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On some Properties of Estimators of a Probability Density

M. A. MIRZACHMEDOV, SH. A. CHASHIMOV

Let X_1, X_2, \dots, X_n be mutually independent and identically distributed s -dimensional random vectors with a probability density $p(x) \in L_2$.

Consider the class of estimators $\hat{p}_n(x)$ of the form

$$\hat{p}_n(x) = \frac{1}{n} \sum_{k=1}^n p_n(x - X_k)$$

for estimating of the density $p(x)$, where $p_n(x) \in L_2$. In this paper some consistency properties of estimators in the sense of mean integral square error are studied.

1. INTRODUCTION

Let

$$X_k = (X_{k1}, \dots, X_{ks}), \quad k = 1, 2, \dots, n,$$

be a random sample with a sample size n from a distribution $F(x)$, $x \in R^s$, with the probability density $p(x) \in L_2$, i.e. X_1, \dots, X_n are mutually independent and identically distributed s -dimensional random vectors with the density $p(x) \in L_2(R^s)$.

Consider the class of estimators $\hat{p}_n(x)$ of the form

$$\hat{p}_n(x) = \frac{1}{n} \sum_{k=1}^n p_n(x - X_k)$$

for estimating of the density $p(x)$, where $p_n(x) \in L_2$. The quality of the estimators is defined by the mean integral square error (MISE)

$$I_n = E \int_{R^s} [\hat{p}_n(x) - p(x)]^2 dx.$$

Let

$$f_1(t) = \int_{R^s} e^{i(t,x)} p(x) dx, \quad f_2(t) = \int_{R^s} e^{i(t,x)} \hat{p}_n(x) dx,$$

$$f_3(t) = \int_{R^s} e^{i(t,x)} p_n(x) dx,$$

where $(t, x) = \sum_{k=1}^s t_k \cdot x_k$ is the inner product of two elements t and x of R^s .

It is proved in [1] and [2] that

$$(1) \quad I_n = \frac{1}{(2\pi)^s} \int_{R^s} \left\{ \frac{1}{n} |f_3(t)|^2 \cdot [1 - |f_1(t)|^2] + |f_1(t)|^2 \cdot |1 - f_3(t)|^2 \right\} dt$$

and if

$$f_3(t) = f_3^*(t) = \frac{n|f_1(t)|^2}{1 + (n-1)|f_1(t)|^2},$$

$$f_3^*(t) = \int_{R^s} e^{i(t,x)} p_n^*(x) dx,$$

where

$$p_n^*(x) = \frac{1}{(2\pi)^s} \int_{R^s} e^{-i(t,x)} f_3^*(t) dt,$$

then the minimum of I_n is reached and

$$(2) \quad I_n^* = \min I_n = \frac{p_n^*(0, 0, \dots, 0)}{n} - O\left(\frac{1}{n}\right).$$

2. ASYMPTOTIC BEHAVIOUR OF I_n^*

For the sake of simplicity only the bivariate case is considered and asymptotic behaviour (if $n \rightarrow \infty$) of I_n^* is studied.

We say (see [3]), that the characteristic function $f_1(t)$ decreases exponentially with respect to coefficients $p_i > 0$ and with respect to t_i ($i = 1, 2$) if the following conditions are satisfied:

(a) $|f_1(t)| \leq A e^{-(p, |t|)}$, for some constant $A > 0$ and for all t ,

$$(b) \quad \lim_{\substack{v_1 \rightarrow \infty \\ v_2 \rightarrow \infty}} \int_0^1 \int_0^1 [1 + e^{(p, v)} \cdot |f_1(v_1 t_1, v_2 t_2)|^2]^{-1} dt = 0,$$

where

$$(p, |t|) = p_1 \cdot |t_1| + p_2 \cdot |t_2|, \quad (p, v) = p_1 v_1 + p_2 v_2.$$

The case when the characteristic function $f_1(t)$ decreases algebraically of degree $p_i > 0$ with respect to arguments t_i ($i = 1, 2$) was considered in [1], [2].

Definition. Let $\varphi(n)$ be some function of n and let $\lim_{n \rightarrow \infty} \varphi(n) = \infty$. The estimator $\hat{p}_n^*(x)$ is said to be *integrally consistent of order $\varphi(n)$* if

$$\lim_{n \rightarrow \infty} \varphi(n) \cdot I_n = a \quad \text{where } 0 < a < \infty$$

(see [4]).

Theorem 1. Let the characteristic function $f_1(t)$ decrease exponentially with respect to coefficients $p_i > 0$ and arguments t_i ($i = 1, 2$). Then the estimator $\hat{p}_n^*(x)$ is integrally consistent of order $n/\ln^2 n$, namely:

$$\lim_{n \rightarrow \infty} \frac{n}{\ln^2 n} \cdot I_n^* = \frac{1}{8\pi^2 p_1 p_2}.$$

Proof.

$$\begin{aligned} & \int_{R^2} \frac{|f_1(t)|^2}{1 + (n-1)|f_1(t)|^2} dt - \int_{R^2} \frac{\exp[-2(p, |t|)]}{1 + (n-1)\exp[-2(p, |t|)]} dt = \\ & = \int_{R^2} \frac{|f_1(t)|^2 - \exp[-2(p, |t|)]}{[1 + (n-1)|f_1(t)|^2][1 + (n-1)\exp[-2(p, |t|)]]} dt = L. \end{aligned}$$

Denote

$$v_1 = \frac{\ln(n-1)}{2p_1}, \quad v_2 = \frac{\ln(n-1)}{2p_2}.$$

Using the condition (a), we have

$$\begin{aligned} |L| & \leq 4(A^2 + 1) \int_0^\infty \int_0^\infty \frac{\exp[-2(p, t)] dt_1 dt_2}{[1 + (n-1)|f_1(t)|^2][1 + (n-1)\exp[-2(p, t)]]} \leq \\ & \leq 4(A^2 + 1) \left\{ \int_0^{v_1} \int_0^{v_2} \frac{dt_1 dt_2}{(n-1)[1 + (n-1) \cdot |f_1(t)|^2]} + \right. \\ & \left. + \int_{v_1}^\infty \int_{v_2}^\infty e^{-2(p, t)} dt_1 dt_2 + \int_0^{v_1} \int_{v_2}^\infty e^{-2(p, t)} dt_1 dt_2 + \int_{v_1}^\infty \int_0^{v_2} e^{-2(p, t)} dt_1 dt_2 \right\} = L_1. \end{aligned}$$

Then, denoting

$$t_1 = v_1 s_1, \quad t_2 = v_2 s_2,$$

$$\begin{aligned}
L_1 &= 4(A^2 + 1) \left\{ \frac{1}{n-1} \int_0^1 \int_0^1 \frac{v_1 v_2 ds_1 ds_2}{1 + \exp[(p, v)] \cdot |f_1(v_1 s_1, v_2, s_2)|^2} + \right. \\
&\quad + \int_1^\infty \int_1^\infty v_1 v_2 e^{-2(p, v s)} ds_1 ds_2 + \int_0^1 \int_1^\infty v_1 v_2 e^{-2(p, v s)} ds_1 ds_2 + \\
&\quad \left. + \int_1^\infty \int_0^1 v_1 v_2 e^{-2(p, v s)} ds_1 ds_2 \right\} = \\
&= 4(A^2 + 1) \left\{ \frac{\ln^2(n-1)}{4p_1 p_2 (n-1)} \int_0^1 \int_0^1 \frac{ds_1 ds_2}{1 + \exp[(p, v)] \cdot |f_1(v_1 s_1, v_2 s_2)|^2} + \right. \\
&\quad \left. + \frac{1}{2p_1 p_2 (n-1)} - \frac{1}{4p_1 p_2 (n-1)^2} \right\}.
\end{aligned}$$

Therefore, $L \rightarrow 0$ if $n \rightarrow \infty$.

Let us consider the following integral

$$\begin{aligned}
K_n &= \int_{R^2} \frac{\exp[-2(p, |t|)]}{1 + (n-1) \exp[-2(p, |t|)]} dt = \\
&= -\frac{2}{p_2(n-1)} \int_0^\infty dt_1 \int_0^\infty d \ln [1 + (n-1) e^{-2p_1 t_1} e^{-2p_2 t_2}] = \\
&= \frac{2}{p_2(n-1)} \int_0^\infty \ln [1 + (n-1) e^{-2p_1 t_1}] dt_1 = \frac{2}{p_2(n-1)} Q_n.
\end{aligned}$$

Let $x = (n-1) e^{-2p_1 t_1}$, then

$$Q_n = \frac{1}{2p_1} \int_0^{n-1} \frac{\ln(1+x)}{x} dx = \frac{1}{2p_1} \int_0^1 \frac{\ln(1+x)}{x} dx + \frac{1}{2p_1} \int_1^{n-1} \frac{\ln(1+x)}{x} dx.$$

Calculations of the two last integral give (see [5])

$$Q_n = \frac{\pi^2}{24p_1} + \frac{\ln^2(n-1)}{4p_1} + \frac{1}{2p_1} \cdot \psi(n),$$

where

$$\psi(n) = \sum_{k=1}^{\infty} \frac{1}{(2k)^2 \cdot (n-1)^{2k}} - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 \cdot (n-1)^{2k-1}}.$$

It is easy to verify that

$$|\psi(n)| < \frac{1}{n-2}.$$

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$$K_n = \frac{\ln^2(n-1)}{2p_1 p_2 (n-1)} + \frac{\pi^2}{12p_1 p_2 (n-1)} + o\left(\frac{1}{n^2}\right).$$

Therefore if $n \rightarrow \infty$, then

$$\frac{n}{\ln^2 n} \int_{\mathbb{R}^2} \frac{|f_1(t)|^2}{1 + (n-1) \cdot |f_1(t)|^2} dt \rightarrow \frac{1}{2p_1 p_2}.$$

Thus, from expression (2) with $s = 2$ the statement of the Theorem 1 follows.

Hence, Theorem 1 is proved.

3. ASYMPTOTIC BEHAVIOUR OF I_n

The estimator $\hat{p}_n(x)$ is said to be of an *exponential type* if

$$(3) \quad f_3(t) = h(A_n \cdot e^{\alpha|t_1|}, A_n \cdot e^{\alpha|t_2|}),$$

where

$$\lim_{n \rightarrow \infty} A_n = 0, \quad |h(t)| \leq B, \quad h(t) \in L_2, \quad t \in \mathbb{R}^2.$$

Let us consider the following condition:

$$(4) \quad |1 - h(t)| \leq B_1 |t_1 t_2| \quad \text{for all } |t_i| \leq 1, \quad i = 1, 2.$$

Theorem 2. Let $f_1(t)$ satisfy the condition (a), let the estimator $\hat{p}_n(x)$ be of an exponential type, let the function $h(t)$ satisfy (4) and let

$$A_n = D \cdot n^{-b}, \quad b > \frac{1}{2}, \quad \alpha \leq p_i \quad (i = 1, 2).$$

Then the estimator $\hat{p}_n(x)$ is integrally consistent of order $n/\ln^2 n$, namely:

$$\lim_{n \rightarrow \infty} \frac{n}{\ln^2 n} \cdot I_n = \left(\frac{b}{\pi\alpha}\right)^2.$$

First we formulate two lemmas.

Lemma 1. Under the conditions of Theorem 2 we have

$$\lim_{n \rightarrow \infty} \frac{1}{\ln^2 n} \int_{D_n^{-b}} \int_{D_n^{-b}} \frac{h^2(t_1, t_2)}{t_1 \cdot t_2} dt_1 dt_2 = b^2.$$

Proof.

$$\begin{aligned}
 (5) \quad & \frac{1}{\ln^2 n} \int_{Dn^{-b}}^{\infty} \int_{Dn^{-b}}^{\infty} \frac{h^2(t_1, t_2)}{t_1 \cdot t_2} dt_1 dt_2 = \\
 & = \frac{1}{\ln^2 n} \int_{Dn^{-b}}^1 \int_{Dn^{-b}}^1 \frac{dt_1 dt_2}{t_1 \cdot t_2} - \frac{1}{\ln^2 n} \int_{Dn^{-b}}^1 \int_{Dn^{-b}}^1 \frac{1 - h^2(t_1, t_2)}{t_1 \cdot t_2} dt_1 dt_2 + \\
 & + \frac{1}{\ln^2 n} \int_1^{\infty} \int_1^{\infty} \frac{h^2(t_1, t_2)}{t_1 \cdot t_2} dt_1 dt_2 + \frac{1}{\ln^2 n} \int_{Dn^{-b}}^1 \int_1^{\infty} \frac{h^2(t_1, t_2)}{t_1 \cdot t_2} dt_1 dt_2 + \\
 & + \frac{1}{\ln^2 n} \int_1^{\infty} \int_{Dn^{-b}}^1 \frac{h^2(t_1, t_2)}{t_1 \cdot t_2} dt_1 dt_2.
 \end{aligned}$$

The first term of (5) tends to b^2 if $n \rightarrow \infty$. It follows from conditions (4) and $h(t_1, t_2) \in L_2$, that the other terms of (5) tend to zero if $n \rightarrow \infty$.

Lemma 1 is proved.

Lemma 2. Under the conditions of Theorem 2 we have

$$\int_{\mathbb{R}^2} |f_1(t)|^2 \cdot |1 - f_3(t)|^2 dt = o\left(\frac{\ln^2 n}{n}\right).$$

Proof.

$$\begin{aligned}
 & \frac{n}{\ln^2 n} \int_{\mathbb{R}^2} |f_1(t)|^2 \cdot |1 - f_3(t)|^2 dt = \\
 & = \frac{4n}{\ln^2 n} \int_0^{\infty} \int_0^{\infty} |f_1(t_1, t_2)|^2 \cdot |1 - f_3(t_1, t_2)|^2 dt_1 dt_2 = Q.
 \end{aligned}$$

Denote

$$x_1 = Dn^{-b}e^{\alpha t_1}, \quad x_2 = Dn^{-b}e^{\alpha t_2}.$$

Then we have

$$\begin{aligned}
 Q & = n \left(\frac{2}{\ln n} \right)^2 \int_{Dn^{-b}}^{\infty} \int_{Dn^{-b}}^{\infty} \left| f_1 \left[\frac{1}{\alpha} \ln \left(\frac{n^b}{D} x_1 \right), \frac{1}{\alpha} \ln \left(\frac{n^b}{D} x_2 \right) \right] \right|^2 \\
 & \cdot \frac{|1 - h(x_1, x_2)|^2}{\alpha^2 x_1 x_2} dx_1 dx_2 \leq n \left(\frac{2}{\ln n} \right)^2 \int_{Dn^{-b}}^1 \int_{Dn^{-b}}^1 A^2 \\
 & \cdot \exp \left\{ -2 \left[\frac{p_1}{\alpha} \ln \left(\frac{x_1}{Dn^{-b}} \right) + \frac{p_2}{\alpha} \ln \left(\frac{x_2}{Dn^{-b}} \right) \right] \right\} B_1^2 \frac{x_1 x_2}{\alpha^2} dx_1 dx_2 + \\
 & + n \left(\frac{2}{\ln n} \right)^2 \int_1^{\infty} \int_1^{\infty} (1 + B)^2 A^2 \exp \left\{ -2 \left[\frac{p_1}{\alpha} \ln \left(\frac{x_1}{Dn^{-b}} \right) + \frac{p_2}{\alpha} \ln \left(\frac{x_2}{Dn^{-b}} \right) \right] \right\}.
 \end{aligned}$$

$$\begin{aligned} & \cdot \frac{dx_1 dx_2}{\alpha^2 x_1 x_2} + n \left(\frac{2}{\ln n} \right)^2 \int_1^\infty \int_{Dn^{-b}}^1 (t+B)^2 A^2 \exp \left\{ -2 \left[\frac{p_1}{\alpha} \ln \left(\frac{x_1}{Dn^{-b}} \right) + \right. \right. \\ & \quad \left. \left. + \frac{p_2}{\alpha} \ln \left(\frac{x_2}{Dn^{-b}} \right) \right] \right\} \frac{dx_1 dx_2}{\alpha^2 x_1 x_2} + n \left(\frac{2}{\ln n} \right)^2 \int_{Dn^{-b}}^1 \int_1^\infty (1+B)^2 A^2 \cdot \\ & \cdot \exp \left\{ -2 \left[\frac{p_1}{\alpha} \ln \left(\frac{x_1}{Dn^{-b}} \right) + \frac{p_2}{\alpha} \ln \left(\frac{x_2}{Dn^{-b}} \right) \right] \right\} \frac{dx_1 dx_2}{\alpha^2 x_1 x_2} = Q_1 + Q_2 + Q_3 + Q_4 . \end{aligned}$$

Since $p_i \geq \alpha, 2b > 1$ we have

$$\begin{aligned} Q_1 & \leq n \left(\frac{2}{\ln n} \right)^2 \frac{A^2 B_1^2}{\alpha^2} \int_{Dn^{-b}}^1 \exp \left[-2 \ln \left(\frac{x_1}{Dn^{-b}} \right) \right] x_1 dx_1 \cdot \\ & \cdot \int_{Dn^{-b}}^1 \exp \left[-2 \ln \left(\frac{x_2}{Dn^{-b}} \right) \right] x_2 dx_2 = n \left(\frac{2}{\ln n} \right)^2 \frac{A^2 B_1^2 D^4}{\alpha^2 n^{4b}} (b \ln n - \ln D)^2 \xrightarrow{n \rightarrow \infty} 0 , \\ Q_2 & \leq n \left(\frac{2}{\ln n} \right)^2 \frac{(1+B)^2 A^2}{\alpha^2} \int_1^\infty \exp \left[-2 \ln \left(\frac{x_1}{Dn^{-b}} \right) \right] dx_1 \cdot \\ & \cdot \int_1^\infty \exp \left[-2 \ln \left(\frac{x_2}{Dn^{-b}} \right) \right] dx_2 = n \left(\frac{2}{\ln n} \right)^2 \frac{(1+B)^2 A^2 D^4}{\alpha^2 n^{4b}} \xrightarrow{n \rightarrow \infty} 0 , \\ Q_3 & \leq n \left(\frac{2}{\ln n} \right)^2 \frac{(1+B)^2 A^2}{\alpha^2} \int_{Dn^{-b}}^1 \exp \left[-2 \ln \left(\frac{x_1}{Dn^{-b}} \right) \right] \frac{dx_1}{x_1} \cdot \\ & \cdot \int_1^\infty \exp \left[-2 \ln \left(\frac{x_2}{Dn^{-b}} \right) \right] dx_2 = n \left(\frac{2}{\ln n} \right)^2 \frac{(1+B)^2 A^2 D^4}{2\alpha^2} \frac{(n^{2b} - 1)}{n^{4b}} \xrightarrow{n \rightarrow \infty} 0 . \end{aligned}$$

Also

$$Q_4 \xrightarrow{n \rightarrow \infty} 0 .$$

Lemma 2 is proved.

Proof of Theorem 2. It follows from (1) that

$$\begin{aligned} n \left(\frac{2\pi}{\ln n} \right)^2 I_n & = \frac{1}{\ln^2 n} \int_{R^2} |f_3(t)|^2 [1 - |f_1(t)|^2] dt + \frac{n}{\ln^2 n} \int_{R^2} |f_1(t)|^2 |1 - f_3(t)|^2 dt = \\ & = \frac{1}{\ln^2 n} \int_{R^2} h^2(Dn^{-b} e^{\alpha|t|}, Dn^{-b} e^{\alpha|t|}) dt - \frac{1}{\ln^2 n} \int_{R^2} h^2(Dn^{-b} e^{\alpha|t|}, Dn^{-b} e^{\alpha|t|}) \cdot \\ & \cdot |f_1(t)|^2 dt + \frac{n}{\ln^2 n} \int_{R^2} |f_1(t)|^2 |1 - f_3(t)|^2 dt . \end{aligned}$$

The first term of the last expression tends to $(2b/\alpha)^2$ if $n \rightarrow \infty$ according to Lemma 1.

The second term tends to zero since $|h(t)| \leq B$ for all (t_1, t_2) .

The third term tends also to zero according to Lemma 2.

Thus

$$\lim_{n \rightarrow \infty} \frac{n}{\ln^2 n} I_n = \left(\frac{b}{\pi\alpha}\right)^2.$$

Hence Theorem 2 is proved.

4. EXAMPLES

The following examples show that the order of consistency of an estimator can be reached.

Example 1. Let us consider the probability density of the form

$$p(x, y) = \frac{1}{\pi^2(1+x^2)(1+y^2)}.$$

Obviously,

$$f(t_1, t_2) = e^{-|t_1| - |t_2|}.$$

Then

$$\begin{aligned} p_n^*(0, 0) &= \frac{4n}{(2\pi)^2} \int_0^\infty \int_0^\infty \frac{\exp[-2t_1 - 2t_2]}{1 + (n-1)\exp[-2t_1 - 2t_2]} dt_1 dt_2 = \\ &= \frac{n \ln^2(n-1)}{8\pi^2(n-1)} + \frac{n}{48(n-1)} + O\left(\frac{1}{n}\right). \end{aligned}$$

Thus if $n \rightarrow \infty$

$$I_n^* = \frac{1}{8\pi^2} \frac{\ln^2 n}{n} + \frac{1}{48} \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Example 2. Now, we also consider the density of Cauchy distribution

$$p(x, y) = \frac{1}{(2\pi)^2} \frac{1}{(1+x^2+y^2)^{3/2}}.$$

In this case

$$f(t_1, t_2) = \frac{1}{(2\pi)^2} \int_{R^2} \exp[i(t_1 x + t_2 y)] p(x, y) dx dy = \exp[-\sqrt{(t_1^2 + t_2^2)}],$$

and

$$p_n^*(0, 0) = \frac{4n}{(2\pi)^2} \int_0^\infty \int_0^\infty \frac{\exp[-2\sqrt{(t_1^2 + t_2^2)}]}{1 + (n-1)\exp[-2\sqrt{(t_1^2 + t_2^2)}]} dt_1 dt_2.$$

250 Using the polar coordinates $t_1 = \rho \cos \theta$, $t_2 = \rho \sin \theta$ we have

$$p_n^*(0, 0) = \frac{n}{4\pi(n-1)} \left[\frac{\pi^2}{24} + \frac{\ln^2(n-1)}{4} + o\left(\frac{1}{n}\right) \right].$$

Hence

$$I_n^* = \frac{1}{16\pi} \frac{\ln^2 n}{n} + \frac{\pi}{96} \frac{1}{n} + o\left(\frac{1}{n^2}\right).$$

Example 3. At last we consider an example where the characteristic function decreases more rapidly than a characteristic function with properties (a), (b). Denote the density of the bivariate normal distribution by $p(x, y)$, i.e.

$$p(x, y) = \frac{1}{2\pi\sqrt{(1-r^2)}} \exp\left\{-\frac{1}{2(1-r^2)}[x^2 - 2rxy + y^2]\right\}, \quad r^2 \neq 1.$$

Obviously

$$f(t_1, t_2) = \exp\left\{-\frac{1}{2}[t_1^2 + 2rt_1t_2 + t_2^2]\right\}.$$

Then

$$\begin{aligned} p_n^*(0, 0) &= \frac{4n}{(2\pi)^2} \int_0^\infty \int_0^\infty \frac{|f(t_1, t_2)|^2}{1 + (n-1)|f(t_1, t_2)|^2} dt_1 dt_2 = \\ &= \frac{n}{\pi^2} \int_0^{\pi/2} d\theta \int_0^\infty \frac{\rho \exp[-(\rho^2 + r\rho^2 \sin 2\theta)]}{1 + (n-1) \exp[-(\rho^2 + r\rho^2 \sin 2\theta)]} d\rho. \end{aligned}$$

Thus (see [5])

$$I_n^* = \frac{1}{2\pi^2\sqrt{(1-r^2)}} \left[\frac{\pi}{2} - \operatorname{arctg} \frac{r}{\sqrt{(1-r^2)}} \right] \frac{\ln n}{n} + o\left(\frac{1}{n}\right).$$

The example 3 shows that the order of consistency of the estimator is the best possible in this case.

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