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OPTIMUM DAMPING DESIGN FOR AN ABSTRACT WAVE EQUATION

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In this paper we address the question of "optimal" damping design in an abstract setting and precisely define and analyze various design criteria which are of importance in applications. We formulate two abstract optimization problems and discuss the necessary optimality conditions for the problems. We will further illustrate our results in application to a one-dimensional damped wave equation, and will present numerical results for different damping designs for this example.

1. INTRODUCTION

In recent years stabilization of flexible structures through active or passive feedback techniques has received much attention. In this regard, analysis of damping to achieve stabilization of these systems is highly important. In mathematical literature where PDE models of flexible structures are used, damping terms are introduced either in the equations (distributed damping), or in the boundary conditions (boundary damping). In formulation of these models, one is faced with the daunting task of proper mathematical definition of the damping operator in order to achieve the "appropriate" notion of stability for the motion of the system. For infinite time horizon Linear Quadratic Regulator problems applied to distributed parameter systems, uniform exponential stability or stabilizability of the system is essential. In [11] and [3] the authors have demonstrated the viability of feedback stabilization of the wave equation through dissipative boundary conditions. In some recent research effort (see [4], [5], [6]), the authors have considered a variable coefficient viscous damping term in the wave equation and have proposed a set of sufficient conditions on the damping term in order to achieve uniform exponential decay of energy.

Motivated by these efforts, our goal in this work is to go one step beyond and consider 'optimum" designs for the damping operator to not only achieve exponential stability but moreover obtain better and faster rates of decay for the energy of the system. This effort can be of special value in applications where the damping mechanism is not given or modeled a priori, and the issue of choosing the best design in order to obtain a desired specific response from the system is a pertinent one. From another point of view, the problem of optimum damping design is closely related to the design of active controls, where such design parameters such as location, mass, or number of these controls are to be decided in an optimal way.

Our goal in this study is to formulate an abstract and general framework for study of these different design problems applied to an abstract second order wave equation and consider important issues such as possible choices of the cost criteria for performing the optimization task, and well-posedness of the mathematical model and the optimization problem. In order to illustrate the theoretical issues, one could consider two specific, simple flexible structures such as the following one-dimensional wave equation with viscous damping on the interval $(-1,1)$,

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + a(x)\,\frac{\partial u(t,x)}{\partial t} = 0, \quad -1 < x < 1,\ t > 0,$$

$$\text{with } u(-1,t) = u(1,t) = 0,$$

$$\text{and initial conditions } u(0) = u_0, \quad \frac{\partial u}{\partial t}(0) = u_1,$$

and the one dimensional Euler–Bernoulli beam with damping

$$\frac{\partial^2}{\partial t^2}\,y(t,x) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2}\,y(t,x) + d(x)\,\frac{\partial^2}{\partial t \partial x^2}\,y(t,x)\right) = 0, \quad 0 < x < 1, \quad t > 0,$$

with boundary conditions at $x = 0, 1$, and initial conditions at $t = 0$.

In this work, we will concentrate on the one dimensional wave equation and will pursue the possible extension of the theory to the beam or the two-dimensional example of a plate in future work. In the ensuing sections, first we present the general framework of the abstract second order damped wave equation, and then discuss the possible design criteria for finding the optimum damping design. For the optimization problem we will present results regarding existence of a minimizer, necessary optimality conditions and the sensitivity equations for both the abstract formulation and the more specific example of the wave equation. In the last section, we will illustrate our theoretical results by numerical examples of different damping designs for the wave equation.

2. MATHEMATICAL MODEL

The equations of motion of many examples of flexible structures such as the one dimensional wave equation or the Euler–Bernoulli beam as presented in the previous section can be formulated as the following second order abstract wave equation

$$M\ddot{u} + D_a\dot{u} + A_0 u = 0 \tag{1}$$

in a Hilbert space H where A_0 is an elliptic operator and M and D_a represent the mass and damping operators, respectively. To cast the problem in the weak form we will follow the theoretical framework as outlined in [2], and assume a Hilbert space $V \subset H$ that is densely and continuously embedded in H. Define a bounded, symmetric sesquilinear form $\sigma_1(u,\hat{u}) : V \times V \to \mathcal{C}$ which is continuous and coercive on V. This sesquilinear form defines a densely defined operator, the stiffness operator A_0, in H where

$$\sigma_1(u,\hat{u}) = \langle A_0 u, \hat{u}\rangle_H$$

for $u \in \text{dom}\,(A_0)$, and $\hat{u} \in V$. Similarly, we can define the following bounded symmetric sesquilinear forms on H, $\rho(u,\hat{u}) = \langle Mu, \hat{u}\rangle_H$ and $\mu_a(u,\hat{u}) = \langle D_a u, \hat{u}\rangle_H$. In order to insure uniform exponential stability of the system, we take the operator D_a to consist of two parts:

$$D_a = D_0 + \hat{D}_a.$$

The first operator, D_0, can be either of the form γA_0, Kelvin-Voigt damping, or of the form $\gamma A_0^{1/2}$, the structural damping. Operator $\hat{D}_a$ is the damping operator that is to be designed.

In order to write equation (1) in the first order weak form, we define the following product spaces, $\mathcal{H} = V \times H$ with product norm $||.||_{\mathcal{H}}$ and $\mathcal{V} = V \times V$, and a sesquilinear $\sigma : \mathcal{V} \to \mathcal{C}$ in the following way:

$$\sigma((u,\hat{u}),(\phi,\hat{\phi})) = -\langle \hat{u},\phi\rangle_V + \sigma_1(u,\hat{\phi}) + \mu_a(\hat{u},\hat{\phi}).$$

Define $w = (u,\hat{u})$, $\chi = (\phi,\hat{\phi}) \in V$, and write equation (1) in the weak form as

$$\langle \dot{w}(t),\chi\rangle_{\mathcal{H}} + \sigma(w(t),\chi) = 0.$$

The above weak form gives rise to the following first order state equation in $\mathcal{H}$

$$\dot{w}(t) = \mathcal{A}_a w(t)$$

where

$$\mathcal{A}_a = \begin{bmatrix} 0 & I \\ -M^{-1}A_0 & -M^{-1}D_a \end{bmatrix}$$

with its domain defined as

$$\text{dom}\,(\mathcal{A}_a) = \{(\phi,\psi) \in \mathcal{H} : \psi \in V \text{ and } A_0\phi + D_a\psi \in H\}.$$

By Lumer–Phillips theorem, one can show that $\mathcal{A}_a$ generates a C_0 semigroup, $S_a(t)$, in the state space $\mathcal{H} = V \times H$ if D is a bounded self-adjoint, and non-negative operator on V. If in addition, D_a satisfies the following condition, (H semi-coercivity)

$$\langle D_a\phi,\phi\rangle_H \geq b|\phi|_H^2 \text{ for some } b > 0,$$

then one can show that $S_a(t)$ is also uniformly exponentially stable, i.e.,

$$||S_a(t)||_{\mathcal{H}} \leq Me^{-\omega t} \text{ for some } M \geq 1, \omega > 0,\ \forall t \geq 0.$$

It can be shown that the two possible choices for D_0 do satisfy the semi-coercivity condition, which means in the presence of additional internal damping, $\hat{D}_a$, exponential stability of the system is guaranteed. If $D_0 = 0$, then the design goal is to model D_a which is dependent on the design parameter(s) $a \in (\mathcal{Q}_{ad} =$ the Design Space) in such a way so that the norm of the semigroup solution of the equation above decays to zero in a desired manner.

3. DESIGN CRITERIA

To formulate a performance index that is based on the dynamical behavior of the solutions one can consider the following three possibilities.

The first one is based on minimizing $\|S_a(\tau)\|_{\mathcal{H}}$, given $\tau > 0$. While this criterion is useful in many applications where the performance measure is based on a decay factor for a desired time interval, mathematical characterization of this problem does not yield an easily implementable performance index.

A second frequently used criterion in the engineering literature is maximization of decay rates of solutions, i. e.,

$$\min_{a \in \mathcal{Q}_{ad}} \quad \sup \operatorname{Re} \sigma(\mathcal{A}_a)$$

where $\sigma(\mathcal{A}_a)$ is the spectrum of the operator $\mathcal{A}_a$. While this criterion is widely used in the finite dimensional models, its use for the infinite dimensional wave equation presents us with several problems: The first problem is related to characterization of $\sigma(\mathcal{A}_a)$, which is difficult to do in many cases, specially in higher dimensional problems. But even in cases where $\sigma(\mathcal{A}_a)$ is easily defined, we still need to have the *spectrum determined growth condition* satisfied, (see [13]):

$$\inf \omega = \{\|S_a(t)\| \le Me^{\omega t} \quad \omega \in R\} = \sup \operatorname{Re} \sigma(\mathcal{A}_a).$$

It has been shown (see [2]) that if the damping operator μ_a is uniformly coercive then $S_a(t)$ is an analytic semigroup and $\sigma(\mathcal{A}_a)$ is sectorial and the spectrum determined growth condition is satisfied. But in general the vertical asymptote of $\sigma(\mathcal{A}_a)$ is difficult to examine. Even in cases where the first two problems are circumvented, maximization of the slowest decay rate which the criterion amounts to does not result in overall reduction of the energy in a finite amount of time.

The third criterion which is based on minimizing the total energy of the system over a long time interval is more easily characterized and realized in actual physical systems than the other two criteria. This criterion for our problem can be defined as

$$\min_{a \in \mathcal{Q}_{ad}} \int_0^\infty \|\mathcal{R}^{1/2} S_a(t) u\|_{\mathcal{H}}^2 \, dt,$$

where $\mathcal{R}$ is a coercive, self-adjoint operator on $\mathcal{H}$. Minimization of the total energy is realized by the characterization of the Datko Lemma [7], [12] which basically states that if $\mathcal{A}_a$ is exponentially stable on $\mathcal{H}$ then the minimum of the total energy is given in terms of the solution to a Lyapunov equation. In other words the following are equivalent:

- $\mathcal{A}_a$ is exponentially stable on $\mathcal{H}$.
- $\int_0^\infty \|S_a(t)u\|^2 \, dt$ is finite for all $u \in \mathcal{H}$.
- There exists a bounded nonnegative, and self-adjoint operator $\Pi_a \in \mathcal{L}(\text{dom } (\mathcal{A}_a), \text{dom } (\mathcal{A}_a^*))$ such that it satisfies the following Lyapunov equation

$$(\mathcal{A}_a^* \Pi_a + \Pi_a \mathcal{A}_a + \mathcal{R})\, u = 0 \tag{2}$$

for all $u \in \text{dom } (A_a)$.

Then we have

$$\int_0^\infty \|\mathcal{R}^{1/2} S_a(t) u\|_{\mathcal{H}}^2 \, dt = \langle \Pi_a u, u \rangle_{\mathcal{H}}.$$

In order to develop a criterion that is independent of the state vector u, we consider the following performance measures that are based on minimizing the total energy

$$\min_{a \in \mathcal{Q}_{ad}} \|\Pi_a\| = \sup_{|u|=1} \langle \Pi_a u, u \rangle_{\mathcal{H}}, \tag{3}$$

and

$$\min_{a \in \mathcal{Q}_{ad}} \operatorname{tr} \Pi_a Q = E(\Pi_a u, u). \tag{4}$$

For criterion (4), we assume that the initial data u is a random vector with normal distribution of zero mean and covariance Q, a nuclear operator, and E denotes the expectation over the initial condition. In general Π_a is not compact in $\mathcal{H}$, therefore it is not always possible to define the trace norm of Π_a. In this sense, the second criterion is the weighted trace norm of Π_a with respect to Q, which in engineering applications is chosen to be the subspace spanned by the dominant eigenfunctions for the nominal plant. If Π_a is compact, then the first criterion amounts to minimizing the L^∞ norm of Π_a, and the second criterion is equivalent to minimizing L^1 norm of Π_a.

Our goal is to solve the optimization problems based on (3) and (4) subject to some constraints on the parameter a. In the following section we consider a specific example and will present the optimization problems in the context of the example.

4. A ONE DIMENSIONAL DAMPED WARE EQUATION

In this section we consider the following one dimensional wave equation on the interval $(-1, 1)$

$$u_{tt} = u_{xx} - a(x) u_t \quad \text{with} \quad u(-1,t) = u(1,t) = 0. \tag{5}$$

The underlying Hilbert space for the abstract formulation is $\mathcal{H} = H_0^1(-1,1) \times L^2(-1,1)$, with the inner-product

$$\left\langle \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle_{\mathcal{H}} = \int_{-1}^{1} (w_1' \bar{v_1}' + w_2 \bar{v}_2) \, dx.$$

The first order form of equation (5) is given by

$$\dot{y} = \mathcal{A}_a y, \quad \text{where}$$

$$\mathcal{A}_a = \begin{bmatrix} 0 & I \\ \partial_x^2 & -a(x) \end{bmatrix}, \qquad y = \begin{bmatrix} u \\ u_t \end{bmatrix}.$$

For this problem we are interested in finding the optimal spatial distribution of damping subject to some constraints on the distribution as well as on the total amount of damping material available. One can consider the following two possible formulations where in both cases $f(a)$ denotes the desired performance index for optimal design. The non-parametric formulation can be stated as

$$\text{minimize } f(a) \text{ over } a(x) \in BV(-1,1), \tag{6}$$

where $BV(-1,1)$ denotes the space of functions of bounded variations on $(-1,1)$, and $a(x)$ satisfies the additional constraints

$$\left\{ a(x) \geq 0, \int_{-1}^{1} a(x)\, \mathrm{d}x = a_{tot} = \text{ Total Mass}, |a|_{BV} \leq \gamma \right\}. \tag{7}$$

For the parametrized optimization problem, we consider the following finite dimensional parameterization. If one models $a(x)$ as a piecewise constant function over ns number of subintervals

$$a(x) = \sum_{i=1}^{ns} a_i \chi_{[x_i - x_{i-1}]}(x)$$

$$-1 = x_0 < x_1 < \ldots < x_{ns} = 1$$

where a_i represents the amount of damping distribution over the ith subinterval (x_{i-1}, x_i), then the goal is to find the optimal values for $a_i \geq 0$ subject to the constraint

$$\int_{-1}^{1} a(x)\, \mathrm{d}x = a_{tot} = \text{ Total Mass.}$$

In both of these formulations, we take the expressions used in Section 3 for $f(a)$:

$$f(a) = \|\Pi_a\| = \sup_{|u|=1} \langle \Pi_a u, u \rangle_{\mathcal{H}}, \text{ or}$$

$$f(a) = \text{tr } \Pi_a Q = E(\Pi_a u, u).$$

In this paper we concentrate on the parametric formulation, and will present results regarding the non-parametric formulation which involves nonsmooth optimization techniques in a forthcoming paper, [9].

5. NECESSARY OPTIMALITY CONDITIONS

In order to show that the optimization problems discussed in the previous section are well-posed, we need to show for each criterion the existence of an optimal parameter and discuss the necessary optimality conditions that characterize the optimal solutions.

We consider the following general constrained optimization problem

$$\text{Minimize } f(a) \quad \text{over} \quad a \in \mathcal{Q}_{ad}, \tag{8}$$

where $\mathcal{Q}_{ad}$, the admissible design space, is assumed to be a weakly sequentially compact set in X, a normed space for the design parameters with norm $\|\cdot\|_X$, and f is a convex functional. To prove existence of a minimizer, we will refer to the following well-known result, (see [8]):

Theorem 5.1. Let $f : X \to R^1$ be a weakly sequentially lower semicontinuous functional on M, a weakly compact subset of the normed linear space X, i.e., for every $u_0 \in M$, and for any sequence $\{u_n\}_{n=1}^{\infty}$ in M such that u_n converges weakly to u_0

$$f(u_0) \leq \liminf_{n\to\infty} f(u_n),$$

then $\exists\, u^* \in M$ such that

$$f(u^*) = \min_{u\in M} f(u).$$

5.1. The first criterion

For criterion (3)

$$\min_{a\in \mathcal{Q}_{ad}} f(a) = \|\Pi_a\| = \sup_{|u|=1} \langle \Pi_a u, u\rangle_{\mathcal{H}}, \text{ over } a \in \mathcal{Q}_{ad},$$

we need to establish weakly lower semicontinuity of $f(a)$ which by the following theorem is contingent upon pointwise continuity of Π_a in a sense that will be made precise below.

Theorem 5.2. The functional $f(a) = \|\Pi_a\|$ is weakly sequentially lower semicontinuous if for any sequence a_n converging weakly to $\hat{a} \in \mathcal{Q}_{ad}$ in X, the following convergence condition holds:

$$\langle \Pi_{a_n} u, u\rangle \to \langle \Pi_{\hat{a}} u, u\rangle \quad \text{for each} \quad u \in \mathcal{H}.$$

P r o o f. From definition of the operator norm, for any given $\epsilon > 0$ we have

$$\|\Pi_{a_n}\| - \epsilon < \langle \Pi_{a_n} x_\epsilon, x_\epsilon\rangle, \qquad \text{for } x_\epsilon \in \mathcal{H} \text{ with } \|x_\epsilon\|_{\mathcal{H}} = 1.$$

Similarly, we have

$$\|\Pi_{\hat{a}}\| - \epsilon < \langle \Pi_{\hat{a}} x_\epsilon, x_\epsilon\rangle, \qquad \text{for } x_\epsilon \in \mathcal{H} \text{ with } \|x_\epsilon\|_{\mathcal{H}} = 1.$$

Note that in these inequalities x_ϵ depends on a_n in general. From the above we obtain the following

$$\begin{aligned} \|\Pi_{\hat{a}}\| &< \epsilon + \langle \Pi_{\hat{a}} x_\epsilon, x_\epsilon\rangle - \langle \Pi_{a_n} x_\epsilon, x_\epsilon\rangle + \langle \Pi_{a_n} x_\epsilon, x_\epsilon\rangle \\ &< \epsilon + \langle (\Pi_{\hat{a}} - \Pi_{a_n}) x_\epsilon, x_\epsilon\rangle + \|\Pi_{a_n}\|. \end{aligned}$$

The second term on the right hand side of the inequality goes to zero as a_n converges weakly to $\hat{a}$ in $\mathcal{Q}_{ad}$ by the assumption of the theorem. Therefore, we have

$$\|\Pi_{\hat{a}}\| < \epsilon + \liminf \|\Pi_{a_n}\|$$

which for arbitrary ϵ gives us the weakly lower semicontinuity of $\|\Pi_a\|$. □

The pointwise convergence of operator Π_a with respect to a can be verified for a general class of problems, and the results will be presented in [9]. But here, we can prove Lipschitz continuity of Π_a with respect to parameter a:

Theorem 5.3. Suppose dom $(\mathcal{A}_a)$ and dom $(\mathcal{A}_a^*)$ are independent of $a \in \mathcal{Q}_{ad}$, a compact set, and $\delta\mathcal{A}_a = \mathcal{A}_a - \mathcal{A}_{\hat{a}}$ satisfies

$$||(\mathcal{A}_a - \mathcal{A}_{\hat{a}})\,x|| \le K||a-\hat{a}||_X\,||x||_{\mathcal{H}}, \text{ for all } x \in \text{dom}\,(\mathcal{A}_a), \text{ and } a, \hat{a} \in \mathcal{Q}_{ad}. \tag{9}$$

Then we have

$$||\Pi_a - \Pi_{\hat{a}}||_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le M||a-\hat{a}||_X \tag{10}$$

where the constant $M > 0$, and Π_a is the solution of the Lyapunov equation

$$\mathcal{A}_a^*\Pi_a + \Pi_a\mathcal{A}_a + \mathcal{R} = 0.$$

Proof. Consider Π_a and $\Pi_{\hat{a}}$ which are solutions of the following Lyapunov equations:

$$(\mathcal{A}_a^*\Pi_a + \Pi_a\mathcal{A}_a + \mathcal{R})\;x = 0, \text{ for all } x \in \text{dom}\,(\mathcal{A}_a),$$

$$(\mathcal{A}_{\hat{a}}^*\Pi_{\hat{a}} + \Pi_{\hat{a}}\mathcal{A}_{\hat{a}} + \mathcal{R})\;x = 0, \text{ for all } x \in \text{dom}\,(\mathcal{A}_{\hat{a}}).$$

By subtracting one equation from the other and adding and subtracting $\mathcal{A}_{\hat{a}}^*\Pi_a$ and $\Pi_a\mathcal{A}_{\hat{a}}$, we obtain the following equation:

$$[\mathcal{A}_{\hat{a}}^*(\Pi_a - \Pi_{\hat{a}}) + (\Pi_a - \Pi_{\hat{a}})\,\mathcal{A}_{\hat{a}} + (\mathcal{A}_a^* - \mathcal{A}_{\hat{a}}^*)\,\Pi_a + \Pi_a(\mathcal{A}_a - \mathcal{A}_{\hat{a}})]\,x = 0 \tag{11}$$

for all $x \in$ dom $(\mathcal{A}_a)$ = dom $(\mathcal{A}_{\hat{a}})$. The equation above is a well-defined Lyapunov equation since $\mathcal{A}_{\hat{a}}$ is exponentially stable. Therefore, we have the following integral representation for the operator $\Pi_a - \Pi_{\hat{a}}$:

$$(\Pi_a - \Pi_{\hat{a}})\,x = \int_0^\infty S_{\hat{a}}^*(t)\,[(\mathcal{A}_a^* - \mathcal{A}_{\hat{a}}^*)\,\Pi_a + \Pi_a(\mathcal{A}_a - \mathcal{A}_{\hat{a}})]\,S_{\hat{a}}(t)\,x\,\mathrm{d}t$$

where $S_{\hat{a}}(t)$ is the C_0 semigroup generated by $\mathcal{A}_{\hat{a}}$ and satisfies the following exponential stability condition:

$$||S_{\hat{a}}(t)|| \le M_1 e^{-\omega_1 t} \quad \text{for some } M_1 \ge 1, \quad \omega_1 \ge 0, \ \forall\, t \ge 0.$$

Now, we have

$$\langle(\Pi_a - \Pi_{\hat{a}})\,x, x\rangle \le \frac{M_1^2}{2\omega_1}||(\mathcal{A}_a^* - \mathcal{A}_{\hat{a}}^*)\,\Pi_a + \Pi_a(\mathcal{A}_a - \mathcal{A}_{\hat{a}})||\,||x||^2. \tag{12}$$

From boundedness of $||\Pi_a||$, and condition (9) we conclude that

$$||\Pi_a - \Pi_{\hat{a}}|| \le C||(\mathcal{A}_a^* - \mathcal{A}_{\hat{a}}^*)\,\Pi_a + \Pi_a(\mathcal{A}_a - \mathcal{A}_{\hat{a}})|| \le M||a-\hat{a}||_X. \tag{13}$$

□

We can immediately obtain the following continuity result.

Corollary 5.4. Suppose dom $(\mathcal{A}_a)$ and dom $(\mathcal{A}_a^*)$ are independent of $a \in \mathcal{Q}_{ad}$, a compact set, and $\delta\mathcal{A}_a = \mathcal{A}_a - \mathcal{A}_{\hat{a}}$ satisfies

$$||\delta\mathcal{A}_a||_{\mathcal{L}(\mathcal{H},\mathcal{H})} \to 0 \qquad \text{as } a \to \hat{a} \text{ strongly in } \mathcal{Q}_{ad}, \tag{14}$$

then

$$||\Pi_a - \Pi_{\hat{a}}||_{\mathcal{L}(\mathcal{H},\mathcal{H})} \to 0 \quad \text{as } a \to \hat{a} \text{ strongly in } \mathcal{Q}_{ad}.$$

For the specific example of the one dimensional wave equation we have:

Example 5.1. For the wave equation (5), the operator Π_a is Lipschitz continuous with respect to parameter a in the following sense:

$$||\Pi_a - \Pi_{\hat{a}}|| \leq C|a - \hat{a}|_\infty \tag{15}$$

where

$$(\mathcal{A}_a - \mathcal{A}_{\hat{a}}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ (a - \hat{a})v \end{bmatrix}. \tag{16}$$

Note: In this example, $X = L^\infty(-1, 1)$.

The following theorem characterizes the equation that the Fréchet derivative, Σ, of the mapping $a \in X \to \Pi_a \in \mathcal{L}(\mathcal{H})$ of Π_a satisfies:

Theorem 5.5. The sensitivity operator Σ satisfies

$$\mathcal{A}_{\hat{a}}^*\Sigma(h) + \Sigma(h)\mathcal{A}_{\hat{a}} + \delta\mathcal{A}_a^*\Pi_{\hat{a}} + \Pi_{\hat{a}}\delta\mathcal{A}_a = 0, \tag{17}$$

where $\delta\mathcal{A}_a = \mathcal{A}_a - \mathcal{A}_{\hat{a}}$ satisfies

$$||\delta\mathcal{A}_a||_{\mathcal{L}(\mathcal{H},\mathcal{H})} \to 0 \text{ as } a \to \hat{a} \text{ strongly in } \mathcal{Q}_{ad}, \tag{18}$$

and $h(x) = a(x) - \hat{a}(x) \in \mathcal{Q}_{ad}$.

Proof. We need to show that Σ is indeed the Fréchet derivative of Π_a, i.e.,

$$\frac{||\Pi_a - \Pi_{\hat{a}} - \Sigma(h)||_{\mathcal{L}(\mathcal{H})}}{||h||_X} \to 0 \text{ as } ||h||_X \to 0 \text{ in } \mathcal{Q}_{ad}.$$

By subtracting equation (17) from (11), we obtain:

$$\mathcal{A}_{\hat{a}}^*\Delta + \Delta\mathcal{A}_{\hat{a}} + \delta\mathcal{A}_a^*(\Pi_a - \Pi_{\hat{a}}) + (\Pi_a - \Pi_{\hat{a}})\delta\mathcal{A}_a = 0,$$

where $\Delta = \Pi_a - \Pi_{\hat{a}} - \Sigma(h)$. By arguments similar to the ones in Theorem 5.3, we have the following bound on $||\Delta||$:

$$||\Delta|| \leq C||\,\delta\mathcal{A}_a||\,||\Pi_a - \Pi_{\hat{a}}||.$$

From conditions on operators $\mathcal{A}_a$ and Π_a, the right hand side of the inequality goes to zero as a converges strongly to $\hat{a}$ in $\mathcal{Q}_{ad}$. □

Again we can immediately obtain the following result for the specific case of the wave equation:

Example 5.2. For the wave equation, (5), the sensitivity operator Σ satisfies

$$\mathcal{A}_{\hat{a}}^{*}\Sigma(h)+\Sigma(h)\mathcal{A}_{\hat{a}}+\delta\mathcal{A}_{a}^{*}\Pi_{\hat{a}}+\Pi_{\hat{a}}\delta\mathcal{A}_{a}=0. \tag{19}$$

where $\delta\mathcal{A}_a=\mathcal{A}_a-\mathcal{A}_{\hat{a}}$ satisfies

$$(\mathcal{A}_a-\mathcal{A}_{\hat{a}})\begin{bmatrix} u \\ v \end{bmatrix}=\begin{bmatrix} 0 \\ hv \end{bmatrix}, \quad \text{and} \quad h(x)=a(x)-\hat{a}(x)\in L^{\infty}. \tag{20}$$

Proof. The arguments are basically the same as the ones in Theorem 5.5, and we have the following estimate on the norm of Δ

$$||\Delta|| \le \quad C||\,\delta\mathcal{A}_a||\,||\Pi_a-\Pi_{\hat{a}}|| \le \hat{M}|a-\hat{a}|_{\infty}^{2}=\hat{M}|h|_{\infty}^{2}.$$ □

5.2. The second criterion

For the second criterion

$$\min_{a\in\mathcal{Q}_a} f(a)=\min_{a\in\mathcal{Q}_{ad}} \operatorname{tr}\Pi_a Q$$

we take Q to be a nuclear operator such that it is a compact self-adjoint operator on the Hilbert space $\mathcal{H}$, whose eigenvalues are summable. In fact, in most practical applications Q can be taken to be a self-adjoint finite rank operator. Having defined Q more specifically, we can now proceed to show that $f(a)=\operatorname{tr}\Pi_a Q$ is indeed continuous with respect to the parameter a. Therefore, by compactness of $\mathcal{Q}_{ad}$, we can show the existence of a minimizer for the criterion.

Theorem 5.6. For the criterion $f(a)=\operatorname{tr}\Pi_a Q$, where Q is a nuclear operator, if for any weakly convergent sequence a_n converging to $\hat{a}\in\mathcal{Q}_{ad}$, the operator Π_a satisfies

$$\langle\Pi_{a_n}u,u\rangle\to\langle\Pi_{\hat{a}}u,u\rangle \quad \text{for each} \quad u\in\mathcal{H} \tag{21}$$

then we have the following

$$f(a_n)\to f(\hat{a}) \quad \text{as} \quad a_n\to\hat{a} \quad \text{weakly in} \quad \mathcal{Q}_{ad}.$$

Proof. From definition of Q and the trace class to which the bounded linear operator $\Pi_a Q$ belongs, for ϕ_i, a complete orthonormal set of eigenfunctions of operator Q, we have

$$f(a_n)=\operatorname{tr}\Pi_{a_n}Q=\sum_i\langle Q\phi_i,\Pi_{a_n}\phi_i\rangle=\sum_i\langle\lambda_i\phi_i,\Pi_{a_n}\phi_i\rangle$$

where $\lambda_i Q=Q\phi_i$. Now by taking the limit as $a_n\to\hat{a}$, from (21) we have

$$f(a_n)=\sum_i\langle\lambda_i\phi_i,\Pi_{a_n}\phi_i\rangle\longrightarrow\sum_i\langle\lambda_i\phi_i,\Pi_{\hat{a}}\phi_i\rangle=f(\hat{a}).$$ □

Now we can prove the following result for the differentiability of the cost function $f(a)=\operatorname{tr}\Pi_a Q$.

Theorem 5.7. Suppose there exists $\delta\mathcal{A}_a \in \mathcal{L}(\mathcal{H},\mathcal{H})$ such that

$$\frac{||\mathcal{A}_{a+h} - \mathcal{A}_a - \delta\mathcal{A}_a||_{\mathcal{L}(\mathcal{H},\mathcal{H})}}{||h||_X} \to 0 \quad \text{as} \quad ||h||_X \to 0, \quad \text{in } \mathcal{Q}_{ad}$$

and dom $(\mathcal{A}_a)$ and dom $(\mathcal{A}_a{}^*)$ are independent of $a \in \mathcal{Q}_{ad}$, a compact set . Then

$$f'(a)\,(h) = \text{tr } (((\delta\mathcal{A}_a)^*\Pi_a + \Pi_a(\delta\mathcal{A}_a))\Lambda)\,, \tag{22}$$

where $\delta\mathcal{A}_a = \mathcal{A}'_a h$, and the Lagrange multiplier Λ to the constraint

$$\mathcal{A}^*_a\Pi_a + \Pi_a\mathcal{A}_a + \mathcal{R} = 0$$

satisfies the adjoint equation

$$\mathcal{A}_a\Lambda + \Lambda\mathcal{A}^*_a + Q = 0. \tag{23}$$

Proof. By differentiability of Π_a from Theorem 5.5, we have

$$f'(a)\,(h) = \text{tr } \Sigma Q,$$

where Σ satisfies:

$$(\mathcal{A}^*_{\hat{a}}\Sigma(h) + \Sigma(h)\,\mathcal{A}_{\hat{a}} + \delta\mathcal{A}^*_a\Pi_{\hat{a}} + \Pi_{\hat{a}}\delta\mathcal{A}_a)\,x = 0, \quad \text{for all} \quad x \in \text{dom }(\mathcal{A}_a).$$

From definition of operators of trace class, for ϕ_i, an arbitrary complete orthonormal system of $\mathcal{H}$, we can write

$$f'(a)\,(h) = \text{tr } \Sigma Q = \sum_i \langle \Sigma\phi_i, Q\phi_i\rangle.$$

From (23), we have

$$\begin{aligned}
\sum_i \langle \Sigma\phi_i, Q\phi_i\rangle &= -\sum_i \langle \Sigma\phi_i, \mathcal{A}_a\Lambda\phi_i + \Lambda\mathcal{A}^*_a\phi_i\rangle \\
&= -\sum_i \langle \mathcal{A}^*_a\Sigma\phi_i + \Sigma\mathcal{A}_a\phi_i, \Lambda\phi_i\rangle \\
&= \sum_i \langle ((\delta\mathcal{A}^*_a)\Pi_a + \Pi_a(\delta\mathcal{A}_a))\phi_i, \Lambda\phi_i\rangle \\
&= \text{tr } (((\delta\mathcal{A}_a)^*\Pi_a + \Pi_a(\delta\mathcal{A}_a))\Lambda)\,.
\end{aligned}$$

Note: For the wave equation (5), all the above results are valid by taking into account that convergence of the parameters in $\mathcal{Q}_{ad}$ is in the sense of the L^∞ norm.

6. FINITE DIMENSIONAL APPROXIMATIONS

In this section, we carry out numerical approximations for solving the parametric optimization problem as suggested in Section (4) for the following one dimensional wave equation:

$$u_{tt} = u_{xx} - a(x)\, u_t \quad \text{with} \quad u(-1,t) = u(1,t) = 0,$$

where the damping term $a(x)$ is modeled as a piecewise constant function over ns number of subintervals

$$a(x) = \sum_{i=1}^{ns} a_i \chi_{[x_i - x_{i-1}]}(x)$$

$$-1 = x_0 < x_1 < \ldots < x_{ns} = 1$$

with a_i representing the amount of damping distribution over the ith subinterval.

For performing the numerical approximations, we employ the Legendre–Tau method which is a variation of the well-known Galerkin technique. In this method the approximate solution is expanded in terms of the Legendre polynomials, $L_n(x)$, which are orthogonal with respect to the $L^2(-1,1)$ norm. These basis elements do not individually satisfy the boundary conditions as in Galerkin method. The boundary conditions are imposed on the approximate solution by use of a non-orthogonal projection operator. For more details on implementations of the Legendre-tau method to the wave equation, see [1], [10].

For a second order wave equation we seek an approximate solution in the form

$$u_n(t,x) = \sum_{j=0}^{n} \xi_j(t)\, L_j(x).$$

The vector $\xi(t) = (\xi_0, \xi_1, \ldots, \xi_{n-2})$ satisfies

$$M^n \ddot{\xi}(t) + D^n \dot{\xi}(t) + K^n \xi(t) = 0 \tag{24}$$

and ξ_{n-1} and $,\xi_n$ are determined as linear combinations of $\xi_0, \xi_1, \ldots, \xi_{n-2}$ by applying the boundary conditions on the solution u_n. The mass matrix M^n, damping matrix D^n, and the stiffness matrix K^n are given by

$$\begin{aligned}
(M^n)_{i,j} &= \langle L_i, L_j \rangle_{L_2(-1,1)} = \frac{2}{2i+1}\delta_{ij},\\
(D^n)_{i,j} &= \sum_{k=1}^{ns} \int_{x_{k-1}}^{x_k} a_k L_i L_j \, dx,\\
(K^n)_{i,j} &= ((H^n)^T M^n H^n)_{i,j}.
\end{aligned}$$

In the expression for K^n, H^n is the matrix representation of the first order differential operator with respect to the Legendre polynomials which also imposes the Dirichlet boundary conditions at the two ends on the approximate solution.

The first order form of (24) for $\eta = [\xi, \dot{\xi}]^T$ is

$$\dot{\eta} = \mathcal{A}^n \eta$$

where

$$\mathcal{A}^n = \begin{bmatrix} 0_{n-1\times n-1} & I_{n-1\times n-1} \\ -M^{-n}K^n & -M^{-n}D^n \end{bmatrix}.$$

In the above, M^{-n} denotes the inverse of the mass matrix M^n. For approximating the total energy we take $\mathcal{R}$ in (2) to be the identity, and we write its matrix representation as

$$\mathcal{R}^n = \begin{bmatrix} K^n & 0_{n-1\times n-1} \\ 0_{n-1\times n-1} & M^n \end{bmatrix}.$$

Assuming $(\mathcal{A}^n, \mathcal{R}^n)$ is detectable, then the total energy in the finite dimensional space is given by

$$E^n(u) = \int_0^\infty \eta^T \mathcal{R}^n \eta \, \mathrm{d}t = \eta_0^T \Pi^n \eta_0 \qquad \eta(0) = \eta_0$$

where Π^n is the matrix representation of the finite-dimensional approximation to Π and is equal to $\mathcal{R}^{-n}\tilde{\Pi}^n$ where $\tilde{\Pi}^n$ satisfies the following Lyapunov equation

$$(\mathcal{A}^n)^T \tilde{\Pi}^n + \tilde{\Pi}^n \mathcal{A}^n + \mathcal{R}^n = 0.$$

The finite dimensional approximation of the first performance index (3) can be written as

$$\min_{a\in\mathcal{Q}_{ad}} \max \operatorname{eig} (\mathcal{R}^{-n}\tilde{\Pi}^n). \tag{25}$$

To calculate the approximate performance index (4), we consider operator Q to be the projection onto a space spanned by the m dominant undamped eigenfunctions of the equation. If Φ_{mn} denotes the matrix representation of the orthogonal projection that projects the finite-dimensional solution space to the m-dimensional space of range of Q, then the matrix representation of the finite-dimensional performance index becomes:

$$\min_{a\in\mathcal{Q}_{ad}} \operatorname{tr} \left((\Phi_{mn}\mathcal{R}^n\Phi_{mn}^T)^{-1} (\Phi_{mn}\tilde{\Pi}^n\Phi_{mn}^T) \right). \tag{26}$$

7. NUMERICAL RESULTS

To perform numerical experiments for various damping designs, we took the number of Legendre polynomials in our approximations to be 20, and the number of subdivisions for distribution of the damping material to be 40. Also, to calculate the second performance criterion, we took m, the number of undamped dominant modes to be 7. We first experimented with a few damping designs and calculated the value of performance indices (25) and (26) in each case. The following figures demonstrate the distribution of a_i's over the $(-1, 1)$ interval for these examples:

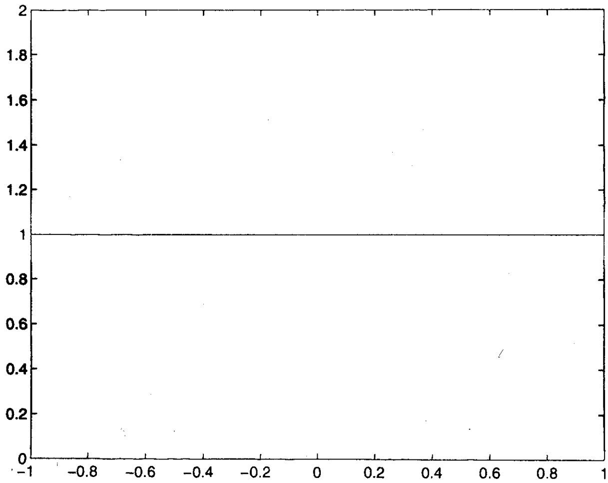

Fig. 1. Uniform distribution.

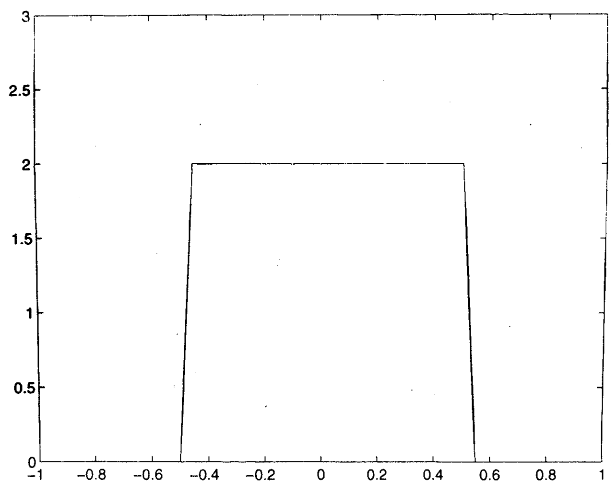

Fig. 2. Center distribution.

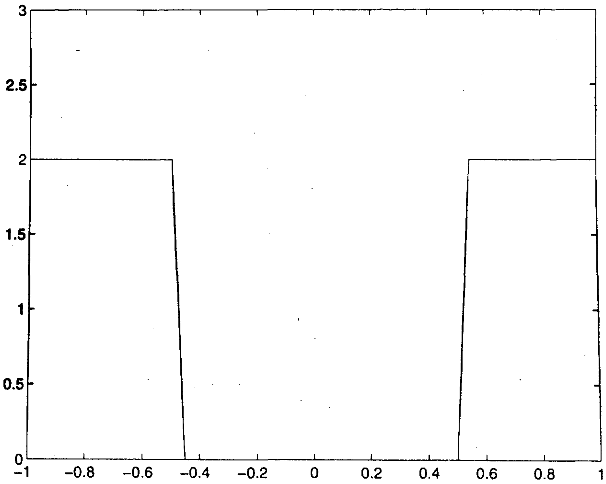

Fig. 3. Corner distribution.

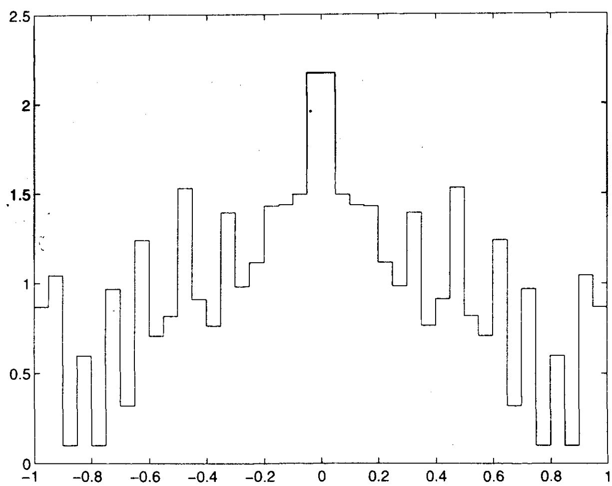

Fig. 4. First optimal distribution with min max eig $(\Pi^n) = 1.2959$.

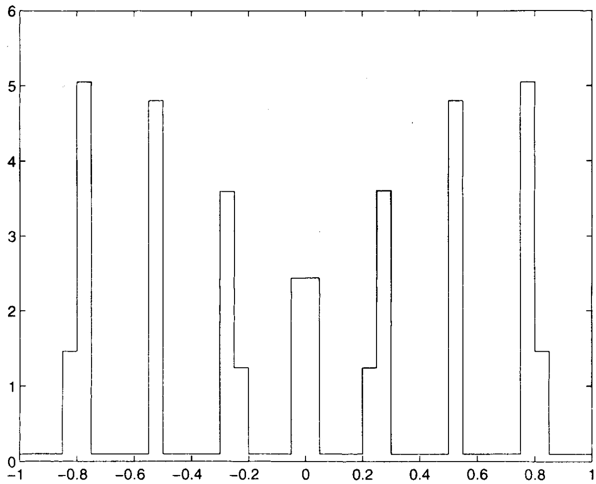

Fig. 5. Second optimal distribution with min tr $= 12.3611$.

The following table compares the different designs and the corresponding values of the two performance indices.

Table 1. Comparison of different designs.

Damping designs	Min(max(eig))	Min (tr)
Uniform Distribution	1.4354	14.3064
Center Distribution	499.1038	15.3697
Corner Distribution	3.2690	18.6927
Optimal Distribution 1	1.2959	14.6744
Optimal Distribution 2	9.3611	12.6311

From this table, one can see that different performance criteria yield different optimal damping designs, and a design that performs well with respect to one criterion, may perform poorly with respect to the others, (compare the results for the center and corner distributions). But overall, the uniform damping design seems to perform quite well with respect to either criterion. The results also indicate that much is to be gained by performing the optimization. From these results one can observe that the key point in optimizing damping designs is to carefully choose the performance criterion that is most suited to the problem in hand. Practical and theoretical considerations should both be taken into account in choosing the proper criterion. For example, depending on the amount of information on the physical modeling of the initial state vector or the number of dominant vibrational modes that need to be suppressed one may choose the criterion that fits the requirements of the problem.

One last important observation in these numerical experiments is the issue of convergence of the optimal design with respect to the number of mesh points. In order to investigate dependence of these designs on ns the number of damping subintervals, we carried out the optimization for both criterion for $ns = 80$.

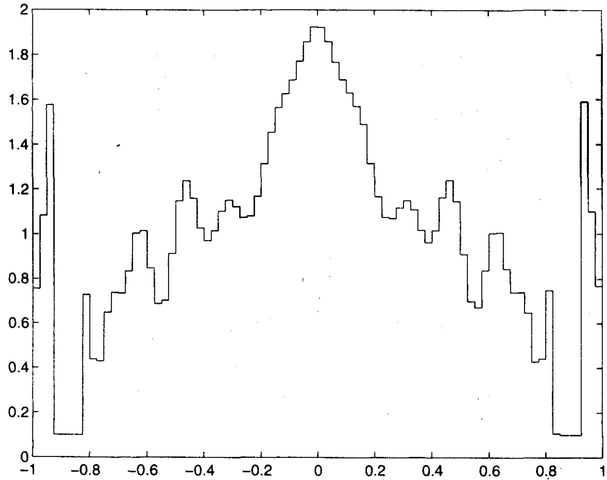

Fig. 6. First optimal distribution with min max eig $(\Pi^n) = 1.2922$, $ns = 80$.

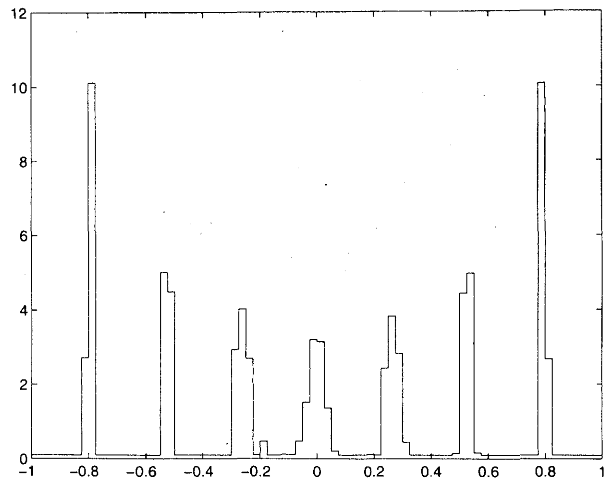

Fig. 7. Second optimal distribution with min tr $= 12.5661$, $ns = 80$.

Comparing these graphs to the ones obtained for $ns = 40$, we see that the damping designs do not converge for increasing number of damping subintervals. Therefore, we need to pursue another formulation for the optimization where this convergence can be obtained. Consideration of BV-regularization or the nonparametric formulation as stated in Section 4 is one avenue for resolution of this problem. The numerical results we have presented here are only preliminary efforts in optimizing damping designs and our future efforts will address numerous issues concerning the numerical and theoretical optimization of these designs and their extensions to the beam and plate equations.

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