* and (3.6.1) are equivalent to the Kuhn-Tucker constraint qualification and they hold, in particular, if the gradients of active constraints at x^* (i.e. components h_i of h with $h_i(x^*) = 0$) are linearly independent (see [16] Craven p. 666). The Mangasarian constraint qualification in [17] Martos p. 127 yields, after some transformations, the local solvability, assumption (3.6.1) and the Slater constraint qualification required in our theorem.

3.8. Corollary. Suppose X is a Banach space, X_0 is an open, convex set and x^*

is an optimal solution of program $(\mathscr{P})$. Let $\varphi(\cdot, w)$ $(w \in W_0)$, $g(x)$ and $h(x)$ be continuous and concave on X_0. Let h be locally solvable at x^*. Let assumptions (3.6.1) and (3.6.2) hold. Then there exist $\mu^* \in W_+^* \setminus \{0\}$, $\eta^* \in Y_+^*$ and $\tau^* \in T^*$ such that $(x^*, w^*, \mu^*, \eta^*, \tau^*)$, where $w^* = f(x^*)$, is an optimal solution of program $(\mathscr{D})$.

Proof. Obviously directional pseudoconcavity of function

$$k(x) = \mu\, \varphi(x, w) + \eta\, g(x) + \tau\, h(x)$$

for all $w \in W_0$, $\mu \in W_+^* \setminus \{0\}$, $\eta \in Y_+^*$ and $\tau \in T^*$ is guaranteed by concavity of functions $\varphi(\cdot, w)$, $g(x)$ and $h(x)$. Remark 1.9 shows that other assumptions required for Theorem 3.6 are also fulfilled. The assertion is then a consequence of Theorem 3.6.

4. DUALITY IN VECTOR FRACTIONAL PROGRAMMING

4.1. Introduction. Some decision problems in management science as well as other extremum problems gives rise to the optimization of ratios. Constrained ratio optimization problems are commonly called fractional programs. They may involve more than one ratio in the objective function. Many works (about 500 according to Schaible [18]) have already appeared in this field. One may find a relatively complete survey on fractional programming in Schaible [18], [19]. We shall now develop a duality theory for vector fractional programming (V. F.P.), which is still les investigated. For the scalar fractional programming there are several approaches to define duals, see [18]–[25], and the most known of them is the transformation one. On the basis of this method one can transform a fractional program, under certain conditions, to a concave maximization program and then apply the known duality theory for concave maximization. As regards V. F. P., these approaches are not applicable, since it is not generally possible to reduce simultaneously all components of objective function to a concave or convex function. That is why one should find a new method to define dual programs for V. F. P. In [10] the author has presented a dual concept for vector quadratic-affine and vector quadratic fractional programs. In the present paper, on the basis of the duality theory developed in Sections 2 and 3 we shall define dual programs for a widely class of V. F. P.

It should be stressed that the results given in this paper are valid for an arbitrary Banach space, whereas the results concerning this problem, which have been published up to this time, were proved only for finite dimensional spaces.

4.2. Definitions. Suppose X is a locally convex space, f_i, g_i $(i = 1, \ldots, p)$ and h_k $(k = 1, \ldots, m)$ are real valued functions, which are defined on a nonempty subset $X_0 \subset X$. We consider the ratio

(4.2.1) $$q_i(x) = f_i(x)/g_i(x) \quad i = 1, \ldots, p$$

over the set

(4.2.2) $$D = \{x \in X_0 \mid h_k(x) \geqq 0 \ \forall k = 1, \ldots, m\}$$

We assume that $g_i(x)$, $i = 1, \ldots, p$, are positive on X_0. If $g_i(x)$ is negative then $q_i(x) = (-f_i(x))/(-g_i(x))$ may be used instead. Put

$$(4.2.3) \qquad Q(x) = (q_1(x), \ldots, q_p(x))^{\mathrm{T}}$$

where T indicates transposed matrix.

The program

$$(4.2.4) \qquad \begin{array}{c} Q(x) \to \sup \\ x \in D \end{array} \qquad (p)$$

is called a vector fractional program (V. F. P.).

In some applications more than one ratio appear in components of objective function. Here we consider the following program. Suppose, in addition, $f_{ij}(x)$, $g_{ij}(x)$ ($i = 1, \ldots, p; j = 1, \ldots, p_i$) are real valued functions on X_0 such that $g_{ij}(x)$ are positive on X_0.

Put

$$(4.2.5) \qquad \tilde{q}_i(x) = \min_{1 \leqq j \leqq p_i} f_{ij}(x)/g_{ij}(x)$$

and

$$(4.2.6) \qquad \tilde{Q}(x) = (\tilde{q}_i(x), \ldots, \tilde{q}_p(x))^{\mathrm{T}} .$$

Then program

$$(4.2.7) \qquad \begin{array}{c} \tilde{Q}(x) \to \sup \\ x \in D \end{array} \qquad (\tilde{p})$$

is sometimes referred to as a generalized vector fractional program (G. V. F. P.).

The focus in fractional programming has been directed to the objective function and not to the constraint set D. As far as D is concerned, in most of the references D is assumed to be a convex set. Accordingly, we will require in this paper that the domain X_0 of all functions in programs (p) and $(\tilde{p})$ is a nonempty convex set and the constraints h_k ($k = 1, \ldots, m$) are concave on X_0. This implies convexity of the feasible region D. In many applications the ratios $q(x) = f(x)/g(x)$ satisfy the the following assumption.

4.2.8. Concavity-Convexity Assumption:
(i) f is concave and g is convex
(ii) f is positive if g is not affine (linear plus constant).

4.2.9. Program (p) resp. $(\tilde{p})$ are called vector concave fractional program (V. C. F. P.) resp. generalized vector concave fractional program (G. V. C. F. P.) if all the ratios appearing in the objective function satisfies the concavity-convexity assumption.

In the following we shall establish a nondifferentiable dual for a G. V. C. F. P. and a quasidifferentiable dual for a V. C. F. P., in particular for vector quadratic fractional programs.

4.3. Nondifferentiable dual

Consider G. V. C. F. P. $(\tilde{p})$. Put

$$r_i = \begin{cases} -\infty & \text{if } g_{ij} \text{ are affine for all } j = 1, \ldots, p_i \\ 0 & \text{otherwise} \end{cases}$$

and $\tilde{W} = R^s$, $W_0 = ((r_1, +\infty) \times \ldots \times (r_p, +\infty)) \cup \{0\}$, where $s = \sum_{i=1}^{p} p_i$ and 0 indicates the zero element of R^p.

We define the function $\varphi : X_0 \times W_0 \to \tilde{W}$ as follows

$$(4.3.1) \qquad \begin{aligned} \varphi(x, w) = [&f_{11}(x) - w_1 g_{11}(x), \ldots, f_{1p_1}(x) - w_1 g_{1p_1}), \ldots \\ &\ldots, f_{i1}(x) - w_i g_{i1}(x), \ldots, f_{ip_i}(x) - w_i g_{ip_i}(x), \ldots \\ &\ldots, f_{p1}(x) - w_p g_{p1}(x), \ldots, f_{pp_p}(x) - w_p g_{pp_p}(x)]^{\mathrm{T}} \end{aligned}$$

for all $x \in X_0$ and $w = (w_1, \ldots, w_p) \in W_0$.

Obviously the function $\varphi(x, w)$ satisfies condition (2.2.1). So according to Section 2 the following program

$$(4.3.2) \qquad \left.\begin{aligned} & w \to \inf \\ & \sup_{x \in X_0} \Big(\sum_{i=1}^{p} \sum_{j=1}^{p_i} u_{ij} f_{ij}(x) + \sum_{k=1}^{m} v_k h_k(x) - \sum_{i=1}^{p} w_i \sum_{j=1}^{p_i} u_{ij} g_{ij}(x) \Big) \leqq 0 \\ & w \geqq r (= (r_1, \ldots, r_p)^{\mathrm{T}}) \\ & u_{ij} \geqq 0 \quad \forall i = 1, \ldots, p\,; \quad j = 1, \ldots, p_i \sum_{i=1}^{p} \sum_{j=1}^{p_i} u_{ij}^2 > 0 \\ & v_k \geqq 0 \quad \forall k = 1, \ldots, m\,. \end{aligned}\right\} \qquad (\tilde{d})$$

is a nondifferentiable dual of G. V. C. F. P. $(\tilde{p})$.

As a consequence of Theorem 2.4 and 2.5 we have

4.3.3. Theorem. For the dual pair $(\tilde{p})$ and $(\tilde{d})$ the weak duality holds. If constraints $h_k(x) \geqq 0$, $k = 1, \ldots, m$, satisfy Slater's constraint qualification then the partially strong duality holds i.e.

$$\mathscr{S}(\tilde{p}) \cap W_0 \subset \mathscr{I}(\tilde{d})$$

4.3.4. Remark. The dual program $(\tilde{d})$ is a generalization of the dual for one-objective fractional program established in Schaible [18], p. 48. Indeed, if $p = 1$ then program $(\tilde{p})$ becomes

$$\sup \{ \min_{1 \leqq i \leqq p} f_i(x)/g_i(x) \mid x \in X_0 \,\&\, h_k(x) \geqq 0 \ \forall k = 1, \ldots, m \} \qquad (\tilde{p}_1)$$

and its dual, as a particular case of $(\tilde{d})$, is

$$\inf \{ \sup_{x \in X_0} \Big(\sum_{i=1}^{p} u_i f_i(x) + \sum_{k=1}^{m} v_k h_k(x) \Big) / \sum_{i=1}^{p} u_i g_i(x) \mid u_i, v_k \geqq 0 \,\&\, \sum_{i=2}^{p} u_i^2 > 0 \} \qquad (\tilde{d}_1)$$

From Theorem 4.3.3 it follows that if the Slater constraint qualification holds then

$$\sup(\tilde{p}_1) = \inf(\tilde{d}_1)$$

where $\sup(\tilde{p}_1)$ and $\inf(\tilde{d}_1)$ are supremum of $(\tilde{p}_1)$ and infimum of $(\tilde{d}_1)$ respectively. Note that in Schaible [18] in order to vanish the dual gap, instead of the Slater constraint qualification, lower semicontinuity of functions f_i, g_i, h_k and compactness of the set X_0 are required to be satisfied.

4.4. Quasidifferentiable duality

Consider the V. C. F. P. (p). Suppose X is a Banach space, X_0 is a nonempty convex and open set, f_i, g_i and $h_k(\forall i, k)$ are continuous.

Put

$$r_i = \begin{cases} -\infty & \text{if } g_i \text{ is affine} \\ 0 & \text{if } g_i \text{ is not affine} \end{cases}$$

$W = R^p$ and $W_0 = (r_1, +\infty) \times \ldots \times (r_p, +\infty) \cup \{0\}$, where 0 is the zero element of R^p.

Define the function $\varphi : X_0 \times W_0 \to R^p$ as follows

(4.4.1) $$\varphi(x, w) = (f_1(x) - w_1 g_1(x), \ldots, f_p(x) - w_p g_p(x))^{\mathrm{T}}$$

for all $x \in X_0$ and $w = (w_1, \ldots, w_p) \in W_0$.

Obviously the function $\varphi(x, w)$ satisfies conditions (3.2.1) and (3.2.2). So applying results of Section 3 we obtain a quasidifferentiable dual of program (p) in the following form

(4.4.2) $$\left.\begin{array}{c} w \to \inf \\ \sum_{i=1}^{p} u_i f_i(x) + \sum_{k=1}^{m} v_k h_k(x) - \sum_{i=1}^{p} w_i u_i g_i(x) \leqq 0 \\ 0 \in \sum_{i=1}^{p} u_i \partial^- f_i(x) + \sum_{k=1}^{m} v_k \partial^- h_k(x) - \sum_{i=1}^{p} w_i u_i \partial^- g_i(x) \\ x \in X_0 \;\&\; w \in R^p : w \geqq r \\ u_i, v_k \geqq 0 \;\; \forall i = 1, \ldots, p\,; \quad k = 1, \ldots, m \;\&\; \sum_{i=1}^{p} u_i^2 > 0 \end{array}\right\} \quad (d)$$

If f_i, g_i and h_k are differentiable for all i, k the dual (d) becomes

(4.4.3) $$w \to \inf$$

$$\left.\begin{array}{ll} \text{(i)} & \sum_{i=1}^{p} u_i f_i(x) + \sum_{k=1}^{m} v_k h_k(x) - \sum_{i=1}^{p} w_i u_i g_i(x) \leqq 0 \\ \text{(ii)} & 0 = \sum_{i=1}^{p} u_i \nabla f_i(x) + \sum_{k=1}^{m} v_k \nabla h_k(x) - \sum_{i=1}^{p} w_i u_i \nabla g_i(x) \\ & x \in X_0 \;\&\; w \in R^p : w \geqq r \\ & u_i, v_k \geqq 0 \quad \forall i = 1, \ldots, p; \quad k = 1, \ldots, m \;\&\; \sum_{i=1}^{p} u_i^2 > 0 \end{array}\right\} \quad (d')$$

As consequences of Theorem 3.4 and 3.5 one obtains

4.4.4. Theorem (Weak Duality). For any feasible solutions x' and (x, w, u, v) of programs (p) and (d) ((d') in differentiable cases) we have

$$f(x') - w \notin \operatorname{int} R^p_+ .$$

4.4.5. Theorem (Strict Duality). Let x^* be an optimal solution of program (p). Let the constraint $h(x) \geqq 0$, where $h(x) = (h_1(x), \ldots, h_m(x))^T$, have the following form

$$h^1(x) \geqq 0 \;\&\; h^2(x) = 0$$

where $h^i : X_0 \to R^{m_i}$, $i = 1, 2$, $m_1 + m_2 = m$. Suppose $h^1(x)$ satisfies the Slater constraint qualification and $h^2(x)$ is locally solvable with

$$(h^2)'(x^*, X) = R^{m_2} .$$

Then there exist $u^* = (u_1^*, \ldots, u_p^*)^T$, $\sum_{i=1}^{p} u_i^{*2} > 0$ and $v^* = (v_1^*, \ldots, v_m^*)^T$ such that (x^*, w^*, u^*, v^*), where $w^* = Q(x^*)$, is an optimal solution of program (d) (respectively of (d') if the concerned functions are differentiable on X_0).

4.4.6. Remark. Obviously the above assertion is still valid if Mangasarian's constraint qualification (see [17] Martos, p. 127) is required instead.

If $p = 1$ and concerned functions are differentiable then dual program (d') reduces to the dual (D_1) of Schaible [20]. There, in order to get strong duality, Schaible has required some constraint qualification to be fulfilled. It is easy seen that our dual (d') is a generalization of Schaible's one.

4.4.7. Vector quadratic fractional program

Suppose C_i and D_i, $i = 1, \ldots, p$, are real symmetric $n \times n$ matrices negatively and positively semidefinite respectively, c_i, $d_i \in R^n$ and α_i, $\beta_i \in R$ for $i = 1, \ldots, p$, A is an $m \times n$ matrix and $b \in R^m$. Let $X_0 \subset R^n$ be a nonempty open and convex set, on which $x^T D_i x + d_i^T x + \beta_i$ are positive for all $i = 1, \ldots, p$. Put

$$f_i(x) = x^T C_i x + c_i^T x + \alpha_i$$

$$g_i(x) = x^T D_i x + d_i^T x + \beta_i$$

and

$$q(x) = (f_1(x)/g_1(x), \ldots, f_p(x)/g_p(x))^T$$

Program

$$q(x) \to \sup$$
$$x \in X_0 \;\&\; Ax \leqq b \qquad (qp)$$

is called a vector quadratic fractional program (V. Q. F. P.).

Since program (qp) is evidently a vector concave fractional program, one can apply the differentiable dual (d') for (qp).

We have

$$\nabla f_i(x) = 2C_i x + c_i \quad i = 1, \ldots, p$$
$$\nabla g_i(x) = 2D_i x + d_i \quad i = 1, \ldots, p$$

and

$$\nabla h_k(x) = -a_k \quad k = 1, \ldots, m$$

where a_k is the kth row of the matrix A and $h_k(x) = b_k - a_k^T x$.

Constraint (ii) of program (d') becomes

$$\text{(ii)} \qquad 0 = \sum_{i=1}^{p} 2u_i(C_i - w_i D_i)\, x + \sum_{i=1}^{p} u_i(c_i - w_i d_i) - A^T v$$

where $v = (v, \ldots, v_m) \in R^m$.

Constraint (i) of (d') becomes

$$0 \geqq \sum_{i=1}^{p} u_i[(x^T C_i x + c_i^T x + \alpha_i) - w_i(x^T D_i x + d_i^T x + \beta_i)] + (b - Ax)^T v$$

and after replacing

$$A^T v = \sum_{i=1}^{p} 2u_i(C_i - w_i D_i)\, x + \sum_{i=1}^{p} u_i(c_i - w_i d_i)\,,$$

what follows from (ii), we obtain

$$\text{(i)} \qquad 0 \geqq - \sum_{i=1}^{p} u_i x^T(C_i - w_i D_i)\, x + \sum_{i=1}^{p} u_i(\alpha_i - w_i \beta_i) + b^T v\,.$$

So a differentiable dual of program (qp) is

$$\left.\begin{array}{c} w \to \inf \\ \sum_{i=1}^{p} 2u_i(C_i - w_i D_i)\, x + \sum_{i=1}^{p} u_i(c_i - w_i d_i) - A^T v = 0 \\ - \sum_{i=1}^{p} u_i x^T(C_i - w_i D_i)\, x + \sum_{i=1}^{p} u_i(\alpha_i - w_i \beta_i) + b^T v \leqq 0 \\ x \in X_0\,, \quad w \in R^p \quad w_i \geqq 0 \quad \text{if} \quad D_i \neq 0 \\ u = (u_1, \ldots, u_p) \in R_+^p \setminus \{0\}\,, \quad v = (v_1, \ldots, v_p) \in R_+^m \end{array}\right\} \qquad (qd)$$

Since $h(x) = b - Ax$ is affine all constraints qualifications required in Theorem 4.4.5 are fulfilled. Hence we have

4.4.8. Theorem (Strict Duality). If x^* is an optimal solution of program (qp) then there exist u^*, v^* such that (x^*, w^*, u^*, v^*), where $w^* = q(x^*)$, is an optimal solution of program (qd).

4.4.9. Remark. Our differentiable dual (qd) is a generalization of the scalar one given in Schaible [20]. Indeed, if $p = 1$, program (qd) is reduced to program (10) of Schaible [20]. Schaible has there assumed that

$$\{x \in R^n \mid Ax \leqq b\} \subset X_0$$

in order to guarantee existence of an optimal solution of the primal program.

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