Kybernetika

Karel Čulík II

Sequential machines with several input and output tapes

Kybernetika, Vol. 3 (1967), No. 6, (540)–559

Persistent URL: http://dml.cz/dmlcz/125014

KYBERNETIKA ČÍSLO 6, ROČNÍK 3/1967

Sequential Machines with Several Input and Output Tapes

Karel Čulík II

In the present paper the concept of Moore's sequential machine (with one input and one output tape) has been generalized to obtain the concept of a multitape sequential machine operating with m input and n output tapes. Results concerning the corresponding (m, n)-mapping are derived, the most essential of which concern the conditions that are to be satisfied by a given (m, n)-mapping so that there may be a multitape sequential machine of a certain type inducing or realizing them. Further, the problem of synthesis and minimization is being solved.

1. INTRODUCTION

The paper is an extension of Section 3 of the author's CSc-thesis [3]. Moore's sequential machines are generalized for m and n input and output tapes respectively. In this case, however, not the commonly used procedure is concerned in which several input (output) tapes are also considered, but they are being read and written upon simultaneously and therefore the abstract input symbols are coded by an m-tuple (n-tuple) of symbols of the so called structural alphabet.

In the paper sequential machines that read the input symbol from one input tape (determined by the internal state of the machine) and those writing the output symbol in each step of their operation on one output tape as well (determined by the internal state of the machine) will be considered. At first sequential machines that read one input symbol in each step and write one output symbol will be examined. Further examination will then concern sequential machines that can but not necessarily need read from tape or yield any output.

An actual example of the sequential machines described is a digital computer having several input and output channels.

We shall now introduce several less common terms and denotations or specify the meaning they will be assigned in this paper. Let Σ be a finite alphabet (a set of symbols). The set of all words over the alphabet will be denoted Σ^*, the n-ary relation over Σ is the set of n-tuples over the alphabet Σ. The set of all n-tuples over Σ will be denoted $(\Sigma)^n$.

If $p, q \in \Sigma^*$, $p = a_1, a_2, \ldots, a_s$, $q = b_1, b_2, \ldots, b_t$ $(a_i \in \Sigma,\ i = 1, 2, \ldots, s;\ b_i \in \Sigma,\ i = 1, 2, \ldots, t)$, then by concatenation of the words p and q the word $a_1 a_2 \ldots a_s b_1 b_2 \ldots \ldots b_t$ is meant being denoted $p \,.\, s$ or only juxtaposed ps.

Let $\mathbf{u} \in (\Sigma)^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} \in (\Sigma)^n$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, then by concatenation of the n-tuples $\mathbf{u}$ and $\mathbf{v}$ the n-tuple $\mathbf{uv} = (u_1 v_1, u_2 v_2, \ldots, u_n v_n)$ is meant.

Further an auxiliary operation $\oplus$ will be introduced generates from the m-tuple of words $\mathbf{u} \in (\Sigma)^m$, $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ and the n-tuples of words $\mathbf{v} \in (\Sigma)^n$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ an $(m + n)$-tuple $\mathbf{u} \oplus \mathbf{v} = (u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n)$. If in one and the same expression both the operation $\oplus$ and that of concatenation will be used, the operation $\oplus$ is assumed to be of higher priority. An empty word is denoted e, so is denoted the so called "empty" symbol $(e \notin \Sigma)$. The k-tuple formed by k empty words or k empty symbols is denoted e^k. Thus,

$$e^k = \underbrace{e \oplus e \oplus \ldots \oplus e}_{k\text{-times}} .$$

If $pq = r$, $p, q \in \Sigma^*$, then p is said to be the initial section of the word r and we write $p \leqq r$; if in addition $q \neq e$, then p is said to be the proper initial section of the word r and we write $p \prec r$.

The k-tuple of words $\mathbf{u} = (u_1, u_2, \ldots, u_k)$ is said to be the initial section of the k-tuple of words $\mathbf{v} = (v_1, v_2, \ldots, v_k)$ and written $\mathbf{u} \leqq \mathbf{v}$ if the word u_i is the initial section of the word v_i for $i = 1, 2, \ldots, k$. In the addition $\mathbf{u} \neq \mathbf{v}$, then u is said to be the proper initial section and we write $\mathbf{u} \prec \mathbf{v}$.

The length of an empty word is zero, the length of a non-empty word $p = a_1 a_2 \ldots \ldots a_s$ $(a_i \in \Sigma,\ i = 1, 2, \ldots, s)$ is s. $\lambda(p)$ denotes the length of p. The length of a k-tuple $\mathbf{u} = (u_1, u_2, \ldots, u_k)$ is $\lambda(\mathbf{u}) = \lambda(u_1) + \lambda(u_2) + \ldots + \lambda(u_k)$.

Moore's sequential machine with the input alphabet Σ_1 and the output alphabet Σ_2 is a system (S, φ, ψ, s_0), where S is a finite set of states, φ is a transition function (generally partial) from $S \times \Sigma_1$ to S, ψ is the output function from S to Σ_2, s_0 the initial state $(s_0 \in S)$. If the input as well as the output alphabet is Σ the sequential machine is said to be over the alphabet Σ. Since we admit the existence of sequential machines with partially defined transition function the sequential machine having the input alphabet Σ_1 and the output alphabet Σ_2 respectively can be considered a sequential machine over the alphabet $\Sigma_1 \cup \Sigma_2$ or over any extension of it. Further on, unless explicitly stated otherwise, all sequential machines will be considered to be over some definitely chosen alphabet Σ.

Definition 1. A multitape sequential machine $\mathscr{M}$ of the class $\mathfrak{A}_{mn}$ will be the term for a system $(S, \varphi, \psi, s_0, V_1, \ldots, V_n, W_1, \ldots, W_m)$, where (S, φ, ψ, s_0) is a common Moore's sequential machine, S being a final set of states, φ a single-valued (general partial) transition function from $S \times \Sigma$ to S, ψ a single-valued output function from S to Σ and $V_1, \ldots, V_n, W_1, \ldots, W_m$ two decompositions of the set S of states, therefore $V_i \cap V_j = \emptyset$ for $i \neq j$; $i, j = 1, 2, \ldots, n$; $\bigcup_{i=1}^{n} V_i = S$; $W_i \cap W_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, m$; $\bigcup_{i=1}^{m} W_i = S$. The above decompositions mean the following: If $s_k \in V_i$, $s_k \in W_j$, then the sequential machine being in the state s_k yields an output symbol on the i-th output tape and accepts the next output symbol from the j-th output tape.

Definition 2. By the mapping from the set of m-tuples of words into the set of n-tuples of words over an alphabet Σ briefly (m, n)-mapping induced by a sequential machine $M \in \mathfrak{A}_{mn}$, $\mathscr{M} = (S, \varphi, \psi, s_0, V_1, \ldots, V_n, W_1, \ldots, W_m)$ we mean a single-valued (m, n)-mapping f with the domain P defined in the following way:

(i) $e^m \in P$, $f(e^m) = e^n$;

(ii) if $\boldsymbol{u} \in (\Sigma^*)^m$, $\boldsymbol{u} \neq e^m$, $\boldsymbol{v} \in (\Sigma^*)^n$, then $\boldsymbol{u} \in P$ and $f(\boldsymbol{u}) = \boldsymbol{v}$ if and only if there exists a sequence $\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_t$ $(\boldsymbol{u}_i \in (\Sigma^*)^m)$ and a sequence $s_1, s_2, \ldots, s_t$ $(s_i \in S)$ such that $\boldsymbol{u}_0 = e^m$; $\boldsymbol{v}_0 = e^n$; $\boldsymbol{u}_t = \boldsymbol{u}$; $\boldsymbol{v}_t = \boldsymbol{v}$, $\boldsymbol{u}_i = \boldsymbol{u}_{i-1} \,.\, e^{k_{i-1}-1} \oplus a_i \oplus e^{m-k_{i-1}}$, $\boldsymbol{v}_i = \boldsymbol{v}_{i-1} \,.\, e^{p_i-1} \oplus b_i \oplus e^{n-p_i}$, $s_i \in W_{k_i}$, $s_i \in V_{p_i}$, $\varphi(s_{i-1}, a_i) = s_i$, $\psi(s_i) = b_i$ for $i = 1, 2, \ldots, t$.

Note 1. When comparing the sequential machines of class $\mathfrak{A}_{mn}$ with the automata of class G_n of [4] we can see that if the output function of the machine $\mathscr{M} \in \mathfrak{A}_{mn}$ is neglected and all its states are held for terminal ones, we obtain an automaton of class G_n which defines a relation formed by the very sum of all m-tuples falling within the domain P of the mapping induced by the machine $\mathscr{M}$.

Definition 3. The m-tuples of P will be formed admissible input m-tuples of the sequential machine $\mathscr{M}$.

The term sequential mapping introduced for (1,1)-mapping in [5] will now be generalized for the (m, n)-mapping.

Definition 4. By the sequential (m, n)-mapping the (m, n)-mapping satisfying the following conditions is considered.

1. The sequential (m, n)-mapping is a single-valued (generally partial) mapping from the set of m-tuples over the alphabet Σ into a set of n-tuples over the alphabet Σ.

2. The domain of the (m, n)-mapping has to satisfy the following conditions:

(i) $e^m \in P$ (e^m being an m-tuple of empty words)

(ii) If $\boldsymbol{u} \in P$, $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ and $\lambda(\boldsymbol{u}) = d$, then there exists k, $1 \leqq k \leqq m$

such that for all m-tuples $\mathbf{v} = (v_1, v_2, \ldots, v_m)$, $\mathbf{v} \in P$, for which $\mathbf{u} \prec \mathbf{v}$, $\lambda(\mathbf{v}) = d + 1$ it is valid that $u_i = v_i$ for all $i \neq k$. (All the m-tuples are formed by the extension of the same word.)

(iii) If $\mathbf{u} \in P$ and $\lambda(\mathbf{u}) = d$, $d > 0$, then there exists only one m-tuple $\mathbf{v}$ for which $\mathbf{v} \in P$, $\mathbf{v} \prec \mathbf{u}$, $\lambda(\mathbf{v}) = d - 1$.

3. Let $\mathbf{u} \in P$, then $\lambda(f(\mathbf{u})) = \lambda(\mathbf{u})$. (The (m, n)-mapping keeps the length of the k-tuples.)

4. If $u_1, u_2 \in P$ and $\mathbf{u}_1 \prec \mathbf{u}_2$, then $f(\mathbf{u}_1) \prec f(\mathbf{u}_2)$.

Theorem 1. *The (m, n)-mapping induced by an arbitrary multitape sequential machine of class $\mathfrak{A}_{mn}$ is sequential (it satisfies the conditions 1–4).*

Proof. Consider an (m, n)-mapping f induced by sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$.

The first condition is satisfied as only deterministic sequential machines are considered.

The fulfilment of the condition 2 (i) results from the definition itself concerning the (m, n)-mapping induced by the sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$. The sequential machine $\mathcal{M}$ is deterministic, the state s_i to which the sequential machine is transferred by the m-tuple of words is, therefore, uniquely determined. The state s_i falls into a single class W_k that states from which tape the next input symbol is to be read. Any m-tuple in P having the initial section $\mathbf{u}$ and the length of which is by one unit longer than $\mathbf{u}$ must have the form $\mathbf{v} = \mathbf{u} \,.\, e^{k-1} \oplus a \oplus e^{m-k}$, where $a \in \Sigma$. Thus, the condition 2 (ii) is satisfied.

The condition 2 (iii) follows from the deterministic character of the sequential machine $\mathcal{M}$. The fulfilment of the conditions 3 and 4 is obvious.

Definition 5. (m, n)-mapping f is said to be realized by the sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$ if it is sequential and is a partial mapping of the (m, n)-mapping induced by the sequential machine $\mathcal{M}$.

Theorem 1 states the necessary conditions the (m, n)-mapping has to meet to be induced by a sequential machine of class $\mathfrak{A}_{mn}$. Now we shall try to find a condition satisfactory for the fact that the (m, n)-mapping could be realized or induced by a sequential machine of class $\mathfrak{A}_{mn}$.

Let an arbitrary sequential (m, n)-mapping f with the domain P be assigned a directed labelled graph G_f (edges and vertices being labelled as well) in the following way: Let each m-tuple $\mathbf{u} \in P$ be assigned one vertex (including the m-tuple e^m), the vertices be identified with them. The vertices of the graph G_f will be labelled by triples from $\Sigma \times P_n \times P_m$ and edges by symbols taken from Σ where P_k is a set of natural numbers being less or equal to k. If $\mathbf{u}, \mathbf{v} \in P$, $a, b \in \Sigma$, $\mathbf{v} = \mathbf{u} \,.\, e^{k-1} \oplus a \oplus \oplus e^{m-k}$ and $f(\mathbf{v}) = f(\mathbf{u}) \,.\, e^{p-1} \oplus b \oplus e^{n-p}$, then in the graph G_f there is an edge connecting the vertex $\mathbf{u}$ with vertex $\mathbf{v}$, this edge being labelled with the symbol a; the element at the end of the triple labelling the vertex $\mathbf{u}$ is the number k, the element at the beginning of the triple labelling the vertex $\mathbf{v}$ is the symbol b, the middle element in the triple being the number p. It should be mentioned that graph G_f is a tree.

From the conditions $1-4$ of the sequential mapping it follows that in this way each sequential (m, n)-mapping is uniquely assigned a directed labelled graph; the labelling, however, of all vertices within this graph is not complete. The vertex e^m (the root of the tree) has been labelled only with the element at the end of the triple; vertices from which no edge goes out (corresponding to the m-tuples from P no extension of which falls into P any longer) have only two elements of their label that are defined.

The labels of two vertices will be termed compatible if the respective elements of the labels are either equal or at least one of them is not defined.

The graph G_f will be assigned an undirected graph G'_f in the following way: The graph G'_f will have the same vertices as the graph G_f. Vertices $\mathbf{v}$ and $\mathbf{w}$ in the graph G'_f will not be connected by an edge if and only if the following conditions is fulfilled:

(i) Vertices $\mathbf{v}$ and $\mathbf{w}$ have compatible labels;

(ii) If from the vertices $\mathbf{v}$ and $\mathbf{w}$ (within the graph G_f) the paths $d = h_1, h_2, \ldots, h_s$ and $d' = h'_1, h'_2, \ldots. h'_s$ come out respectively the labels of the edge h_i and h'_i being equal for $i = 1, 2, \ldots, s$, then if the path d passes successively through the vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ as well as the path d' does through the vertices $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s$ respectively, the vertices $\mathbf{v}_i$ and $\mathbf{w}_i$ have compatible labels for $i = 1, 2, \ldots, s$.

Definition 6. Let the undirected graph G'_f have the finite chromatic number r (see [1]), then the sequential (m, n)-mapping f is said to have the finite weight r, otherwise it is said to have an infinite weight.

The above definition is a generalization of the operator weight of the $(1,1)$-mapping introduced by B. A. Trachtenbrot [8].

Theorem 2. *The sequential (m, n)-mapping can be realized by the sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$ with r number of states if and only if it has finite weight which is less or equal to r.*

Proof. 1. Let the sequential (m, n)-mapping f have the finite weight r. Then there exists a chromatic decomposition of the graph G'_f with r classes $R_1, \ldots, R_r$. Let a sequential machine $\mathcal{K} \in \mathfrak{A}_{mn}$, $\mathcal{K} = (\{R_1, \ldots, R_r\}, \varphi, \psi, R_s, V_1, \ldots, V_n, W_1, \ldots. W_m)$, be constructed where R_i, $i = 1, \ldots, r$ are the classes of the chromatic decomposition of the graph G'_f, the initial state of the sequential machine $\mathcal{K}$ being the class R_s of the decomposition R_i, which includes the vertex e^m of the graph G'_f. The transition function φ, the output function ψ and decompositions $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ have been chosen in the following way: If in the graph G'_f there is an edge from the vertex $\mathbf{v} \in R_i$ to the vertex $\mathbf{v}' \in R_j$, the edge being labelled with the symbol a, then $\varphi(R_i, a) = R_j$. If the vertex $\mathbf{v} \in R_i$ is labelled with (b, p, k), then $\psi(R_i) = b$, $R_i \in V_p$, $R_i \in W_k$. The choice of the graph G'_f and the properties of the chromatic decomposition guarantee that the transition function φ chosen as well as the output function ψ

are single-valued and the definitions of the decompositions $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ are meaningful.

Let us prove that the sequential machine $\mathscr{K}$ realizes the (m, n)-mapping f. Let the m-tuple $\mathbf{v}$ of length t fall within domain of the mapping f and $f(\mathbf{u}) = \mathbf{v}$. According to the condition 2 (iii) for the sequential mapping, there exists one and only one initial section of the m-tuple of length d for $d = 0, 1, \ldots, t - 1$, falling within the domain f. The sequence of these initial sections (vertices of graph G_f) be denoted by $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{t-1}$. Let $\mathbf{u}_t = \mathbf{u}$. Let a_i, b_i $(a_i, b_i \in \Sigma, i = 1, 2, \ldots, t)$ denote symbols and k_i, p_i integers for which the following conditions hold

$$\mathbf{u}_{i+1} = \mathbf{u}_i \,.\, e^{k_i - 1} \quad \oplus a_{i+1} \oplus e^{n-k_i} \qquad \text{for} \quad i = 0, 1, \ldots, t - 1 \,;$$

$$\mathbf{v}_{i+1} = \mathbf{v}_i \,.\, e^{p_{i+1}-1} \oplus b_{i+1} \oplus e^{m-p_{i+1}} \quad \text{for} \quad i = 0, 1, \ldots, t - 1 \,.$$

Then let us denote by $R_0, R_1, \ldots, R_t$ the sequence of classes of the chromatic decomposition of graph G_f (the states of the sequential machine $\mathscr{K}$) for which $\mathbf{u}_i \in R_i$. According to the definition of the sequential machine $\mathscr{K}$ $R_i \in V_{k_i}$ and $R_i \in W_{p_i}$ are valid for $i = 0, 1, \ldots, t - 1$ and $i = 1, 2, \ldots, t$ respectively; $\varphi(R_{i-1}, a_i) = R_i$; $\psi(R_i) = b_i$ for $i = 1, 2, \ldots, t$. If g denotes the (m, n)-mapping induced by the sequential machine $\mathscr{K}$, the conditions (i) and (ii) are satisfied so that the n-tuple $\mathbf{u}$ may fall within the domain of the (m, n)-mapping g and $g(\mathbf{u}) = \mathbf{v}$ be valid. The (m, n)-mapping f is, therefore, a partial mapping of the (m, n)-mapping g and is realized by the sequential machine $\mathscr{K}$.

2. Let there be a sequential machine $\mathscr{L} = (\{s_1, \ldots, s_q\}, \varphi, \psi, s_1, V_1, \ldots, V_n, W_1, \ldots \ldots, W_m)$, that realizes the (m, n)-mapping f.

Let an arbitrary vertex in graph G_f be assigned that state of the sequential machine $\mathscr{L}$ to which it is transferred from the initial state by an n-tuple of words over the alphabet corresponding to the vertex $\mathbf{v}$. Since the sequential machine is a deterministic one, each vertex of graph G_f is assigned one and only one state of the sequential machine $\mathscr{L}$ and thus the decomposition $M_1, \ldots, M_q$ of vertices of the graph G_f as well as those of the graph G_f' can be chosen in such a way that the arbitrary class M_i of decompositions will be generated by all the vertices that are assigned the state s_i of the sequential machine $\mathscr{L}$. For each two vertices of G_f' that belong to the same class of such a decomposition chosen conditions (i) and (ii) are clearly satisfied. The vertices, therefore, are not connected with an edge. Therefore, the decomposition $M_1, \ldots, M_q$ is a chromatic decomposition of graph G_f'. Thus the (m, n)-mapping f has the finite weight less or equal to q. From this it follows that the (m, n)-mapping having the finite weight r cannot be realized by a sequential machine $\mathscr{L} \in \mathfrak{A}_{mn}$ with the number of states less than r and an (m, n)-mapping having an infinite weight cannot be realized by any sequential machine of class $\mathfrak{A}_{mn}$.

Corollary 1. *All the (m, n)-mappings realized by the sequential machines from class $\mathfrak{A}_{mn}$ are sequential (m, n)-mappings having finite weights.*

Now, let us try to find a satisfactory condition for the possibility of inducing an (m, n)-mapping by a sequential machine of class $\mathfrak{A}_{mn}$ having r states.

Let us assume an arbitrary sequential mapping f. Then let us construct, by the above procedure a directed labelled graph G_f and in turn an undirected graph G''_f to it in the following way:

The graph G''_f has the same vertices as has the graph G_f. The vertices $\mathbf{v}$ and $\mathbf{w}$ within G''_f will not be connected by an edge if and only if the following conditions are satisfied:

(i′) the vertices $\mathbf{v}$ and $\mathbf{w}$ have compatible labels;

(ii′) for each path (within G_f) leading from the vertex $\mathbf{v}$ or $\mathbf{w}$ there exists a path leading from the vertices $\mathbf{w}$ or $\mathbf{v}$ respectively whose edges have the same label (even as to the order) and whose vertices have compatible labels.

Definition 7. If the graph G''_f constructed in the above manner has the finite chromatic number r (see [1]), the sequential (m, n)-mapping f is said to have the finite strong weight r, otherwise it is said to have an infinite strong weight.

Theorem 3. *If the sequential (m, n)-mapping f has the finite strong weight r, it also has the finite weight p and $p \leqq r$ is valid.*

Proof. It follows from the uniqueness of the (m, n)-mapping f (the deterministic character of the sequential machine) that the condition (ii′) is stronger than the condition (ii) thus G'_f being a subgraph (see [1]) of the graph G''_f. Therefore the chromatic number of G'_f is less or equal to the chromatic number of the graph G''_f.

Example 1. Let us assume an (m, n)-mapping having a finite weight $(r = 1)$ but an infinite strong weight.

Let us choose the alphabet $\Sigma = \{a, b\}$ and the (1,1)-mapping (further only "mapping") f with the domain P generated by all words over the alphabet Σ such that in each initial section of theirs more symbols a than symbols b occur. The mapping f map each word $\sigma \in P$ onto a word of the same length generated only by symbols a.

The mapping f is apparently of the weight 1 and is realized by a common Moore's sequential machine $\mathcal{M}$ ($\mathcal{M} \in \mathfrak{A}_{11}$), $\mathcal{M} = (\{s_0\}, \varphi, \psi, s_0)$, where $\varphi(s_0, a) = \varphi(s_0, b) = s_0$, $\psi(s_0) = a$.

At the same time the mapping f is of infinite strong weight since e.g. all the vertices in the graph G''_f which are corresponding to words generated solely by a are connected by edges.

By this example we can see that the notion of strong weight is nontrivial even for common sequential mappings (operators).

Theorem 4. *A sequential (m, n)-mapping f can be induced by a sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$ with r states if and only if it has finite strong weight which is less or equal to r.*

The proof is quite analogous to that of Theorem 2.

From Theorems 1 and 4 it immediately follows:

Corollary 2. *The (m, n)-mappings induced by sequential machines of class $\mathfrak{A}_{mn}$ are exactly all the sequential (m, n)-mappings that have finite strong weights.*

3. MINIMIZATION OF STATES OF MULTITAPE SEQUENTIAL MACHINES

The results known for common sequential machines (see [6]) will now be generalized for the sequential machines of class $\mathfrak{A}_{mn}$.

A system of sets $K_1, K_2, \ldots, K_p$ of states of a given sequential machine $\mathfrak{A}_{mn}$, $\mathcal{M} = (S, \varphi, \psi, s_0, V_1, \ldots, V_n, W_1, \ldots, W_m)$ will be termed a system of invariant classes of states of the sequential machine $\mathcal{M}$ if they satisfy the following conditions:

(i) Each state of the sequential machine $\mathcal{M}$ falls in one of the classes $K_1, K_2, \ldots$ $\ldots, K_p$ at least;

(ii) If two states, s_i and s_j, fall in one and the same class K_q, then they fall in the same class V_l and the same class W_k as well and $\psi(s_i) = \psi(s_j)$;

(iii) If s_i, s_j fall in the same class K_q and for any a $\varphi(s_i, a)$ and $\varphi(s_j, a)$ are defined, then $\varphi(s_i, a)$ and $\varphi(s_j, a)$ fall in the same class K_r (depending on the choice of class K_q and the input signal a).

Theorem 5. *To each system $K_1, K_2, \ldots, K_p$ of invariant classes of states of the sequential machine $\mathcal{M}$ a sequential machine $\mathcal{N}$ can be constructed which realizes a mapping induced by the sequential machine $\mathcal{M}$ and has p states.*

Proof. Let us choose $\mathcal{N} = (\{K_1, \ldots, K_p\}, \varphi^{\mathcal{N}}, \psi^{\mathcal{N}}, K_z, V_1^{\mathcal{N}}, \ldots, V_n^{\mathcal{N}}, W_1^{\mathcal{N}}, \ldots, W_m^{\mathcal{N}})$ where K_z is one of those classes to which the state s_0 belongs. The transition function $\varphi^{\mathcal{N}}$ is chosen in the following way: For an arbitrary class K_i and an arbitrary $a \in \Sigma$ the value $\varphi^{\mathcal{N}}(K_i, a)$ is not defined if the values $\varphi(s_j, a)$ for all states s_j falling in K_i are not defined. If for any state $s_j \in K_i$ the value $\varphi(s_j, a)$ is defined, so is defined the value $\varphi^{\mathcal{N}}(K_i, a)$. For this value any of the classes K_k, including all the states $\varphi(s_j, a)$ for $s_j \in K$ can be chosen.

The transition function $\psi^{\mathcal{N}}$ is chosen in the following way: If $s_j \in K_i$, $\psi(s_j) = b$, then $\psi^{\mathcal{N}}(K_i) = b$.

The decompositions $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ can be chosen as follows: If $s_j \in K_i$, $s_j \in V_p$, $s_j \in W_r$, then $K_i \in V_p^{\mathcal{N}}$, $K_i \in W_r^{\mathcal{N}}$. From the above conditions (i)–(iii) that are fulfilled by a system of invariant classes it follows that $\varphi^{\mathcal{N}}$, $\psi^{\mathcal{N}}$, $\{V_i : i = 1, 2, \ldots, n\}$, $\{W_i : i = 1, 2, \ldots, m\}$ are uniquely defined.

Analogous as in the case of a common sequential machine (see [6]) the sequential machine $\mathcal{N}$ can be proved to realize a mapping induced by the sequential machine $\mathcal{M}$.

4. ASSOCIATED SEQUENTIAL MACHINES AND MAPPINGS

In this section we shall be restricted, for technical reasons, to the sequential machines $\mathcal{M} \in \mathfrak{A}_{mn}$ for which $s_0 \in W_1$, i.e. such that read the first input symbol from the first input tape. In one actual case it does not mean any limitation of generality as the input tapes of the sequential machine can be, in any case, adequately renumbered. The class of sequential machines satisfying this condition will be denoted by A^1_{mn} ($A^1_{mn} \subset A_{mn}$). As for as sequential (m, n)-mappings are considered we assume them to satisfy an analogous condition, i.e. each m-tuple of length 1 falling within the definition field of the sequential (m, n)-mapping has one single nonempty symbol in the place of the first word (having the form $(a, e, e, \ldots, e)$).

In the graph G_f assigned to the (m, n)-mapping f (see section 2) the vertex e^m will be no longer labelled (by the section 2 in the given case, it would be always labelled with a the triple $(0, 0, 1)$.

Definition 8. Let $\mathcal{M} \in \mathfrak{A}^1_{mn}$, $\mathcal{M} = (S^{\mathcal{M}}, \varphi^{\mathcal{M}}, \psi^{\mathcal{M}}, s_0^{\mathcal{M}}, V_1, \ldots, V_n, W_1, \ldots, W_m)$ over the alphabet Σ. The sequential machine $\mathcal{M}$ and a common Moore's sequential machine $\mathcal{N}$ ($\mathcal{N} \in \mathfrak{A}_{11}$, $\mathfrak{A}_{11} = \mathfrak{A}^1_{11}$) $\mathcal{N} = (S^{\mathcal{N}}, \varphi^{\mathcal{N}}, \psi^{\mathcal{N}}, s_0^{\mathcal{N}})$ with the input alphabet Σ and the output alphabet $\Sigma \times P_n \times P_m$ (P_k being a set of natural numbers less or equal to k) are said to be mutually associated if the following is valid:

(i) $$S^{\mathcal{M}} = S^{\mathcal{N}} ;$$

(ii) $$\varphi^{\mathcal{M}} = \varphi^{\mathcal{N}} ;$$

(iii) $$s_0^{\mathcal{M}} = s_0^{\mathcal{N}} ;$$

(iv) $$s_i \in V_j, \; s_i \in W_k, \; \psi^{\mathcal{M}}(s_i) = b \quad \text{if and only if} \quad \psi^{\mathcal{N}}(s_i) = (b, j, k) .$$

Note 2. It is obvious that to a given sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$ one can easily find a uniquely defined associated Moore's sequential machine and vice versa.

Definition 9. The sequential (m, n)-mapping f over the alphabet Σ (Σ being both input and output alphabet) and the sequential mapping (1,1)-mapping g having the input alphabet Σ, and the output alphabet $\Sigma \times P_n \times P_m$ are said to be mutually associated if the directed graphs G_f and G_g assigned to then respectively are isomorfic the corresponding vertices being compatibly labelled and the corresponding edges being equally labelled.

Note 3. To a given (m, n)-mapping f an associated mapping g can be found and vice versa. The word $a_1, \ldots, a_t$ falls within the domain of mapping g and

$$g(a_1, \ldots, a_t) = (b_1, p_1, k_2)(b_2, p_2, k_3) \ldots (b_{t-1}, p_{t-1}, k_t)(b_t, p_t, \cdot)$$

if and only if the m-tuple $\mathbf{u}_i$ ($i = 1, 2, \ldots, t$), where

$$\mathbf{u}_0 = e^n , \; \mathbf{u}_i = \mathbf{u}_{i-1} \cdot e^{k_i - 1} \oplus a_i \oplus e^{m-k_i} \quad \text{for} \quad i = 1, 2, \ldots, t$$

falls within the definition field of the mapping f and the following is valid:

$$f(\mathbf{u}_i) = f(\mathbf{u}_{i-1}) \cdot e^{p_i - 1} \oplus b_i \oplus e^{n-p_i} \quad \text{for} \quad i = 1, 2, \ldots, t .$$

Theorem 6. *Let $\mathcal{M} \in \mathfrak{A}_{mn}$, $\mathcal{N} \in \mathfrak{A}_{11}$ be mutually associated sequential machines. Then the (m, n)-mapping f induced by the sequential $\mathcal{M}$ and the mapping g induced by the sequential machine $\mathcal{N}$ are mutually associated.*

Proof. Let an m-tuple $\mathbf{u}$ fall within the definition field of the (m, n)-mapping f, $f(\mathbf{u}) = \mathbf{v}$ being valid. Let us use denotations from Definition 9 and Note 3. By Definition 8 $S^{\mathcal{M}} = S^{\mathcal{N}}$, if therefore the sequence of states $s_1, \ldots, s_t$ from Definition 9 is considered as a sequence of states of the sequential machine $\mathcal{N}$, then it follows from Definitions 8 and 9 that the word $a_1 a_2 \ldots a_t$ falls within the domain of mapping g and $g(a_1 a_2 \ldots a_t) = b_1 b_2 \ldots b_t$.

Let us now assign the vertices of graphs G_f and G_g in such a way that to the vertex $\mathbf{u}$ of G_f the vertex $a_1 a_2 \ldots a_t$ of G_g corresponds (using the denotation of Definition 9).

There is an edge from the vertex $\mathbf{u}$ to the vertex $\mathbf{v}$ in G_f if and only if $\mathbf{v} = \mathbf{u} \, . \, e^{k_{t+1}-1} \oplus a_{t+1} \oplus e^{m-k_{t+1}}$ and it is labelled with a_{t+1}. In that particular case, however, the edge leads in G_g from the vertex $a_1 a_2 \ldots a_t$ to the vertex $a_1 a_2 \ldots a_{t+1}$ and is also labelled with a_{t+1}. From the condition (iv) of Definition 8 it follows that the labels of mutually corresponding vertices are compatible.

There is a good reason why we have introduced the notion of associated sequential machines and mappings: they make it possible to transfer the solution of problems concerning the analysis, synthesis and minimization of multitape sequential machines to the solution of analogous problems for the common Moore's sequential machines. Thus e.g. a sequential (m, n)-mapping f given and we search for the sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$ by which it is induced or realized, we can proceed in the following way: 1. We construct the mapping g associated to the (m, n)-mapping f. 2. We find the Moore's sequential machine $\mathcal{N}$ inducing or realizing the mapping g (the problem of ordinary synthesis). We construct the sequential machine $\mathcal{M} \in \mathfrak{A}_{mn}$ the associated sequential machine to which is the sequential machine $\mathcal{N}$.

Example 2. Let us now present an example of synthesis of a sequential machine realizing a sequential (2,2)-mapping f over the alphabet a, b defined by the Table 1. The associated mapping g is defined by the Table 2.

Table 1.

$\mathbf{u}$	$f(\mathbf{u})$	$\mathbf{u}$	$f(\mathbf{u})$
(b, e)	(e, b)	(ab, a)	(bba, e)
(b, a)	(e, ba)	(aa, e)	(ba, e)
(ba, a)	(b, ba)	(b, b)	(a, b)
(baa, a)	(ba, ba)	(bb, a)	(e, bab)
(a, e)	(b, e)	(bb, ab)	(a, bab)
(ab, e)	(bb, e)	(bb, aa)	$(e, baba)$

Table 2.

p	$g(p)$
b	$(b, 2, .)$
ba	$(b, 2, 2)\,(a, 2, .)$
baa	$(b, 2, 2)\,(a, 2, 1)\,(b, 1, .)$
$baaa$	$(b, 2, 2)\,(a, 2, 1)\,(b, 1, 1)\,(a, 1, .)$
a	$(b, 1, .)$
ab	$(b, 1, 1)\,(b, 1, .)$
aba	$(b, 1, 1)\,(b, 1, 2)\,(a, 1, .)$
aa	$(b, 1, 1)\,(a, 1, .)$
bb	$(b, 2, 2)\,(a, 1, .)$
bab	$(b, 2, 2)\,(a, 2, 1)\,(b, 2, .)$
$babb$	$(b, 2, 2)\,(a, 2, 1)\,(b, 2, 2)\,(a, 1, .)$
$baba$	$(b, 2, 2)\,(a, 2, 1)\,(b, 2, 2)\,(a, 2, .)$

Applying the already known methods we can construct a Moore's sequential machine $\mathscr{B}$ realizing the mapping g.

$$\mathscr{B} = \left(\{s_0, s_1, s_2, s_3, s_4\}, \varphi, \psi, s_0\right),$$

where φ and ψ are defined by Tables 3 and 4 respectively.

Table 3.

s	$\varphi(s, a)$	$\varphi(s, b)$
s_0	s_1	s_2
s_1	s_4	s_3
s_2	s_0	s_4
s_3	s_4	—
s_4	—	—

Table 4.

s	$\psi(s)$
s_0	$(a, 2, 1)$
s_1	$(b, 1, 1)$
s_2	$(b, 2, 2)$
s_3	$(b, 1, 2)$
s_4	$(a, 1, 1)$

Now we can easily find the (m, n)-sequential machine to which the sequential machine $\mathscr{B}$ is associated. It is sequential machine $\mathscr{A} = (\{s_0, s_1, s_2, s_3, s_4\}, \varphi, \psi^{\mathscr{A}}, s_0, \{s_1, s_3, s_4\}, \{s_0, s_2\}, \{s_0, s_1, s_4\}, \{s_2, s_3\})$, where the transition function is the same as that with the sequential machine $\mathscr{B}$ and the output function $\psi^{\mathscr{A}}$ is defined by the Table 5.

Table 5.

s	$\psi^{\mathscr{A}}(s)$
s_0	a
s_1	b
s_2	b
s_3	b
s_4	a

5. MULTITAPE SEQUENTIAL MACHINES OF CLASS $\mathfrak{B}_{mn}$

Definition 10. The multitape sequential machine of class $\mathfrak{B}_{mn}$ will be the term for a system $(S, \varphi, \psi, s_0, V_0, V_1, \ldots, V_n, W_0, W_1, \ldots, W_m)$, where S is a finite set of states, φ a single-valued (general partial) transition function, ψ a single-valued output function (from $S - V_0$ to Σ), $V_0, V_1, \ldots, V_n$; $W_0, W_1, \ldots, W_m$ being two decompositions of the set of states S, namely $V_i \cap V_j = \emptyset$ for $i \neq j$, $i, j = 0, 1, \ldots, n$; $\bigcup_{i=1}^{n} V_i = S$; $W_i \cap W_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \ldots, m$; $\bigcup_{i=1}^{m} W_i = S$. The decomposition have the following meaning: If $s_k \in V_i$ and $i \neq 0$, then the sequential machine being in the state s_k yields the output symbol $\psi(s_k)$ on the i-th output tape; if $s_k \in V_0$, then the sequential machine being in the state s_k does not yield any output symbol. If $s_k \in W_i$ and $i \neq 0$, then the sequential machine being in the state s_k reads the next input symbol from the i-th input tape; if $s_k \in W_0$, then the sequential machine being in the state s_k does not read any input symbol (the transition function is, for the states of W_0, defined independently of the input).

Note 4. The sequential machines of class $\mathfrak{B}_{mn}$ can be interpreted by assuming that there exist a ficticious input as well as a ficticious output tapes. If $s_k \in W_0$ the sequential machine is said to read an empty symbol from a fictitious input tape; if $s_k \in V_0$ the sequential machine is said to write a empty output symbol on a fictitious output tape. With this kind of interpretation the transition function φ is mapping from $S \times (\Sigma \cup \{e\})$ to S, the output function ψ is mapping of the set S to $\Sigma \cup \{e\}$.

Note 4 can be made use of in defining the (m, n)-mapping (generally multivalued) induced by the sequential machine $\mathscr{M} \in \mathfrak{B}_{mn}$.

In general case multivalued mapping f will be described by the relation F. We write $(\mathbf{u}, \mathbf{v}) \in F$, if $\mathbf{v}$ is one of the n-tuples assigned to the m-tuple $\mathbf{u}$.

The sequential machine $\mathscr{M} \in \mathfrak{B}_{mn}$ will be considered a sequential machine with $m + 1$ input and $n + 1$ output tapes of class $\mathfrak{A}_{mn}$ in the sense of Note 4; the $(m + 1, n + 1)$-mapping (over the alphabet $\Sigma \cup \{e\}$) induced by it will be denoted by g.

Definition 11. The (m, n)-mapping induced by the sequential machine $\mathscr{M} \in \mathfrak{B}_{mn}$ is generally multivalued (m, n)-mapping f given by the relation F in the following way:

Let $\mathbf{u} = (u_1, u_2, \ldots, u_m)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, $u_i \in \Sigma^*$, $v_i \in \Sigma^*$, then $(\mathbf{u}, \mathbf{v}) \in F$ if and only if there exist $u_0, v_0 \in \{e\}^*$ such that $g((u_0, u_1, \ldots, u_m)) = (v_0, v_1, \ldots, v_n)$.

Example 3. There will be an example of a $(2,1)$-mapping f over the alphabet $\Sigma = \{a, b, c, \ldots, z, \oplus, *\}$ with the domain P. For the pair of words (p, q) is $(p, q) \in P$ if and only if the number of occurrences of the symbol $\oplus$ in the word p is twice as large as that in the word q and

(i) $p \in L_1$, where the language $L_1 = L(G)$, G being a grammar (see [5]) $G = (\{a, b, \ldots, z, \oplus, *\}, \{S, A, Q, R, T\}, \{S \to SR, S \to SQ, S \to ST, T \to A, T \to TA, Q \to \oplus T \oplus, Q \to \oplus \oplus, R \to * T *, R \to * *, A \to a, A \to b, \ldots, A \to z\}, S)$;

(ii) $q = q_1 \oplus$, $q_1 \in (\Sigma - \{*\})^*$ or $q = e$.

Thus, the words $q \neq e$ can be written as follows: $q = q_1 \oplus q_2 \oplus \ldots \oplus q_t \oplus$, where $q_i \in (\Sigma - \{\oplus, *\})^*$.

The sections R (see grammar G) and Q will be termed comments and corrections respectively. Let the $(2,1)$-mapping f assign to the pair (p, q) a word $r \in (\Sigma - \{*, \oplus\})$ which is formed by omitting all comments in the word p and replacing the individual corrections in their turn by the sections $q_1, q_2, \ldots, q_t$ of the word q (t being the number of corrections). A sequential machine $\mathscr{M} \in \mathfrak{B}_{21}$ will be constructed that induces the $(2,1)$-mapping f. Let us use the denotations: $\Omega = \Sigma - \{\oplus, *\}$.

$$\begin{aligned}\mathscr{M} = (&\{s_0, s_1^\xi, s_2, s_3, s_4^\xi, s_5: \xi \in \Omega\}, \varphi, \psi, s_0, \{s_0, s_2, s_3, s_5\},\\ &\{s_1^\xi, s_4^\xi: \xi \in \Omega\}, \emptyset, \{s_0, s_1^\xi, s_2, s_5: \xi \in \Omega\}, \{s_3, s_4^\xi: \xi \in \Omega\}),\end{aligned}$$

where φ, ψ are defined in the following way:

$$\begin{aligned}
\varphi(s_0, \xi) &= s_1^\xi \quad \text{for} \quad \xi \in \Omega;\\
\varphi(s_0, *) &= s_2;\\
\varphi(s_0, \oplus) &= s_3;\\
\varphi(s_1^\xi, \eta) &= s_1^\eta \quad \text{for} \quad \xi, \eta \in \Omega;\\
\varphi(s_1^\xi, *) &= s_2 \quad \text{for} \quad \xi \in \Omega;\\
\varphi(s_1^\xi, \oplus) &= s_3 \quad \text{for} \quad \xi \in \Omega;\\
\varphi(s_2, \xi) &= s_2 \quad \text{for} \quad \xi \in \Omega;\\
\varphi(s_2, *) &= s_0;\\
\varphi(s_3, \xi) &= s_4^\xi \quad \text{for} \quad \xi \in \Omega;\\
\varphi(s_3, \oplus) &= s_5;
\end{aligned}$$

$$\begin{aligned}
\varphi(s_4^\xi, \eta) &= s_4^\eta \quad \text{for} \quad \xi, \eta \in \Omega\,;\\
\varphi(s_4^\xi, \oplus) &= s_5 \quad \text{for} \quad \xi \in \Omega\,;\\
\varphi(s_5, \xi) &= s_5 \quad \text{for} \quad \xi \in \Omega\,;\\
\varphi(s_5, \oplus) &= s_0\,;\\
\psi(s_1^\xi) &= \xi \quad \text{for} \quad \xi \in \Omega\,;\\
\psi(s_4^\xi) &= \xi \quad \text{for} \quad \xi \in \Omega\,.
\end{aligned}$$

A diagram of the sequential machine $\mathscr{M}$ for a simplified case $\Omega = \{a, b\}$ is presented in Fig. 1. The diagram of a sequential machine of class $\mathfrak{B}_{mn}$ is drawn in a similar way as that for Moore's sequential machines, each circlet representing a state s_i contains below the denotation of the state of the triple including an indication of the value output function $\psi(s_i)$ (if defined) and indices of the decomposition classes of p, k if $s_i \in V_p$, $s_i \in W_k$.

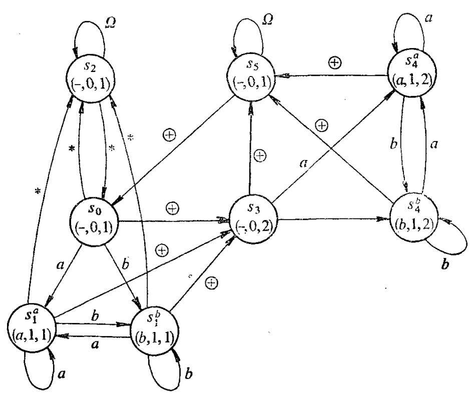

Fig. 1.

Definition 12. The weak sequential (m, n)-mapping will be the term for generally multivalued (m, n)-mapping satisfying the following conditions.

1. Let an (m, n)-mapping be defined by the relation F and have the domain P, $(P \in (\Sigma^*)^n,\ P = \operatorname{dom} F)$; then

(i) $e^m \in P$;

(ii) If $u \in P$, $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ and $\lambda(\mathbf{u}) = d$, then there exists k, $1 \leqq k \leqq m$ such that for all m-tuples $\mathbf{v} = (v_1, v_2, \ldots, v_m)$, for which $\mathbf{u} \prec \mathbf{v}$, $\lambda(\mathbf{v}) = d + 1$, $u_i = v_i$ for all $i \neq k$. (All m-tuples $\mathbf{v}$ are generated by extensions of the same word.)

(iii) If $\mathbf{u} \in P$ and $\lambda(\mathbf{u}) = d$, $d > 0$, then for each integer t, $0 \leqq t \leqq d - 1$ there is exactly one m-tuple $\mathbf{v} \in P$, $\mathbf{v} \prec \mathbf{u}$, $\lambda(\mathbf{v}) = t$.

2. Let be $(\mathbf{u}_1, \mathbf{v}_1) \in F$, $(\mathbf{u}_2, \mathbf{v}_2) \in F$, $(\mathbf{u}_1, \mathbf{v}_1) \neq (\mathbf{u}_2, \mathbf{v}_2)$.

(i) If $\boldsymbol{u}_1 \prec \boldsymbol{u}_2$, then $\mathbf{v}_1 \lesssim \mathbf{v}_2$;

(ii) If $\boldsymbol{u}_1 = \boldsymbol{u}_2$, then $\mathbf{v}_1 \lesssim \mathbf{v}_2$ or $\mathbf{v}_2 \prec \mathbf{v}_1$ and for each n-tuple $\mathbf{v}$, for which either $\mathbf{v}_1 \prec \mathbf{v} \prec \mathbf{v}_2$ or $\mathbf{v}_2 \prec \mathbf{v} \prec \mathbf{v}_1$ holds, also $(\boldsymbol{u}_1, \mathbf{v}) \in F$ is valid.

Theorem 7. *Each sequential (m, n)-mapping is a weak sequential one at the same time.*

Proof. The first condition for the weak sequential (m, n)-mapping is in accord to the second condition for the sequential (m, n)-mapping. The part (i) of the second condition of the weak sequential (m, n)-mapping is weaker than the fourth condition of the sequential (m, n)-mapping; it is not possible for the case (ii) to occur with the sequential (m, n)-mapping since according to the first condition must be single-valued.

Theorem 8. *Any (m, n)-mapping induced by an arbitrary sequential machine of class $\mathfrak{B}_{mn}$ is a weak sequential one.*

Proof. Let us consider a (m, n)-mapping f defined by the relation $F(P = \operatorname{dom} F)$ and induced by a sequential machine $\mathcal{M} \in \mathfrak{B}_{mn}$.

Now, the first condition will be dealt with. The fulfilment of (i) directly follows from the definition of the mapping induced by the sequential machine $\mathcal{M}$.

The sequential machine $\mathcal{M}$ is deterministic therefore the state s_i, to which the machine $\mathcal{M}$ is brought by the m-tuple $\boldsymbol{u}$ is uniquely defined. Let W_k $(k \neq 0)$ be the class of state decomposition, within which either the state s_i or the state $\varphi^p(s_i)$ falls if $s_i \in W_0$ and $\varphi^j(s_i) \in W_0$ for $j = 1, 2, \ldots, p - 1$ ($\varphi^j(s_i)$ denotes $\underbrace{\varphi(\varphi(\ldots \varphi(s_i) \ldots))}_{j\text{-times}}$. Thus, the sequential machine reads the following symbol from the k-th input tape, each m-tuple in P having the initial section $\boldsymbol{u}$ and being of the length by one longer than $\boldsymbol{u}$ must be $\mathbf{v} = \boldsymbol{u} \,.\, e^{k-1} \oplus a \oplus e^{k-m}$ where $a \in \Sigma$. So the part (ii) is satisfied.

The fulfilment of (iii) follows from the deterministic character of the sequential machine $\mathcal{M}$. The fulfilment of the second condition follows from the deterministic character of the sequential machine $\mathcal{M}$ and from the definition of mapping induced by it.

Definition 13. The (m, n)-mapping f is said to be realized by the sequential machine $\mathcal{M} \in \mathfrak{B}_{mn}$ if it is weak sequential and is a partial mapping of the (m, n)-mapping induced by the sequential machine $\mathcal{M}$.

Theorem 8 state the necessary conditions which must be satisfied for the weak sequential (m, n)-mapping to be induced by a sequential machine of class $\mathfrak{B}_{mn}$. Now, we shall find a condition satisfactory for the case (m, n)-mapping is to be realised or induced by a sequential machine of class $\mathfrak{B}_{mn}$.

Let an arbitrary weak sequential (m, n)-mapping f defined by the relation F be assigned a labelled graph G_f (a tree) with the set of vertices F; edges and vertices being labelled as well. From the vertex $(\boldsymbol{u}_1, \boldsymbol{v}_1)$ there leads an edge to the vertex $(\boldsymbol{u}_2, \boldsymbol{v}_2)$ if one of the conditions below is satisfied.

(i) $\boldsymbol{u}_1 = \boldsymbol{u}_2$ and there exist $p \in P_n$ and $b \in \Sigma$ such that

$$\boldsymbol{v}_2 = \boldsymbol{v}_1 \,.\, e^{p-1} \oplus b \oplus e^{n-p} ;$$

(ii) $\boldsymbol{v}_1 = \boldsymbol{v}_2$ and there exist $k \in P_m$ and $a \in \Sigma$ such that

$$\boldsymbol{u}_2 = \boldsymbol{u}_1 \,.\, e^{k-1} \oplus a \oplus e^{m-k} ;$$

(iii) there exist $k \in P_m$, $p \in P_n$, $a \in \Sigma$ such that

$$\boldsymbol{u}_2 = \boldsymbol{u}_1 \,.\, e^{k-1} \oplus a \oplus e^{m-k} , \quad \boldsymbol{v}_2 = \boldsymbol{v}_1 \,.\, e^{p-1} \oplus b \oplus e^{n-p} .$$

The edges of the graph are labelled with symbols from $\Sigma \cup \{e\}$, the vertices of the graph are labelled with triples (b, p, k) where $b \in \Sigma \cup \{e\}$, $p \in P_n$, $k \in P_m$. In the case of (i) the edge is labelled with an "empty" symbol e, the last element in the label of the vertex $(\boldsymbol{u}, \boldsymbol{v})$ being 0 a the first two elements in the label of the vertex $(\boldsymbol{u}_2, \boldsymbol{v}_2)$ being b, p.

In the case of (ii) the edge is labelled with the symbol a, the last element in the label of the vertex $(\boldsymbol{u}_1, \boldsymbol{v}_1)$ being k and the first two elements in the label of the vertex $(\boldsymbol{u}_2, \boldsymbol{v}_2)$ being e, 0.

In the case of (iii) the edge is labelled with the symbol a, the last element in the label of the vertex $(\boldsymbol{u}_1, \boldsymbol{v}_1)$ being k and the first two elements in the label of the vertex $(\boldsymbol{u}_2, \boldsymbol{v}_2)$ being b, p.

By the conditions 1, 2 each weak sequential (m, n)-mapping is uniquely assigned a directed labelled graph whose labels, however, are not completed (analogous as in section 2).

Let the graph G_f be assigned undirected graphs G'_f and G''_f in the same way as it was done in section 2 and let the validity of the definitions of weight and strong weight of the (m, n)-mapping be extended even to the weak sequential mappings.

Theorem 9. *The weak sequential (m, n)-mapping can be realized by the sequential machine $\mathcal{M} \in \mathfrak{B}_{mn}$ with r states if and only if it has finite weight which is less or equal to r.*

Proof. 1. Let a weak sequential mapping f defined by the relation F have the finite weight r. Then there exists a chromatic decomposition of the graph G'_f having r classes $R_1, R_2, \ldots, R_r$. Let us construct a sequential machine $\mathcal{K} \in \mathfrak{B}_{mn}$ interpreted in the sense of Note 4, $\mathcal{K} = (\{R_1, R_2, \ldots, R_r\}, \varphi, \psi, R_s, V_0, V_1, \ldots, V_n, W_0, W_1, \ldots, W_m)$ where R_i, $i = 1, 2, \ldots, r$ are classes of the chromatic decomposition, the initial state of the sequential machine being the class R_s of the decomposition $R_1, \ldots, R_r$ into which the vertex e^m of the graph G'_f falls. The transition function φ, the output function ψ and the decompositions $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ are chosen in the following manner. If in the graph G'_f there is an edge from the vertex $(\boldsymbol{u}_1, \boldsymbol{v}_1) \in R_i$ to the vertex

$(\mathbf{u}_2, \mathbf{v}_2) \in R_j$ and this edge is labelled with the symbol a, then for $a \neq e$ is $\varphi(R_i, a) = R_j$ hold or else $\psi(R_i) = R_j$. If the vertex $(\mathbf{u}_1, \mathbf{v}_2)$ is labelled with the triple (b, p, k), then $R \in V_p$, $R_i \in W_k$ and if $b \neq e$, then $\psi(R_i) = b$. If $b = a$, then $\psi(R_i)$ is not defined.

The choice of the graph G_f' and the properties of the chromatic decomposition guarantee that the transition function φ chosen as well as the output function ψ are single-valued and that the definition of the decomposition $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ is meaningful. Analogically as in the proof of Theorem 2 we can show that the sequential machine $\mathscr{K}$ does realize the (m, n)-mapping f.

2. Let there be a sequential machine $\mathscr{L} = (S, \varphi, \psi, s_1, V_0, V_1, \ldots, V_n, W_0, W_1, \ldots \ldots, W_m)$ having q states which realizes the weak sequential (m, n)-mapping f defined by the relation F. First, let $\mathscr{L}$ be modified to become the sequential machine $\mathscr{L}' = (S', \varphi', \psi', s_1', V_0', V_1', \ldots, V_n', W_0', W_1', \ldots, W_m')$ having q' states $(q' \leqq q)$ such that from the set of its states the states falling within $V_0 \cap W_0$ are skipped.

The initial state s_1', the transition function φ' and the output function ψ' will be defined in the following manner: If $s_1 \notin V_0 \cap W_0$, then $s_1' = s_1$, or else $s_1' = \varphi^k(s_1)$ where k is the smallest natural number for which $\varphi^k(s_1) \notin V_0 \cap W_0$.

If $s_i, s_j \notin V_0 \cap W_0$ and $\varphi(s_i, a) = s_j$ or $\varphi(s_i) = s_j$, then $\varphi'(s_i, a) = s_j$ or $\varphi'(s_i) = s_j$.

If $s_i, s_j \notin V_0 \cap W_0$, $s_{k_1}, s_{k_2}, \ldots, s_{k_p} \in V_0 \cap W_0$, $\varphi(s_i, a) = s_{k_1}$ or $\varphi(s_i) = s_{k_1}$, $\varphi(s_{k_t}) = s_{k_{t+1}}$ for $t = 1, 2, \ldots, p - 1$ and $\varphi(s_{k_p}) = s_j$, then $\varphi'(s_i, a) = s_j$ or $\varphi'(s_i) = s_j$. Let an arbitrary vertex $(\mathbf{u}, \mathbf{v})$ of the graph G_f' be assigned that state of the sequential machine $\mathscr{L}'$ in which $\mathscr{L}'$ is being found after having added the m-tuple $\mathbf{u}$ from the input tapes and written the n-tuple on the output tapes respectively. From the deterministic character of the sequential machine $\mathscr{L}'$ and from the fact it lacks states falling within $V_0 \cap W_0$ it follows that each vertex of the graph G_f is assigned one and only one state of the sequential machine L' and therefore, the decomposition $M_1, \ldots, \ldots, M_q$ of the vertices of the graph G_f and so of the graph G_f' can be chosen in the following manner: The class M_i is composed of exactly all the vertices which are assigned the state s_i of the sequential machine $\mathscr{L}'$. For each pair of vertices which fall within the same class of the decomposition M_i the conditions (i) and (ii) are obviously satisfied so that they be not connected with an edge in the graph G_f'. Thus, the decomposition $M_1, \ldots, M_q$ is a chromatic decomposition of the graph G_f'. The (m, n)-mapping f has, therefore, the finite weight $r \leqq q' \leqq q$. The weak sequential (m, n)-mapping having the finite weight r cannot be so realized by the sequential machine $\mathscr{L} \in \mathfrak{B}_{mn}$ with the number of states less than r and the weak sequential (m, n)-mapping having the infinite weight cannot be realized by any sequential machine of class $\mathfrak{B}_{mn}$.

Corollary 3. *The (m, n)-mappings realized by the sequential machines of class $\mathfrak{B}_{mn}$ are exactly all the weak sequential (m, n)-mappings having infinite weights.*

Theorem 10. *The weak sequential (m, n)-mapping f can be induced by the sequential machine $\mathscr{M} \in \mathfrak{B}_{mn}$ with r states if and only if it has finite strong weight which is less or equal to r.*

The proof is analogous to that of Theorem 9.

From Theorems 8 and 10 it follows:

Corollary 4. *The (m, n)-mappings induced by the sequential machines of class $\mathfrak{B}_{mn}$ are exactly all the weak sequential (m, n)-mappings having finite strong weights.*

Analogically as in the case of the sequential machines of class $\mathfrak{A}_{mn}$ even for the sequential machines of class $\mathfrak{B}_{mn}$ associated Moore's sequential machines can be introduced and analogous results can be proved for them.

6. CLOSURE OF THE (1,1)-FUNCTIONAL TRANSDUCTIONS UNDER INTERSECTION

Let us present one theoretical application of the results derived. In [5] the closure of some classes of binary relations under Boolean and other operations has been examined. From the classes and operations considered only the closure of the so called (1,1)-functional transductions under intersection has not been solved. (The corresponding place in the survey table has been filled by a question mark.)

The relation R is said to be a functional transduction if there is a sequential machine $\mathscr{A} \in \mathfrak{A}_{11}$ (Moore's sequential machine) such that $R = (u, v)$: $f(u) = v$, where f is the mapping induced by the sequential machine $\mathscr{A}$. If the mapping f is, in addition, one-to-one mapping, the relation R is said to be a (1,1)-functional transduction.

Theorem 11. *The class of functional transduction as well as the class of (1,1)-functional transductions are closed under intersection.*

Proof. Let R_1, R_2 be functional transductions. Then there exists sequential machines $\mathscr{A}_1 \in \mathfrak{A}_{11}, \mathscr{A}_2 \in \mathfrak{A}_{11}$ inducing the mappings f_1, f_2 with domains P_1, P_2 such that $R_1 = \{(u, v): u \in P_1, f(u) = v\}$, $R_2 = \{(u, v): u \in P_2, f(u) = v\}$.

Let us choose a mapping f_3 with the domain P_3 where

$$P_3 = \{u: u \in P_1 \cap P_2, f_1(u) = f_2(u)\} ;$$
$$f_3(u) = f_1(u) = f_2(u) \quad \text{for} \quad u \in P_3 .$$

Let us denote by $G''_1 = (X_1, U_1)$, $G''_2 = (X_2, U_2)$, $G''_3 = (X_3, U_3)$ the undirected graphs assigned according to section 2 to the mappings f_1, f_2, f_3 respectively. Obviously $X_3 \subset X_1$, $X_3 \subset X_2$ is valid. Furthermore, from the conditions for the construction of the graphs G''_1, G''_2, G''_3 it follows for $x \in X_3$, $y \in X_3$: If $(x, y) \notin U_1$ and $(x, y) \notin U_2$, then $(x, y) \notin U_3$. By negation of the preceding proposition we obtain: If $(x, y) \in U_3$, then $(x, y) \in U_1$ or $(x, y) \in U_2$.

The graph $G_3 = (X_3, U_3)$ is, therefore, a partial subgraph of the graph $G_4 = (X_1 \cup X_2, U_1 \cup U_2)$. If the chromatic numbers of the graphs G_1, G_2 are denoted by r_1, r_2 respectively, then the chromatic number of the graph G_4 and that of its partial subgraph G_3 as well is less or equal to $r_1 + r_2$.

From the definition of the mapping f_3 it follows that it is a sequential mapping and from the preceding expression it is clear that it has a finite proper weight. By Theorem 4 there exists a sequential machine $\mathscr{A}_3$ that induces the mapping f_3.

The relation $R_3 = R_1 \cap R_2 = \{(u, v): u \in P_3, f(u) = v\}$ is, therefore, a functional one and if, moreover, the relation R_1 and R_2 have been $(1,1)$-functional relations, even this feature has been satisfied.

(Received March 3rd, 1967.)

REFERENCES

[1] Berge C.: Theorie des graphes et ses applications. Paris 1958.

[2] Chomsky N.: On certain formal properties of grammars. Information and Control *2* (1959), 137—167.

[3] Čulík K. II.: Relace, jazyky a zobrazení representované zobecněnými automaty. CSc-Thesis, ÚTIA — Czechoslovak Academy of Sciences, Praha 1966.

[4] Čulík K. II.: Relace representované n-páskovými automaty. Kybernetika *3* (1967), 4, 321 až 345.

[5] Elgot C. C. and Mezei J. E.: On finite relation defined by generalized automata. IBM J. of Research and Development *9* (1965), 47—68.

[6] Глушков В. М.: Синтез цифровых автоматов. Киев 1964.

[7] Moore E. F.: Gedanken — experiments on sequential machines. In: Automata Studies, Princeton, 1956, 129—153.

[8] Трахтенброт Б. А.: Об операторах реализуетых в логических сетях. ДАН СССР *112* (1957), 6.

VÝTAH

Konečné automaty s několika vstupními a výstupními páskami

Karel Čulík II

V práci se zobecňují Mooreho automaty. Uvažují se automaty s m vstupními a n výstupními páskami, které v každém taktu své práce čtou vstupní symbol vždy z jedné vstupní pásky určené vnitřním stavem automatu a zapisují výstupní symbol na jednu z výstupních pásek rovněž určenou vnitřním stavem automatu. Uvažuje se i obecnější případ, kdy automat v některých taktech nemusí číst vstupní symbol nebo vydávat výstup.

Zavádí se pojem (m, n)-zobrazení (zobrazení m-tic slov na n-tice slov) indukovaného resp. realizovaného uvažovanými typy automatů a zkoumají se nutné a postačující podmínky, které musí splňovat takové (m, n)-zobrazení. Tyto podmínky jsou formulovány pomocí pojmu váhy resp. silné váhy (m, n)-zobrazení, která jsou zobecněním Trachtenbrotovy váhy operátoru.

Ukazuje se, jak lze převést úlohy syntézy a minimalizace vícepáskových automatů na obdobné úlohy pro obyčejné Mooreho automaty. Kromě toho se ukazuje i způsob přímé minimalizace stavů vícepáskového automatu.

RNDr Karel Čulík II, CSs., Matematická laboratoř ČVUT, Horská 4, Praha 2.