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Simultaneous Channels Decomposable into Memoryless Components I

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The first part consists of 3 sections. Basic notations are introduced in the Section 1. Properties of probability vectors useful for approximating of the entropy function and the channel probability functions are studied in the Section 2. The Section 3 contains estimates of the maximum length of n-dimensional ε-codes for simultaneous channel decomposable into a finite number of memoryless components.

The simultaneous channel decomposable into memoryless components is defined in the second part of this paper. In that part bounds for the maximum length $S_n(\varepsilon, C)$ of n-dimensional ε-codes for the simultaneous channel C are derived and the paper is closed with theorems on the asymptotic behaviour of $S_n(\varepsilon, C)$.

1. INTRODUCTION

At first we introduce several concepts and notations which are in accordance with [1], [2], [3] and [5]. We shall assume that we are given an input alphabet $B = \{1, \ldots, b\}$ and an output alphabet $A = \{1, \ldots, a\}$ such that $\min\{a, b\} \geqq 2$. We shall use the notation $d = \max\{a, b\}$. Any vector $p = (p_1, \ldots, p_r)$ which has non-negative coordinates and satisfies $\sum p_j = 1$ will be called a probability vector; the set of all probability vectors belonging to R^b will be denoted by P. Any matrix $w = (w(j \mid i);\ i \in B,\ j \in A)$ such that $\{(w(1 \mid i), \ldots, w(a \mid i));\ i \in B\}$ are probability vectors will be called a probability matrix. The set of all probability matrices will be denoted by W. Let sequences y, x belong to B^n, A^n respectively. If we denote for each matrix $w \in W$

$$w(x \mid y) = \prod_{k=1}^{n} w(x_k \mid y_k) \tag{1.1}$$

then the function

$$w(D \mid y) = \sum_{x \in D} w(x \mid y) \tag{1.2}$$

is a probability defined on subsets of A^n by the matrix w under the condition y.

Let $\xi = (\xi_1, \ldots, \xi_m)$ be a probability vector and

$$T_0 = \{t; t : \{1, \ldots, m\} \to W\} . \tag{1.3}$$

If $t \in T_0$, we shall use the notations t_α or w_{t_α} instead of $t(\alpha)$ for the sake of brevity or to stress that $t(\alpha)$ is a matrix. If $y \in B^n$, we shall denote by $w_t(\cdot \mid y)$ the probability

$$w_t(D \mid y) = \sum_{\alpha=1}^{m} \xi_\alpha \, w_{t_\alpha}(D \mid y) , \tag{1.4}$$

where the function $w_{t_\alpha}(D \mid y)$ is defined by means of the matrix w_{t_α} by the formula (1.2).

Let us denote I the set of all positive integers, $A^I = \{\{x_n\}_{n\in I};\ x_n \in A\}$ and $\mathscr{F}_A$ the σ-algebra of subsets of A^I generated by the class of all finite-dimensional cylinders. Given $w \in W$, $\eta \in B^I$ there exists unique probability $w(\cdot \mid \eta)$ defined on $\mathscr{F}_A$ and satisfying

$$w(\{x \in A^I;\ x_k = j_k \ \ k = 1, \ldots, n\} \mid \eta) = \prod_{k=1}^{n} w(j_k \mid \eta_k) . \tag{1.5}$$

All these notations enable us to state

Definition 1.1. Let $W_1, \ldots, W_m$ be non-empty subsets of W and $T = \{t \in T_0;\ t_\alpha \in W_\alpha,\ \alpha = 1, \ldots, m\}$. By a simultaneous channel $C = (B, A, \xi, T)$ decomposable into finitely many memoryless components we shall mean the system of probabilities $\{w_t(\cdot \mid \eta); \eta \in B^I, t \in T\}$, where $w_t(\cdot \mid \eta) = \sum_{\alpha=1}^{m} \xi_\alpha \, w_{t_\alpha}(\cdot \mid \eta)$ and the probability $w_{t_\alpha}(\cdot \mid \eta)$ is defined on $\mathscr{F}_A$ by means of the matrix w_{t_α} as in the formula (1.5).

The notion of the code and of the length of the code which is used in the following definition is defined in [3], p. 116.

Definition 1.2. Let C be the channel described in the preceding definition and $\varepsilon \in (0, 1)$. The code $\{Q(y)\}_{y\in Y}$ will be called (n, N, ε) code for C, if it is n-dimensional code, has the length N and satisfies (cf. (1.4))

$$w_t(Q(y) \mid y) > 1 - \varepsilon$$

for all $y \in Y$, $t \in T$. Such a code will be also called an n-dimensional ε-code for C.

Throughout the whole paper we shall denote by $\log x$ the logarithm to the base 2 and by $\ln x$ the logarithm to the base e. As usual the entropy of a vector p is denoted by $H(p) = -\sum_i p_i \log p_i$ and for $p \in P$, $w \in W$ the transmission rate is defined by the formula

$$R_w(p) = H(wp) - \sum_{i=1}^{b} p_i \, H(w(\cdot \mid i)) , \tag{1.6}$$

where the vector wp has the coordinates $\sum_{i=1}^{b} w(j \mid i)\, p_i$, $j = 1, \ldots, a$.

Let us denote

$$(1.7) \qquad S_n(\varepsilon, C) = \max\{N;\ \text{there exists an } (n, N, \varepsilon) \text{ code for } C\}\ .$$

If $m = 1$ i.e. $T = W_1$, then (cf. [5])

$$(1.8) \qquad \lim_{n\to\infty} \frac{1}{n} \log S_n(\varepsilon, C) = \sup_{p\in P} \inf_{w\in W_1} R_w(p)\ .$$

In the general case, the limit on the left-hand side of (1.8) depends on ε, which is connected with the concept of ε-capacity (cf. [2]). The following result is proved in [3]. Let $\xi = (\xi_1, \ldots, \xi_m)$ be a probability vector. If matrices $w_1, \ldots, w_m$ belonging to W are such that the number $\min\{w_\alpha(j \mid i);\ 1 \leqq \alpha \leqq m,\ j \in A,\ i \in B\}$ is positive, then for any continuity point $\varepsilon \in (0, 1)$ of the function

$$r(\varepsilon) = \sup_{p\in P} \inf\{y;\ \xi\{\alpha;\ R_{w_\alpha}(p) \leqq y\} \geqq \varepsilon\}\ ,$$

where $\xi(\mathscr{A}) = \sum_{\alpha\in\mathscr{A}} \xi_\alpha$, the relation

$$\lim_{n\to\infty} \frac{1}{n} \log S_n(\varepsilon, C) = r(\varepsilon)$$

holds; here the channel $C = (B, A, \xi, T)$ and $W_\alpha = \{w_\alpha\}$ for $\alpha = 1, \ldots, m$.

This paper contains similar results. A theorem on ε-capacity for a general type of simultaneous channel is proved in the part II; this case involves also the channel, defined in [4].

2. BASIC INEQUALITIES

The main assertions of this paper are proved by making use of combinatoric properties of vectors and by approximating channels by some other suitable channels. Basic properties of such approximations are given in this sections.

Definition 2.1. Let $p \in P$ and w be a probability matrix. We shall use the following notations.

(I) $\mu^p(\cdot)$ is a probability on B^n, determined by the vector p, i.e. defined by the formulas

$$(2.1) \qquad \mu^p(\{i_k\}_{k=1}^n) = \prod_{k=1}^n p_{i_k}\ , \quad \mu^p(C) = \sum_{y\in C} \mu^p(y)\ .$$

(II) $\omega(\cdot)$ is a probability on A^n, defined by the formula (cf. (1.1))

$$(2.2) \qquad \omega(x) = \sum_{y\in B^n} \mu^p(y)\, w(x \mid y)\ .$$

Definition 2.2. Let $p \in P$, $w \in W$ and $\varepsilon \in (0, 1)$. A sequence $x \in A^n$ is said to be $\varepsilon(w)$-generated by a sequence $y \in B^n$, if (cf. (2.1) in [3])

$$|N(j, i \mid x, y) - w(j \mid i)\, N(i \mid y)| \leqq d[\varepsilon^{-1} N(i \mid y)\, w(j \mid i)\, (1 - w(j \mid i))]^{1/2}$$

for every $i \in B$, $j \in A$.

In later considerations the following properties of the quantities $H(wp)$, $R_w(p)$ (cf. (1.6)) will be used.

Lemma 2.1. Let $x \in A^n$ be $\varepsilon(w)$-generated by a p-sequence $y \in B^n$ (for the definition of the p-sequence cf. p. 119 in [3]).

(I) For all $j \in A$

$$|1/n\; N(j \mid x) - \omega(j)| \leqq 4\, d^3(\omega(j))^{1/4}\, (n\varepsilon)^{-1/2}\,, \tag{2.3}$$

this inequality being sharp in the case $\omega(j) > 0$.

(II)

$$\exp_2\,[-nH(wp) - 16\, d^4(n\varepsilon^{-1})^{1/2}] < \omega(x) < \tag{2.4}$$
$$< \exp_2\,[-nH(wp) + 16\, d^4(n\varepsilon^{-1})^{1/2}]\,.$$

Let us denote for $p \in P$

$$F(p) = \{y \in B^n;\; y \text{ is a } p\text{-sequence}\}\,, \tag{2.5}$$

and put for $y \in B^n$

$$\Gamma_w(y, \varepsilon) = \{x \in A^n;\; x \text{ is } \varepsilon(w)\text{-generated by } y\}\,. \tag{2.6}$$

Lemma 2.2. (I) If $p \in P$, $y \in B^n$ and ε is a positive number, then

$$\mu^p(F(p)) > \tfrac{3}{4}\,, \tag{2.7}$$

$$w[\Gamma_w(y, \varepsilon) \mid y] > 1 - \varepsilon\,. \tag{2.8}$$

(II) If we denote $C_n(w, y, \varepsilon)$ the number of sequences belonging to A^n, which are $\varepsilon(w)$-generated by a p-sequence $y \in B^n$, then

$$C_n(w, y, \varepsilon) < \exp_2\,[n \sum_1^b p_i H(w(\cdot \mid i)) + 16 d^4(n\varepsilon^{-1})^{1/2}]\,. \tag{2.9}$$

Proofs of these lemmas can be found in [5], Chapter 2. The following assertion is in an explicit form given in [5], p. 38.

Lemma 2.3. Let $p \in P$, $\varepsilon \in (0, 1)$ and $n > 1024 d^6 \varepsilon^{-1}$. If $w, w^* \in W$ and (cf. (2.6)) the set $\Gamma_w(y, \varepsilon) \cap \Gamma_{w^*}(y^*, \varepsilon)$ is non-empty for some p-sequences $y, y^* \in B^n$, then

$$|H(wp) - H(w^*p)| < 70 d^5 (n\varepsilon)^{-1/2} . \tag{2.10}$$

Proof. Let x belong to $\Gamma_w(y, \varepsilon) \cap \Gamma_{w^*}(y^*, \varepsilon)$. The inequality (2.3) implies that

$$|\omega(j) - \omega^*(j)| \leqq 4d^3(n\varepsilon)^{-1/2}\left[\omega(j)^{1/4} + \omega^*(j)^{1/4}\right] .$$

Thus we have to guess the number $|\alpha \log \alpha - \beta \log \beta|$ if we know that

$$0 \leqq \alpha < \beta \leqq 1 , \quad |\alpha - \beta| \leqq 8d^3(n\varepsilon)^{-1/2}\beta^{1/4} .$$

Let $\beta \leqq \mathrm{e}^{-1}$. Let us denote $q = -8d^3(n\varepsilon)^{-1/2}\beta^{1/4} + \beta$. The inequality $q \geqq 0$ together with $-x \log x < 1$ implies

$$|\alpha \log \alpha - \beta \log \beta| < 32 d^3 (n\varepsilon)^{-1/2} . \tag{2.11}$$

Let $q < 0$. If we denote $\gamma = \left[8d^3(n\varepsilon)^{-1/2}\right]^{4/3}$, then $\alpha < \beta < \gamma < \mathrm{e}^{-1}$ which means that

$$|\alpha \log \alpha - \beta \log \beta| < 64 d^4 (n\varepsilon)^{-1/2} . \tag{2.12}$$

Finally, let $\beta > \mathrm{e}^{-1}$. According to the assumptions $\alpha > \mathrm{e}^{-1} - 4^{-1}$ and with the help of Lagrange's theorem on increment of a function we obtain

$$|\alpha \log \alpha - \beta \log \beta| < 64 d^3 (n\varepsilon)^{-1/2} . \tag{2.13}$$

Obviously, the inequalities (2.11), (2.12), (2.13) imply (2.10).

Lemma 2.4. Let $p = (p_1, \ldots, p_a)$ be a probability vector. If $r > a$, then there exists a probability vector $p^* = (p_1^*, \ldots, p_a^*)$ such that

(I) p_k^* is an integer multiple of r^{-1} for $k = 1, \ldots, a - 1$;

(II) $\min\{p_k^*; k = 1, \ldots, a\} \geqq r^{-1}$;

(III) $\|p - p^*\| = \max_k |p_k - p_k^*| < a^2/r$.

Proof. Let us denote

$$p_k = \begin{cases} \dfrac{1}{r} & p_k \leqq (a+1)/r , \\[2ex] \dfrac{s}{r} & p_k > (a+1)/r , \quad s = \max\{n; (a+n)\, r^{-1} \leqq p_k\} , \end{cases}$$

and put $p_a^* = 1 - \sum_1^{a-1} p_k$. If the set $K = \{k < a;\ p_k \leqq (a + 1)/r\}$ contains $a - 1$ elements, then $\sum_1^{a-1} p_k^* < 1 - r^{-1}$, i.e. $p_a^* > r^{-1}$. If K contains $z \leqq a - 2$ elements, then

$$\sum_1^{a-1} p_k^* \leqq zr^{-1} + \sum_{k \in K} (p_k - ar^{-1}) \leqq 1 - 2r^{-1},$$

which implies I and II.

Lemma 2.5. If $r \geqq 16$, $K \geqq 2$ and probability vectors $p = (p_1, \ldots, p_a)$, $p^* = (p_1^*, \ldots, p_a^*)$ are such that $\|p - p^*\| \leqq Kr^{-1}$, then

$$|H(p) - H(p^*)| < 4dK(r)^{-1/2}. \tag{2.14}$$

Proof. Let us denote $f(y) = -(\ln y + 1)/\ln 2$. Taking into account Lagrange's theorem on increment of a function we see that it is sufficient to prove that the inequalities

$$|x \log x| \leqq r^{-1/2}, \quad |f(y)| < r^{1/2} \tag{2.15}$$

hold for all $x \in \langle 0, \exp_2(-r^{1/2})\rangle$, $y \in \langle \exp_2(-r^{1/2}), 1\rangle$. If x belongs to the first interval, then

$$0 \leqq -x \log x \leqq rr^{-1/2} \exp_2(-r^{1/2}),$$

which together with

$$\exp_2(r^{1/2}) - r \geqq 0$$

implies the first inequality in (2.15). If y belongs to the second interval then the inequalities

$$-(\ln 2)^{-1} < f(y) < r^{1/2}$$

complete the proof.

Let $p, p^* \in P$ and $w, w^* \in W$. We shall use the notations

$$\|p - p^*\| = \max\{|p_k - p_k^*|;\ k \in B\},$$
$$\|w - w^*\| = \max\{|w(j \mid i) - w^*(j \mid i)|;\ j \in A,\ i \in B\}.$$

Lemma 2.6. (I) If $\|w - w^*\| \leqq d^2 r^{-1}$ and $r \geqq 16$, then (cf. (1.6))

$$|R_w(p) - R_{w^*}(p)| < 8d^3 r^{-1/2} \tag{2.16}$$

for all vectors $p \in P$.

(II) Let γ be a positive number and $r \geqq \max\{16, (8d^2\gamma^{-1})^2\}$. If $\|p - p^*\| < r^{-1}$, then

$$|R_w(p) - R_w(p^*)| < \gamma \tag{2.17}$$

for all matrices $w \in W$.

Proof. (I) Let $p \in P$ and $\omega \in R^a$ be a probability vector with coordinates

$$\omega(j) = \sum_1^b w(j \mid i)\, p_i\,.$$

If the probability vector ω^* is defined by w^*, p in the same way as ω, then

$$\|\omega - \omega^*\| \leqq \max_j \sum_1^b |w(j \mid i) - w^*(j \mid i)|\, p_i \leqq d^2 r^{-1}\,,$$

and making use of the preceding lemma we obtain (2.16).

(II) The inequality $H(w(\cdot \mid i)) < d$ (cf. (2.2.4) in [5]) together with (2.14) implies (2.17).

Lemma 2.7. If the matrices $w, w^* \in W$ are such that $\|w - w^*\| \leqq d^2 n^{-4}$ then for all sets $D \subset A^n$ and vectors $y \in B^n$ (cf. (1.2))

$$|w(D \mid y) - w^*(D \mid y)| < \delta_n\,, \tag{2.18}$$

where

$$\delta_n = 3d^2[(1 + 2d^2 n^{-2})^n - 1]\,. \tag{2.19}$$

Proof. The proof is analogical as in [5], p. 36. Let $n > d$, $D_n \subset A^n$ and $y \in B^n$. Denote

$$G_n(y) = \{x \in D_n;\ w(x_k \mid y_k) \geqq n^{-2},\ k = 1, \ldots, n\}\,, \quad H_n(y) = D_n - G_n(y)\,.$$

If $w(j \mid i) \geqq n^{-2}$, then

$$\left|\frac{w^*(j \mid i)}{w(j \mid i)} - 1\right| \leqq w(j \mid i)^{-1}\, \|w - w^*\| \leqq d^2 n^{-2}\,,$$

and it follows that for all $x \in G_n(y)$

$$(1 - d^2 n^{-2})^n \leqq \frac{w^*(x \mid y)}{w(x \mid y)} \leqq (1 + d^2 n^{-2})^n\,. \tag{2.20}$$

If we denote $\alpha_n = (1 + d^2 n^{-2})^n - 1$, then both (2.20) and $\alpha_n \geqq 1 - (1 - d^2 n^{-2})^n$ imply

$$1 - \alpha_n \leqq \frac{w^*(x \mid y)}{w(x \mid y)} \leqq 1 + \alpha_n\,,$$

which means that

$$|w^*(G_n(y) \mid y) - w(G_n(y) \mid y)| \leqq \alpha_n\,. \tag{2.21}$$

On the other hand, if $w(j \mid i) < n^{-2}$, then

$$w(\{x \in A^n;\ N(j, i \mid x, y) \geqq 1\} \mid y) \leqq \sum_{k=1}^{N(i|y)} \binom{N(i \mid y)}{k} w(j \mid i)^k < (1 + n^{-2})^n - 1 .$$

Hence

$$w(H_n(y) \mid y) < d^2[(1 + n^{-2})^n - 1] , \tag{2.22}$$

and similarly

$$w^*(H_n(y) \mid y) < d^2[(1 + d^2 n^{-4} + n^{-2})^n - 1] . \tag{2.23}$$

Obviously, the inequality (2.18) follows from (2.21), (2.22) and (2.23).

3. THE LENGTH OF CODES FOR SIMULTANEOUS CHANNELS DECOMPOSABLE INTO FINITELY MANY MEMORYLESS COMPONENTS

Let us denote W_n^* the set of all matrices $w \in W$, which satisfy

$$w(j \mid i) \text{ is an integer multiple of } n^{-4} \text{ for all } j \in A ,\quad i \in B ; \tag{3.1}$$

$$\min \{w(j \mid i);\ j \in A,\ i \in B\} \geqq n^{-4} . \tag{3.2}$$

Following the usual terminology, we shall denote by card Y the number of elements of the set Y.

Theorem 3.1. Let $V \subset W$ be a non-empty set and $p \in P$. If $\varepsilon \in (0, 1)$, $\varepsilon' \in (0, \varepsilon)$ and $n > 1024 d(\varepsilon')^{-1}$ is such that the number δ_n defined by (2.19) satisfies $\varepsilon - \delta_n > \varepsilon'$, then there exists an n-dimensional code $\{Q(y)\}_{y \in Y}$ such that

(I) Y contains only p-sequences and $w(Q(y) \mid y) > 1 - \varepsilon$ for each matrix $w \in V$ and sequence $y \in Y$;

(II) Y satisfies the inequality

$$\frac{1}{n} \log \operatorname{card} Y > \inf_{w \in V} R_w(p) - n^{-1/2} f(n, \varepsilon, \varepsilon', d) , \tag{3.3}$$

where $R_w(p)$ is defined by (1.6) and

$$f(n, \varepsilon, \varepsilon', d) = 100 d^5 (\varepsilon')^{-1/2} + 9 d^2 n^{-1/2} \log n + n^{-3/2}\, 8 d^3 - \\ - n^{-1/2} \log \gamma_n(\varepsilon, \varepsilon') , \tag{3.4}$$

$$\gamma_n(\varepsilon, \varepsilon') = \min \{\varepsilon - \delta_n - \varepsilon',\ 1 - \varepsilon\} .$$

Proof. We shall construct this code similarly as in [5], pp. 37–39. Let us denote

$$V_n^* = \{w^* \in W_n^*;\ \|w^* - w\| < d^2 n^{-4} \text{ for some matrix } w \in V\}$$

and put for $y \in B^n$

$$\Gamma(y) = \bigcup_{w \in V^*_n} \Gamma_w(y, \varepsilon')$$

(cf. (2.6)). The relations (2.7), (2.8) imply that there exists a sequence $\{y_k, Q(y_k)\}_{k=1}^N$ such that

1. $\{y_k\}_{k=1}^N$ are p-sequences, $Q(y_k) = \Gamma(y_k) - \bigcup_{j=1}^{k-1} \Gamma(y_j)$ and $N \geqq 1$;
2. if $w \in V_n^*$, then $w[\Gamma_w(y_k, \varepsilon') \cap Q(y_k) \mid y_k] > 1 - (\varepsilon - \delta_n)$ for $k = 1, \ldots, N$;
3. if $y \in B^n$ is a p-sequence, then for some matrix $w \in V_n^*$

$$w[\Gamma_w(y, \varepsilon') - Q \mid y] \leqq 1 - (\varepsilon - \delta_n),$$

where

$$Q = \bigcup_1^N Q(y_k).$$

If $y \in F(p)$ (cf. (2.5)), then by the property 3 and (2.8) there exists a matrix $w_y \in V_n^*$ such that

$$w_y[\Gamma_{w_y}(y, \varepsilon') \cap Q \mid y] > \gamma_n(\varepsilon, \varepsilon').$$

It follows from (2.7) that there is a set $G \subset F(p)$ and a matrix $w \in V_n^*$ such that

$$\mu^p(G) > \tfrac{3}{4} \exp_n(-4d^2), \tag{3.5}$$

$$w[\Gamma_w(y, \varepsilon') \cap Q \mid y] > \gamma_n(\varepsilon, \varepsilon') \quad \text{for all} \quad y \in G.$$

If we put

$$D = Q \cap (\bigcup_{y \in G} \Gamma_w(y, \varepsilon'))$$

then the inequalities (3.5) and the formula (2.4) imply

$$\text{card } D > \exp_n(-5d^2)\, \gamma_n(\varepsilon, \varepsilon') \exp_2 [nH(wp) - 16d^4(n/\varepsilon')^{1/2}]. \tag{3.6}$$

The relations I, II can be proved by means of (2.18) and (3.6), (2.9), (2.10) and (2.16) similarly as in [5], pp. 37–39.

Let $(B, A, \sum_1^m \xi_\alpha w_\alpha)$ be a channel decomposable into finitely many memoryless components (cf. [3]). Let us denote for $\mathscr{A} \subset \{1, \ldots, m\}$

$$w_{\mathscr{A}}(x \mid y) = \xi(\mathscr{A})^{-1} \sum_{\alpha \in \mathscr{A}} \xi_\alpha w_\alpha(x \mid y), \tag{3.7}$$

$$\omega_{\mathscr{A}}(x) = \xi(\mathscr{A})^{-1} \sum_{\alpha \in \mathscr{A}} \xi_\alpha\, \omega_\alpha(x),$$

where $\xi(\mathscr{A}) = \sum_{\alpha \in \mathscr{A}} \xi_\alpha$ and the probabilities $w_\alpha(\cdot \mid y)$, ω_α are defined by means of the matrix w_α by (1.1) and (2.2).

Lemma 3.1. Let y be a p-sequence and $x \in \bigcup_{\alpha \in \mathscr{A}} \Gamma_{w_\alpha}(y, \varepsilon)$ (cf. (2.6)). Let us denote

$$I_n(x, y; w_{\mathscr{A}}, p) = (1/n) \log \frac{w_{\mathscr{A}}(x \mid y)}{\omega_{\mathscr{A}}(x)} .$$

If the number $w_0 = \min \{w_\alpha(j \mid i);\ \alpha \in \mathscr{A},\ j \in A,\ i \in B\}$ is positive, then for every $\alpha \in \mathscr{A}$ (cf. (1.6))

$$(3.8) \qquad |R_{w_\alpha}(p) - I_n(x, y; w_{\mathscr{A}}, p)| < (1/n) \log (2/\xi_0) - 40 d^4 (n\varepsilon)^{-1/2} \log w_0 ,$$

where

$$(3.9) \qquad \xi_0 = \min \{\xi_\alpha;\ \alpha = 1, \ldots, m\} .$$

Proof. This assertion can be proved in the same way as Lemma 1 in [3] with the only different point that the estimate

$$\log \frac{1}{w_0} \leqq \frac{1}{w_0}$$

has to be omitted.

Let $C = (B, A, \xi, T)$ be a channel described by Definition 1.1. The channel will be called non-singular, if the number

$$(3.10) \qquad \varkappa(C) = \inf \{w(j \mid i);\ w \in \bigcup_{\alpha=1}^{m} W_\alpha,\ j \in A,\ i \in B\}$$

is positive. Let us denote

$$w_{\mathscr{A},t}(Q(y) \mid y) = \frac{1}{\xi(\mathscr{A})} \sum_{\alpha \in \mathscr{A}} \xi_\alpha\, w_{t_\alpha}(Q(y) \mid y) ,$$

where $\mathscr{A} \subset \{1, \ldots, m\}$, $t \in T_0$ and the probability $w_{t_\alpha}(\cdot \mid y)$ is defined by the matrix w_{t_α} as in (1.2). If we denote for $p \in P$, $\varepsilon \in (0, 1)$ by $S_n^*(\varepsilon, C_{\mathscr{A}}, p)$ the maximum length of n-dimensional codes $\{Q(y)\}_{y \in Y}$ which satisfy

$$(3.11) \qquad Y \text{ contains only } p\text{-sequences} ,$$

$$w_{\mathscr{A},t}(Q(y) \mid y) > 1 - \varepsilon \quad \text{for each} \quad t \in T, \quad y \in Y ,$$

then the following theorem holds.

Theorem 3.2. Let us denote

$$r = \max_{\alpha \in \mathscr{A}} \inf_{t \in T} R_{t_\alpha}(p) , \quad r' = \min_{\alpha \in \mathscr{A}} \inf_{t \in T} R_{t_\alpha}(p) ,$$

where $R_{t_\alpha}(p)$ is defined by means of the matrix w_{t_α} as in (1.6).

(I) Let $\varepsilon' \in (0, \varepsilon)$. If $n > 1024d^6/\varepsilon'$ is such that (cf. (2.19)) $\varepsilon - \delta_n > \varepsilon'$, then (cf. (3.4))

$$(3.12) \qquad (1/n) \log S_n^*(\varepsilon, C_{\mathscr{A}}, p) > r' - n^{-1/2} f(n, \varepsilon, \varepsilon', d) .$$

(II) Let $\varepsilon'' \in (0, 1 - \varepsilon)$. If $n > d$ is such that $(1 - \varepsilon) - \delta_n > \varepsilon''$, then (cf. (3.9))

$$(3.13) \qquad (1/n) \log S_n^*(\varepsilon, C_{\mathscr{A}}, p) < r + n^{-1/2} g(n, \varepsilon, \varepsilon'', \xi_0, d) ,$$

where

$$g(n, \varepsilon, \varepsilon'', \xi_0, d) = 9d^3 n^{-3/2} - n^{-1/2} \log (1 - \varepsilon - \delta_n - \varepsilon'') + n^{-1/2} \log \frac{2}{\xi_0} + \gamma_n(C) ,$$

$$\gamma_n(C) = 160 d^4 (\varepsilon'')^{-1/2} \log n .$$

If the channel is non-singular, then for $n^4 > 2d^2/\varkappa(C)$

$$\gamma_n(C) = 40 d^4 (\varepsilon'')^{-1/2} (1 - \log \varkappa(C)) .$$

Proof. Denoting $V = \bigcup_{\alpha \in \mathscr{A}} W_\alpha$ we see that Theorem 3.1 implies I.

(II) Let $\{Q(y)\}_{y \in Y}$ be an n-dimensional code satisfying (3.11). Let us choose $t \in T$ such that

$$(3.14) \qquad R_{t_\alpha}(p) < n^{-2} + \inf_{t \in T} R_{t_\alpha}(p)$$

for every $\alpha \in \mathscr{A}$. Since $n > d$, we can find, by Lemma 2.4, matrices $\{w_\alpha;\ \alpha \in \mathscr{A}\}$ belonging to W_n^* (cf. (3.1), (3.2)) such that

$$(3.15) \qquad \|w_{t_\alpha} - w_\alpha\| < d^2 n^{-4} .$$

The last inequality and Lemma 2.7 imply (cf. (3.7))

$$w_{\mathscr{A}}(Q(y) \mid y) > 1 - (\varepsilon + \delta_n) .$$

Further, taking into account Lemma 3.1, we see that

$$\frac{1}{n} \log \frac{w_{\mathscr{A}}(x \mid y)}{\omega_{\mathscr{A}}(x)} < \max_{\alpha \in \mathscr{A}} R_{w_\alpha}(p) + \lambda_n$$

for each $x \in Q(y) \cap [\bigcup_{\alpha \in \mathscr{A}} \Gamma_{w_\alpha}(y, \varepsilon'')]$. Now, if we make use of both (2.8) and disjointness of the sets $\{Q(y)\}$, we obtain

$$(3.16) \qquad N(1 - \varepsilon - \delta_n - \varepsilon'') < \exp_2 [n(\max_{\alpha \in \mathscr{A}} R_{w_\alpha}(p) + \lambda_n)] .$$

Since $w_0 \geqq n^{-4}$ by (3.2) and since

$$\max_{\alpha \in \mathscr{A}} R_{w_\alpha}(p) < r + 9d^3 n^{-2}$$

by (3.15), (3.14) and Lemma 2.6, the theorem is proved in the case $\varkappa(C) = 0$. The non-singular case can be proved similarly.

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