Esko Turunen

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ALGEBRAIC ANALYSIS OF LPC+Ch CALCULUS

ESKO TURUNEN

The paper deals with Mattila's LPC+Ch Calculus (cf. [2]). This fuzzy inference system is an attempt to introduce linguistic objects to mathematical logic without defining these objects mathematically. LPC+Ch Calculus is analyzed from algebraic point of view and it is demonstrated that suitable factorization of the set of well formed formulae (in fact, Lindenbaum algebra) leads to a structure called ET-algebra and introduced in the beginning of the paper. On its basis, all the theorems presented in [2] and many others can be proved in a simple way which is demonstrated in the Lemmas 1 and 2 and Propositions 1–3. The conclusion critically discusses some other issues of LPC+Ch Calculus, specially that no formal semantics for it is given.

1. INTRODUCTION

Even since Zadeh in 1965 introduced his study 'Fuzzy Sets' [5], various aspects of fuzziness has been investigated in many papers. Since a unique Fuzzy Set Theory does not yet exist, it is necessary to continue research in this field. Different sciences have their special methods and approaches even to same topic. A philologist studies fuzziness from a different point of view to a scientist on engineers. Lakoff's approach [1] is linguistic. Mattila's [2], Rhodes's and Menani's [4] studies are more mathematical in nature. Zadeh himself writes in [6]:

> 'In the spirit as well as in substance, fuzzy logic and approximate reasoning represent a rather sharp departure from the traditional approaches to logic and the mathematization of human reasoning. Thus, in essence, fuzzy logic may be viewed as an attempt to accommodation with the pervasive reality of fuzziness and vagueness in human cognition. In this sense, fuzzy logic represents a retreat from what may well be an unrealizable objective, namely the construction of a rigorous mathematical foundation for human reasoning and rational behavior.'

Literally, he means that fuzzy logic can not be regarded as mathematics at all, since rigorousness is one of the fundamental features in mathematics. Nevertheless, the most important methods in obtaining results in various fuzzy logical inference systems are mathematical; hence, the author tends to assume that fuzzy logic is a new idea rather than a real rival to mathematics and, therefore, can be mathematicized.

Another reason for this conclusion is that mathematical reasoning on fuzzy inference systems, the metamathematics of fuzzy logic, is classical Boolean logic. Statements concerning fuzzy logic are either true or false, they are not vague.

From the mathematical point of view it is relevant to set the question: is fuzzy logic an exact theory of fuzziness or may the theory itself be somehow fuzzy? The deep conviction of the author is that, to be mathematically acceptable, fuzzy logic must be an exact theory. Fuzzy logic is the logic of inexact phenomena, but the metalogic of fuzzy logic is two-valued.

Here we study as an example an investigation on hedges (such as 'very', 'more or less', 'rather', etc.) called LPC+Ch Calculus (cf. [2]), which is an attempt to introduce linguistic objects to mathematical logic without defining them mathematically. Similarly as the problems of classical logic can be reduced to the examination of Boolean algebras, the discussion of any new type of non-classical logic leads to the examination of an adequate type of abstract algebra. We prove that the algebra corresponding to LPC+Ch Calculus is an abstract algebra which we call, for want of anything better, ET-algebra (the letters ET standing for Extra Terrestrial, since LPC+Ch Calculus, being two valued, is somewhat unusual in the fuzzy framework). We give another proof for the results in [2] and establish many others. After our algebraic analysis we compare LPC+Ch Calculus with three other studies on hedges. Because of the lack of mathematical exactness on the foundations of LPC+Ch Calculus, quite odd results arise.

A symbol := which we often use abbreviates a frase 'is defined by' and iff abbreviates 'if and only if'.

2. ON ET-ALGEBRAS

Definition 1. An abstract algebra $A = \langle A, \wedge, \vee, \neg, \rightarrow, \mathbb{I}, \mathbb{F}_1, \mathbb{F}_2, \ldots \rangle$ such that $A = \langle A, \wedge, \vee, \neg, \rightarrow \rangle$ is a Boolean algebra and the operations $\mathbb{I}$, $\mathbb{F}_1$, $\mathbb{F}_2, \ldots$, defined on A, fulfil the axioms

$$\mathbb{I}(a) = a, \tag{1}$$

$$\mathbb{F}_k(a \wedge b) = \mathbb{F}_k(a) \wedge \mathbb{F}_k(b), \tag{2}$$

$$\mathbb{F}_{k+1}(a) \leq \mathbb{F}_k(a) \leq \cdots \leq a, \tag{3}$$

$$\mathbb{F}_k(1) = 1 \quad \text{(the unit of } A\text{)}, \tag{4}$$

for any $a, b \in A$, $k \in N$, is called *ET-algebra*. We omit index and write $\mathbb{F}$ for $\mathbb{F}_k$ when this causes no confusion. The *dual* $\mathbb{F}^*$ of an operation $\mathbb{F}$ is defined by

$$\mathbb{F}^*(a) = \neg\mathbb{F}(\neg a) \quad \text{for any } a \in A. \tag{5}$$

If for an operation $\mathbb{F}$ holds

$$\mathbb{F}(\mathbb{F}(a)) = \mathbb{F}(a) \quad \text{for any } a \in A, \tag{6}$$

we say that $\mathbb{F}$ is *idempotent*.

Remark 1. A direct consequence from (2), (3), (4) and (6) is that for any idempotent operation $\mathbb{F}$, the algebra $A = \langle A, \wedge, \vee, \neg, \rightarrow, \mathbb{F}\rangle$ is a topological Boolean algebra. $\mathbb{F}$ is the *interior operation* and its dual $\mathbb{F}^*$ is the corresponding *closure operation*.

Next, we list some properties of ET-algebras. All of them are trivial or almost trivial but it will turn out that each of them has a counterpart in [2].

Proposition 1. In ET-algebra A hold

$$\mathbb{I}(a) = \mathbb{I}^*(a) \tag{7}$$
$$a \leq \mathbb{F}^*(a), \tag{8}$$
$$\mathbb{F}(a) \leq \mathbb{F}^*(a), \tag{9}$$
$$\text{if } a = 1,\ \mathbb{F}(a) \leq b,\ \text{then } \mathbb{F}(b) = 1, \tag{10}$$
$$\text{if } a = 1,\ \mathbb{F}(a) \leq \mathbb{F}(b),\ \text{then } \mathbb{F}(b) = 1, \tag{11}$$
$$\text{if } a = 1,\ \mathbb{F}(a) \leq b,\ \text{then } b = 1, \tag{12}$$
$$\text{if } \mathbb{F}(a) = 1,\ a \leq \mathbb{F}(b),\ \text{then } \mathbb{F}(b) = 1, \tag{13}$$
$$\mathbb{I}(a) \leq \mathbb{F}^*(a), \tag{14}$$
$$\neg\mathbb{F}^*(\neg a) \leq \mathbb{I}(a), \tag{15}$$
$$\mathbb{F}_k^*(a) \leq \mathbb{F}_n^*(a),\ (k \leq n), \tag{16}$$
$$\mathbb{F} \text{ is isotone, i.e., if } a \leq b,\ \text{then } \mathbb{F}(a) \leq \mathbb{F}(b) \tag{17}$$
$$\mathbb{F}(a \rightarrow b) \leq \mathbb{F}(a) \rightarrow \mathbb{F}(b), \tag{18}$$
$$\mathbb{F}(\neg a) = \neg\mathbb{F}^*(a), \tag{19}$$
$$\neg\mathbb{F}(a) = \mathbb{F}^*(\neg a), \tag{20}$$
$$\neg\mathbb{F}^*(a \vee b) = \neg F^*(a) \wedge \neg\mathbb{F}^*(b), \tag{21}$$
$$\mathbb{F}^*(a) \vee \mathbb{F}^*(b) = \mathbb{F}^*(a \vee b), \tag{22}$$
$$\mathbb{F}(a) \vee \mathbb{F}(b) \leq \mathbb{F}(a \vee b), \tag{23}$$
$$\mathbb{F}^* \text{ is isotone}, \tag{24}$$
$$\mathbb{F}^*(a \wedge b) \leq \mathbb{F}^*(a) \wedge \mathbb{F}^*(b), \tag{25}$$
$$\text{if } a = b,\ \text{then } \mathbb{F}(a) = \mathbb{F}(b),\ \mathbb{F}^*(a) = \mathbb{F}^*(b), \tag{26}$$
$$\text{if } a = 1,\ \text{then } \mathbb{F}(b) = \mathbb{F}(a \wedge b),\ \mathbb{F}^*(b) = \mathbb{F}^*(a \wedge b), \tag{27}$$
$$\mathbb{F}\left(\textstyle\bigwedge_{i\in I} a_i\right) \leq \textstyle\bigwedge_{i\in I}\mathbb{F}(a_i), \tag{28}$$
$$\mathbb{F}^*\left(\textstyle\bigwedge_{i\in I} a_i\right) \leq \textstyle\bigwedge_{i\in I}\mathbb{F}^*(a_i), \tag{29}$$
$$\textstyle\bigvee_{i\in I}\mathbb{F}(a_i) \leq \mathbb{F}\left(\textstyle\bigvee_{i\in I} a_i\right), \tag{30}$$
$$\mathbb{F}(0) = 0 \text{ (the zero of } A), \tag{31}$$
$$\mathbb{F}^n(a) \leq \mathbb{F}^k(a),\ k \leq n, \tag{32}$$
$$(\mathbb{F}^*)^k(a) \leq (\mathbb{F}^*)^n(a),\ k \leq n, \tag{33}$$
$$\mathbb{F}(\mathbb{F}^n(a)) \leq \mathbb{F}(\mathbb{F}^k(a)),\ \mathbb{F}^*(\mathbb{F}^n(a)) \leq \mathbb{F}^*(\mathbb{F}^k(a)),\ k \leq n, \tag{34}$$
$$\mathbb{F}(b) \wedge \mathbb{F}\left(\textstyle\bigvee_{i\in I} a_i\right) = \mathbb{F}\left(\textstyle\bigvee_{i\in I}(b \wedge a_i)\right), \tag{35}$$
$$\mathbb{F}^*(b) \vee \mathbb{F}^*\left(\textstyle\bigwedge_{i\in I} a_i\right) = \mathbb{F}^*\left(\textstyle\bigwedge_{i\in I}(b \vee a_i)\right), \tag{36}$$

where $a, b, a_i \in A$, ($i \in I$, I an index set), $\mathbb{F}$, $\mathbb{F}_k$, $\mathbb{F}_n$ are operations on A and $\mathbb{F}^*$, $\mathbb{F}_k^*$, $\mathbb{F}_n^*$ are their duals, respectively, $\mathbb{F}^n : A \searrow A$ are defined recursively by $\mathbb{F}^1(a) = \mathbb{F}(a), \ldots, \mathbb{F}^{k+1}(a) = \mathbb{F}(\mathbb{F}^k(a))$, ($k, n \in N$) and the infinite joins and meets in (28), (29), (30), (35) and (36) are assumed to exist in A.

Proof. In Boolean algebras hold $a = \neg\neg a$ and $a \leq b$ iff $\neg b \leq \neg a$. Thus, (7) and (8) hold. (9) – (16), (19), (20), (26), (27) and (31) are trivial. Let $a \leq b$. Then $a = a \wedge b$ and hence $\mathbb{F}(a) = \mathbb{F}(a \wedge b) = \mathbb{F}(a) \wedge \mathbb{F}(b) \leq \mathbb{F}(b)$. So (17) holds. By (2) and (17), $\mathbb{F}((a \rightarrow b) \wedge a) = \mathbb{F}(a \rightarrow b) \wedge \mathbb{F}(a) \leq \mathbb{F}(b)$, which is equivalent to (18). In Boolean algebras holds $\neg(a \vee b) = \neg a \wedge \neg b$. This implies (21) and (22). Since $a, b \leq a \vee b$, we have $\mathbb{F}(a), \mathbb{F}(b) \leq \mathbb{F}(a \vee b)$. Therefore, (23) holds. Let $a \leq b$. Then $a \vee b = b$ and $\mathbb{F}^*(a) \leq \mathbb{F}^*(a) \vee \mathbb{F}^*(b) = \mathbb{F}^*(a \vee b) = \mathbb{F}^*(b)$. Hence, (24) holds. Since $a \wedge b \leq a$, b we have $\mathbb{F}^*(a \wedge b) \leq \mathbb{F}^*(a)$, $\mathbb{F}^*(b)$ and so (25) holds. Let $\bigwedge_{i \in I} a_i$ exist in A. Since $\mathbb{F}$ is isotone, we have $\mathbb{F}\left(\bigwedge_{i \in I} a_i\right) \leq \mathbb{F}(a_i)$ for each $i \in I$. If $\bigwedge_{i \in I} \mathbb{F}(a_i)$ exist in A, then trivially (28) holds. By a similar argument also (29) and (30) are valid. (32) follows (3) and (17). (33) we establish by using (8) and (24). Similarly (34). Since $b \wedge \left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I}(b \wedge a_i)$ holds in Boolean algebras, (35) is easy. Similarly (36). The proof is complete. □

3. THE ALGEBRA OF LPC+Ch CALCULUS

In this section have a better look at LPC+Ch Calculus. The set $\mathbb{W}$ of *well formed formulae* of LPC+Ch is constructed such that (i) if $\mathcal{A}$ is a wff of LPC, then $\mathcal{A}$ is in $\mathbb{W}$, (ii) if $\mathcal{A}, \mathcal{B} \in \mathbb{W}$, x is a variable and $\mathcal{F} : \mathbb{W} \searrow \mathbb{W}$ is a *modifier*, then $\mathcal{F}(\mathcal{A})$, $\forall x\, \mathcal{A}$, $\neg \mathcal{A}$ and $\mathcal{A} \Rightarrow \mathcal{B}$ is in $\mathbb{W}$. (iii) There are no other formulae in $\mathbb{W}$ than those defined by (i) and (ii). The set of (*substantiating*) modifiers $\mathcal{I} \leq \mathcal{F}_1 \leq \mathcal{F}_2 \leq \ldots$ is denoted by $\mathbb{O}$ (for the order relation, cf. [2]). Again we omit index and write $\mathcal{F}$ for $\mathcal{F}_k$ when there is no fear of confusion. The connectives $\sqcap$, $\sqcup$, $\Leftrightarrow$, and the quantifier $\exists$ are defined in the usual way. The *dual* $\mathcal{F}^*$ of a modifier $\mathcal{F}$ is a (*weakening*) modifier defined by $\mathcal{F}^*(\mathcal{A}) := \neg\mathcal{F}(\neg\mathcal{A})$, $\mathcal{A} \in \mathbb{W}$. For these modifiers holds $\ldots \leq \mathcal{F}_k^* \leq \cdots \leq \mathcal{F}_1^* \leq \mathcal{I}$. Note that $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\mathbb{I}$, $\mathbb{F}$ are operations on ET-algebra A, while $\sqcap$, $\sqcup$, $\neg$, $\Rightarrow$, $\mathcal{I}$, $\mathcal{F}$ are logical symbols in the language of LPC+Ch. The *axiom schemata* of LPC+Ch are the axiom schemata of LPC enriched with the following schemata

$$\vdash \mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{H}(\mathcal{A}) \text{ where } \mathcal{F}, \mathcal{H} \text{ are modifiers such that } \mathcal{H} \leq \mathcal{F}. \qquad \text{(AxCh)}$$

$$\vdash \mathcal{I}(\mathcal{A}) \Leftrightarrow \mathcal{A}, \text{ where } \mathcal{I} \text{ is the } identity \text{ modifier.} \qquad \text{(AxId)}$$

The *rules of inference* of LPC+Ch are those of LPC and the following

$$\mathcal{A} \Rightarrow \mathcal{B},\ \mathcal{F}(\mathcal{A}) \vdash \mathcal{F}(\mathcal{B}), \text{ where } \mathcal{F} \text{ is a modifier or its dual,} \qquad \text{(MMP)}$$

$$\vdash \mathcal{A} \text{ implies } \vdash \mathcal{F}(\mathcal{A}), \text{ where } \mathcal{F} \text{ is a modifier.} \qquad \text{(RS)}$$

Remark 2. We take here the axiomatization of LPC from [3] and not that mentioned in [2]. This treatment does not cause, of course, any restriction, since a wff $\mathcal{A}$ of LPC is theorem in one axiomatization of LPC iff it is theorem in any other axiomatization of LPC. We proceed as in [3], pp. 469–473.

Define on $\mathbb{W}$ a relation $\leq$ such that

$$\mathcal{A} \leq \mathcal{B} \text{ if } \vdash \mathcal{A} \Rightarrow \mathcal{B} \text{ in LPC+Ch}, \tag{37}$$

and, moreover, a relation $\approx$ such that

$$\mathcal{A} \approx \mathcal{B} \text{ if } \vdash \mathcal{A} \Rightarrow \mathcal{B} \text{ and } \vdash \mathcal{B} \Rightarrow \mathcal{A} \text{ in LPC+Ch}. \tag{38}$$

Then $\approx$ is a congruence in $\mathbb{W}$ with respect to the connectives $\cap, \cup, \neg, \Rightarrow$. Denote by $|\mathcal{A}|$ the equivalence class containing $\mathcal{A} \in \mathbb{W}$ and denote the factor set $\{|\mathcal{A}| \mid \mathcal{A} \in \mathbb{W}\}$ by $\mathbb{W}/\approx$. Then the algebra $\mathbb{A} = \langle \mathbb{W}/\approx, \leq, \wedge, \vee, \neg, \rightarrow \rangle$, where

$$|\mathcal{A}| \leq |\mathcal{B}| \quad \text{iff } \vdash \mathcal{A} \Rightarrow \mathcal{B} \text{ in LPC+Ch}, \tag{39}$$

$$|\mathcal{A}| = |\mathcal{B}| \quad \text{iff } |\mathcal{A}| \leq |\mathcal{B}| \text{ and } |\mathcal{B}| \leq |\mathcal{A}|, \tag{40}$$

and the operations $\wedge$ (g.l.b), $\vee$ (l.u.b), $\neg$ (complement), $\rightarrow$ (residuation) are defined by

$$|\mathcal{A}| \wedge |\mathcal{B}| := |\mathcal{A} \cap \mathcal{B}|, \tag{41}$$

$$|\mathcal{A}| \vee |\mathcal{B}| := |\mathcal{A} \cup \mathcal{B}|, \tag{42}$$

$$\neg|\mathcal{A}| := |\neg\mathcal{A}|, \tag{43}$$

$$|\mathcal{A}| \rightarrow |\mathcal{B}| := |\mathcal{A} \Rightarrow \mathcal{B}|, \tag{44}$$

is a Boolean algebra. Define

$$\bigwedge\nolimits_{t \text{ is a term}} |\mathcal{A}[t]| := |\forall x\, \mathcal{A}|, \tag{45}$$

$$\bigvee\nolimits_{t \text{ is a term}} |\mathcal{A}[t]| := |\exists x\, \mathcal{A}|. \tag{46}$$

Then $\mathbb{A}$ is a Q-algebra (i.e., the infinite joins and meets in (45) and (46) exists in $\mathbb{W}/\approx$). Moreover, $|\mathcal{A}| = |1|$ (the unit of $\mathbb{W}/\approx$) iff $\vdash \mathcal{A}$ in LPC+Ch and $\neg|1| := |0|$ is the minimum of $\mathbb{W}/\approx$.

Thus far everything said can be found in [3]. The new part comes in the following.

Theorem 1. The algebra $\mathbb{A}_{\text{Ch}} = \langle \mathbb{W}/\approx, \leq, \wedge, \vee, \neg, \rightarrow, \mathbb{I}, \mathbb{F}_1, \mathbb{F}_2, \ldots \rangle$, where the operations $\mathbb{I}, \mathbb{F}_k : (\mathbb{W}/\approx) \rightarrow (\mathbb{W}/\approx)$ are defined by

$$\mathbb{I}(|\mathcal{A}|) := |\mathcal{I}(\mathcal{A})| \quad \text{and} \quad \mathbb{F}_k(|\mathcal{A}|) := |\mathcal{F}_k(\mathcal{A})|, \quad \mathcal{A} \in \mathbb{W},\ \mathcal{F}_k \in \mathbb{O},\ k \in N, \tag{47}$$

is an ET-algebra.

Proof. First we have to show that $\approx$ is a congruence also with respect to any $\mathcal{F} \in \mathbb{O}$. Assume $\mathcal{A} \approx \mathcal{B}$, $\mathcal{A}, \mathcal{B} \in \mathbb{W}$. Then $\vdash \mathcal{A} \Rightarrow \mathcal{B}$ and $\vdash \mathcal{B} \Rightarrow \mathcal{A}$ in LPC+Ch. By (RS), $\vdash \mathcal{F}(\mathcal{A} \Rightarrow \mathcal{B})$ and $\vdash \mathcal{F}(\mathcal{B} \Rightarrow \mathcal{A})$ in LPC+Ch. By equation (18) in [2], $\vdash \mathcal{F}(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{F}(\mathcal{B}))$ and $\vdash \mathcal{F}(\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow (\mathcal{F}(\mathcal{B}) \Rightarrow \mathcal{F}(\mathcal{A}))$ hold. By Modus Ponens, $\vdash \mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{F}(\mathcal{B})$ and $\vdash \mathcal{F}(\mathcal{B}) \Rightarrow \mathcal{F}(\mathcal{A})$, hence $\mathcal{F}(\mathcal{A}) \approx \mathcal{F}(\mathcal{B})$. Accordingly, the definition (47) is correct, i.e., the operations $\mathbb{F}$ on $\mathbb{W}/\approx$ are well-defined. Then we establish the ET-axioms.

1^0: By (AxId), $\vdash \mathcal{I}(\mathcal{A}) \Rightarrow \mathcal{A}$, $\vdash \mathcal{A} \Rightarrow \mathcal{I}(\mathcal{A})$, so by (39), $|\mathcal{I}(\mathcal{A})| \leq |\mathcal{A}|$ and $|\mathcal{A}| \leq |\mathcal{I}(\mathcal{A})|$. By (40), $|\mathcal{I}(\mathcal{A})| = |\mathcal{A}|$. This, together with (47), implies $\mathbb{I}(|\mathcal{A}|) = |\mathcal{A}|$ for any $|\mathcal{A}| \in \mathbb{W}/\approx$.

2^0: By equation (20) in [2], it holds that $\vdash (\mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B})) \Rightarrow \mathcal{F}(\mathcal{A} \cap \mathcal{B})$ and $\vdash \mathcal{F}((\mathcal{A} \cap \mathcal{B}) \Rightarrow \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}))$ in LPC+Ch. By (39) and (40) we have

$$|\mathcal{F}(\mathcal{A} \cap \mathcal{B})| = |\mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B})|. \tag{48}$$

Then $\mathbb{F}(|\mathcal{A}| \wedge |\mathcal{B}|) = \mathbb{F}(|\mathcal{A} \cap \mathcal{B}|) = |\mathcal{F}(\mathcal{A} \cap \mathcal{B})| = |\mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B})| = |\mathcal{F}(\mathcal{A})| \wedge |\mathcal{F}(\mathcal{B})| = \mathbb{F}(|\mathcal{A}|) \wedge \mathbb{F}(|\mathcal{B}|)$ holds for any $|\mathcal{A}|$, $|\mathcal{B}| \in \mathbb{W}/\approx$ and for any operation $\mathbb{F}$.

3^0: By (AxCh), $\vdash \mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{I}(\mathcal{A})$ in LPC+Ch, hence $|\mathcal{F}(\mathcal{A})| \leq |\mathcal{I}(\mathcal{A})| = |\mathcal{A}|$. By (47), we have

$$\mathbb{F}(|\mathcal{A}|) \leq |\mathcal{A}| \quad \text{for any } |\mathcal{A}| \in \mathbb{W}/\approx, \text{ any operation } \mathbb{F}. \tag{49}$$

Let $\mathcal{F}_k$, $\mathcal{F}_n \in \mathbb{O}$ be such that $\mathcal{F}_k \leq \mathcal{F}_n$. Then $k \leq n$. By (AxCh), $\vdash \mathcal{F}_n(\mathcal{A}) \Rightarrow \mathcal{F}_k(\mathcal{B})$ iff $|\mathcal{F}_n(\mathcal{A})| \leq |\mathcal{F}_k(\mathcal{A})|$ iff $\mathbb{F}_n(|\mathcal{A}|) \leq \mathbb{F}_k(|\mathcal{A}|)$, $|\mathcal{A}| \in \mathbb{W}/\approx$, which together with (49) implies $\ldots \mathbb{F}_{k+1}(|\mathcal{A}|) \leq \mathbb{F}_k(|\mathcal{A}|) \leq \cdots \leq |\mathcal{A}|$, $|\mathcal{A}| \in \mathbb{W}/\approx$, $k \in N$.

4^0: Let $|\mathcal{A}| = |1|$. Then $\vdash \mathcal{A}$. By (RS), $\vdash \mathcal{F}(\mathcal{A})$. Hence, $|\mathcal{F}(\mathcal{A})| = |1|$, i.e., $\mathbb{F}(|\mathcal{A}|) = |1|$. That is to say $\mathbb{F}(|1|) = |1|$ for any operation $\mathbb{F}$.

The proof is complete. □

Lemma 1. All the (correct) propositions in [2] can be proved using the Theorem 1 and Proposition 1 only.

Proof. Let us prove as a model the LPC+Ch theorem

$$\vdash \mathcal{F}^*(\mathcal{A} \cap \mathcal{B}) \Rightarrow (\mathcal{F}^*(\mathcal{A}) \cap \mathcal{F}^*(\mathcal{B})) \tag{50}$$

Indeed, (50) holds iff $|\mathcal{F}^*(\mathcal{A} \cap \mathcal{B})| \leq |\mathcal{F}^*(\mathcal{A}) \cap \mathcal{F}^*(\mathcal{B})|$ iff $\mathbb{F}^*(|\mathcal{A} \cap \mathcal{B})|) \leq |\mathcal{F}^*(\mathcal{A})| \wedge |\mathcal{F}^*(\mathcal{B})|$ iff $\mathbb{F}^*(|\mathcal{A}| \wedge |\mathcal{B}|) \leq \mathbb{F}^*(|\mathcal{A}|) \wedge \mathbb{F}^*(|\mathcal{B}|)$, which holds true by (25). The rest part can be proved similarly using (7)–(18), (21)–(24), (26), (28)–(34), respectively. □

Lemma 2. The following are LPC+Ch theorems

$\vdash \mathcal{F}(\neg\mathcal{A}) \Leftrightarrow \neg\mathcal{F}^*(\mathcal{A})$,

$\vdash \neg\mathcal{F}(\neg\mathcal{A}) \Leftrightarrow \neg\mathcal{F}^*(\neg\mathcal{A})$,

if $\vdash \mathcal{A}$, then $\vdash \mathcal{F}^*(\mathcal{B}) \Leftrightarrow \mathcal{F}^*(\mathcal{A} \cap \mathcal{B})$,

$\vdash (\mathcal{F}(\mathcal{B}) \cap \mathcal{F}(\exists\, x\, \mathcal{A})) \Leftrightarrow \mathcal{F}(\exists\, x(\mathcal{A} \cap \mathcal{B}))$, x does not occur in $\mathcal{B}$,

$\vdash (\mathcal{F}^*(\mathcal{B}) \cup \mathcal{F}^*(\forall\, x\, \mathcal{A})) \Leftrightarrow \mathcal{F}^*(\forall\, x(\mathcal{A} \cup \mathcal{B}))$, x does not occur in $\mathcal{B}$.

They follow easily by (19), (20), (27), (35) and (36), respectively.

The advantages of the algebraic treatment of LPC+Ch are obvious. For example one needs 36 steps long deduction for

$$\vdash (\mathcal{F}(\mathcal{B}) \cup \mathcal{F}(\mathcal{A})) \Rightarrow \mathcal{F}(\mathcal{A} \cup \mathcal{B}), \tag{51}$$

which is almost trivial; indeed by (23), $\mathbb{F}(|\mathcal{A}|) \vee \mathbb{F}(|\mathcal{B}|) \leq \mathcal{F}(|\mathcal{A}| \vee |\mathcal{B}|)$ iff $|\mathbb{F}(\mathcal{A}) \cup \mathcal{F}(\mathcal{B})| \leq |\mathcal{F}(\mathcal{A} \cup \mathcal{B})|$, which is equivalent to (51).

Proposition 2. Let $\mathcal{A}, \mathcal{B} \in \mathbb{W}$, $\mathcal{F} \in \mathbb{O}$. Then

$$\vdash (\mathcal{F}(\mathcal{A}) \cup \mathcal{F}(\mathcal{B})) \Leftrightarrow \mathcal{F}(\mathcal{A} \cup \mathcal{B}), \tag{52}$$
$$\vdash (\mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{F}(\mathcal{B})) \Leftrightarrow \mathcal{F}(\mathcal{A} \Rightarrow \mathcal{B}), \tag{53}$$
$$\vdash (\mathcal{F}^*(\mathcal{A}) \cap \mathcal{F}^*(\mathcal{B})) \Leftrightarrow \mathcal{F}^*(\mathcal{A} \cap \mathcal{B}) \tag{54}$$

are LPC+Ch theorems if and only if $\mathcal{F}$ is identity modifier.

Proof. $\vdash (\mathcal{I}(\mathcal{A}) \cup \mathcal{I}(\mathcal{B})) \Leftrightarrow \mathcal{I}(\mathcal{A} \cup \mathcal{B})$ iff $\mathbb{I}(|\mathcal{A}|) \vee \mathbb{I}(|\mathcal{B}|) = \mathbb{I}(|\mathcal{A}| \vee |\mathcal{B}|)$, which is the case by (1). Conversely, let (52) hold, $\mathcal{A} \in \mathbb{W}$, $\mathcal{B} = \neg\mathcal{A}$, $\mathcal{F} \in \mathbb{O}$. Then, it holds that $\mathbb{F}(|\mathcal{A} \cup \neg\mathcal{A}|) = |\mathcal{A} \cup \neg\mathcal{A}| = |1|$. By (52), we have that $\vdash (\mathcal{F}(\mathcal{A} \cup \neg\mathcal{A}) \Leftrightarrow (\mathcal{F}(\mathcal{A}) \cup \mathcal{F}(\neg\mathcal{A}))$ iff $\mathbb{F}(|\mathcal{A} \cup \neg\mathcal{A}|) = \mathbb{F}(|\mathcal{A}|) \vee \mathbb{F}(|\neg\mathcal{A}|)$ iff $\mathbb{F}(|\mathcal{A}|) \vee \mathbb{F}(|\neg\mathcal{A}|) = |1|$ iff $\neg\mathbb{F}(\neg|\mathcal{A}|) \wedge \neg\mathbb{F}(|\mathcal{A}|) = |0|$ iff $\mathbb{F}^*(|\mathcal{A}|) \leq \neg\neg\mathbb{F}(|\mathcal{A}|) = \mathbb{F}(|\mathcal{A}|))$. But because $\mathbb{F}(|\mathcal{A}|) \leq |\mathcal{A}| \leq \mathbb{F}^*(|\mathcal{A}|)$, we conclude that $\mathbb{F}(|\mathcal{A}|) = |\mathcal{A}|$. The last condition is equivalent to $\vdash \mathcal{F}(\mathcal{A}) \Leftrightarrow \mathcal{A}$, which is (AxId). Hence, $\mathcal{F}$ is identity modifier.

It is easy to see that if $\mathcal{F}$ is identity modifier, then (53) holds. We have to establish only

$$\text{if } \vdash (\mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{F}(\mathcal{B})) \Rightarrow \mathcal{F}(\mathcal{A} \Rightarrow \mathcal{B}), \text{ then } \mathcal{F} \text{ is identity.} \tag{55}$$

This is because $\vdash \mathcal{F}(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{F}(\mathcal{B}))$ holds by (18). Let $\mathcal{A}, \mathcal{B} \in \mathbb{W}$, $\mathcal{F} \in \mathbb{O}$, be such that $\vdash \neg\mathcal{B}$. Then $\neg|\mathcal{B}| = |\neg\mathcal{B}| = |1|$ and $\mathbb{F}(|\mathcal{B}|) = |\mathcal{B}| = |0|$. We have $\vdash (\mathcal{F}(\mathcal{A}) \Rightarrow \mathcal{F}(\mathcal{B})) \Rightarrow \mathcal{F}(\mathcal{A} \Rightarrow \mathcal{B})$ iff $\mathbb{F}(|\mathcal{A}|) \to \mathbb{F}(|\mathcal{B}|) \leq \mathbb{F}(|\mathcal{A}| \to |\mathcal{B}|)$ iff $\mathbb{F}(|\mathcal{A}|) \to |0| \leq \mathbb{F}(|\mathcal{A}| \to |0|)$ iff $\neg\mathbb{F}(|\mathcal{A}|) \leq \mathbb{F}(\neg|\mathcal{A}|)$ iff $\neg\mathbb{F}(\neg|\mathcal{A}|) \leq \mathbb{F}(|\mathcal{A}|)$ iff $\mathbb{F}^*(|\mathcal{A}|) \leq \mathbb{F}(|\mathcal{A}|)$. Again we conclude that $\mathcal{F}$ fulfils (AxId), i.e., is identity modifier. (54) the the dual statement of (52).

□

Proposition 3. Let $\mathcal{A} \in \mathbb{W}$, $\mathcal{F} \in \mathbb{O}$. Then

$$\vdash \mathcal{A} \text{ if and only if } \vdash \mathcal{F}(\mathcal{A}), \tag{56}$$
$$\text{if } \vdash \mathcal{A}, \text{ then } \vdash \mathcal{F}^*(\mathcal{A}), \tag{57}$$
$$\text{if } \vdash \mathcal{A}, \text{ then } \vdash \mathcal{F}(\mathcal{A}) \cap \mathcal{F}^*(\mathcal{A}), \tag{58}$$
$$\text{if } \vdash \mathcal{A}, \text{ then } \vdash \mathcal{F}(\mathcal{A}) \cap \mathcal{A}, \tag{59}$$
$$\text{if } \vdash \mathcal{A}, \text{ then } \vdash \mathcal{A} \cap \mathcal{F}^*(\mathcal{A}). \tag{60}$$

Proof. Let $\vdash \mathcal{F}(\mathcal{A})$. Then $|1| = |\mathcal{F}(\mathcal{A})| = \mathbb{F}(|\mathcal{A}|) \leq |\mathcal{A}| \leq |1|$. Hence, $|\mathcal{A}| = |1|$, i.e., $\vdash \mathcal{A}$. The other part of (56) is rule (RS). To establish (57), assume $\vdash \mathcal{A}$. Then $|\mathcal{A}| = |1|$, so $|\neg\mathcal{A}| = \neg|\mathcal{A}| = |0|$ and, hence, $\mathcal{F}(\neg|\mathcal{A}|) = |0|$, or, equivalently, $\neg\mathcal{F}(\neg|\mathcal{A}|) = |1|$. The last condition is equal to $|\mathcal{F}^*(\mathcal{A})| = |1|$, which implies $\vdash \mathcal{F}^*(\mathcal{A})$. To demonstrate (58), let $\vdash \mathcal{A}$. By (56), (57), $\vdash \mathcal{F}(\mathcal{A})$, $\vdash \mathcal{F}^*(\mathcal{A})$. Hence, $|\mathcal{A}| = |\mathcal{F}(\mathcal{A})| = |\mathcal{F}^*(\mathcal{A})| = |1|$, which implies $|\mathcal{F}^*(\mathcal{A}) \cap \mathcal{F}(\mathcal{A})| = |\mathcal{F}^*(\mathcal{A})| \wedge |\mathcal{F}(\mathcal{A})| = |1|$. We conclude that $\vdash \mathcal{F}(\mathcal{A}) \cap \mathcal{F}^*(\mathcal{A})$. So (58) holds. Similarly we demonstrate (59) and (60).

□

Corollary. If $\neg$ is a modifier or a dual of a modifier, then LPC+Ch is inconsistent. (Note that the third last line on page 195 in [2] stating that LPC+Ch Calculus is consistent is not correct.)

4. CONCLUSION

We have studied fuzzy logic inference system called LPC+Ch Calculus from a purely mathematical point of view. Theorem 1 states that the algebra of LPC+Ch is an ET-algebra, hence, all the problems in LPC+Ch can be solved algebraically. This does not, however, make LPC+Ch Calculus nonproblematic. Since no formal semantics for LPC+Ch Calculus, nor a list of modifiers (the basic notion!) is given, we do not have any *mathematical* definition for such basic concepts as modifier operator or hedge (this criticism partly applies to Rhodes's and Menani's study [4], too). They rest only on intuitive ideas (for this kind of treatment, see [3], pp. 144). To see what kind of difficulties arise let us take three examples. Lakoff [1] and Zadeh [6] studied hedges, too, 'anything but', 'very likely' and 'very unlikely' among them. Are they modifiers also in LPC+Ch Calculus? If so, then e. g.

'very likely($\mathcal{A}$implies$\mathcal{A}$) and very unlikely ($\mathcal{A}$implies$\mathcal{A}$)',
'($\mathcal{A}$implies$\mathcal{A}$) and anything but($\mathcal{A}$implies$\mathcal{A}$)'

are deducible statements in LPC+Ch, which sounds very odd. In the study of Rhodes and Menani [4] 'not' is considered as a modifier. The acceptance of this modifier into LPC+Ch Calculus has, by the Corollary of Proposition 3, fatal consequences as

'$\mathcal{A}$ and non-$\mathcal{A}$'

is a deducible statement in LPC+Ch in that case. We conclude that LPC+Ch Calculus as well as fuzzy logic in general must have solid mathematical foundations if considered as non-classical mathematical logic.

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Esko Turunen, Lappeenranta University of Technology, P. O. Box 20, 53851, LPR. Finland.