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On Nonblocking and Rearrangeable Communication Graphs

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Strictly nonblocking, widely nonblocking and rearrangeable communication graphs are characterized by means of graph homomorphisms based on Zelinka's tolerance relations on graphs. Channel-regular n-stage communication graphs are considered and some remarks for reducing such graphs are given.

1. INTRODUCTION

The purpose of this paper is twofold: We shall first characterize rearrangeable and nonblocking communication graphs by means of homomorphisms between graphs. Beneš has formerly characterized nonblocking communication graphs by means of semilattice mappings; the characterization here develops Beneš' results [1, 2] further. Secondly we shall consider constructing of communication network from the point of view of graph and lattice theories. In [7] Waller presented a graph-theoretic way of constructing n-stage communication graphs, called the fibred product of graphs. An n-stage graph is determined by three kinds of graphs: channel, terminal and path graphs. In fact we shall try to determine some properties of channel and terminal graphs when the corresponding n-stage graph is nonblocking. This is done by studying relations between the state semilattices of the n-stage graph and of terminal and channel graphs. As stated by Cattermole and Waller [3, 7], a wide class of communication graphs consists of regular n-stage graphs. This motivates the restriction to a narrow class of graphs here.

We shall first introduce the concepts and notations used here as briefly as possible. Secondly we give the characterization of rearrangeable and nonblocking graphs, and thereafter we finally consider interconnections between the state semilattices of an n-stage graph and its channel and terminal graphs.

By graph we shall mean an undirected connected graph $G = (V(G), E(G))$ without loops and multiple edges. Given any two graph projections $p_1 : G_1 \to G_3$ and $p_2 : G_2 \to G_3$, we define their fibred product to be the projection $p : G_1 \circ G_2 \to G_3$, where $G_1 \circ G_2$ has the vertex set $V(G_1 \circ G_2) = \{(v_1, v_2) \in V(G_1) \times V(G_2) : p_1(v_1) = = p_2(v_2)\}$, with adjacency $(v_1, v_2) \sim (v_1', v_2')$ whenever v_1 is adjacent to v_1' in G_1' and v_2 to v_2' in G_2. p is defined by $p(v_1, v_2) = p_1(v_1)$ which equals $p_2(v_2)$. The projections of fibred products are homomorphisms generated by tolerance relations on graphs; for tolerances and homomorphisms see Zelinka [8] and Nieminen [4], respectively.

An n-stage graph G is a graph witn a projection p onto the path graph P_n (the path of length n). Denoting the vertices of P_n by $u_1, u_2, \ldots, u_n$, we call the set $p^{-1}(u_r)$ the rth stage, $r = 1, \ldots, n$. Vertices constructing the 1st stage are inlets and those of the nth stage are outlets. The subgraph of G consisting of all paths of length n from an inlet v_1 to an outlet v_n is called a channel graphs $C(v_1, v_n)$. An n-stage graph G is called channel-regular, if the channel graphs of G are all isomorphic.

A terminal graph is a channel-regular n-stage graph with the connectivity equal to 1. Besides Waller [7] and Cattermole [3], channel-regular n-stage graphs are considered also by Takagi [6]. In [7] Waller characterized channel-regular n-stage graphs as follows:

Lemma 1. Let C be an n-stage channel graph and T an n-stage terminal graph with r inlets and s outlets. Then the fibred product $p : C \circ T \to P_n$ is a channel-regular n-stage graph with r inlets and s outlets. Further, every channel-regular multistage graph can be uniquely expressed as the fibred product of a terminal graph and a channel graph.

With any Hasse diagram of a finite semilattice L one can associate two kinds of directed graphs: one where the edges are directed downward and one where the edges are directed upward. There is a homomorphism α between two digraphs D_1 and D_2 of t-type, if the vertices $V(D_1)$ can be divided into vertex disjoint classes $C_1, C_2, \ldots, C_n$ such that:

(i) If $x, y \in C_i$ then $\Gamma x \cap C_j \neq \emptyset$ if and only if $\Gamma y \cap C_j \neq \emptyset$, $i \neq j$, and $\Gamma x \cap C_i = = \emptyset$ for each i.

(ii) The vertices of D_2 are the classes $C_1, \ldots, C_n$ of D_1, and there is a directed edge $\overrightarrow{C_i, C_j} \in E(D_2)$ if and only if for some $x \in C_i$ in D_1 it holds: $\Gamma x \cap C_j \neq \emptyset$.

This is the directed version of the homomorphisms of graphs based on tolerance relations on graphs introduced by Zelinka in [8].

A state x of a channel-regular n-stage graph G is a set of disjoint paths of length n, each path joining an inlet to an outlet. The set of the states of a G is partially ordered by the set inclusion $\leqq$, where $x \leqq y$ means that the state x can be obtained from the state y by removing zero or more calls. S is a meet-semilattice where $\wedge$ coincides with the set intersection and where $x \vee z$ exists whenever there is a state $y \geqq x, z$. As the states of G consist of sets of pairwise disjoint paths, then, if $y \leqq x$, there is a state $z \in S$ such that $y \wedge z = 0$ and $z \vee y = x$. The least element of S is the empty state.

An assignment is a specification of what inlets should be connected to what outlets in the G under consideration. Thus the set A of assignments represents of all fixed-point-free correspondences from the set of inlets to the set of outlets of G. The set A is partially ordered by the set inclusion, whence it is a meet-semilattice with the empty assignment as the least element. There is a natural map $\gamma : S \to A$ which takes each state $x \in S$ into the assignment it realizes. The mapping γ of S into A has the following properties: (i) $x \geqq y \Rightarrow \gamma(x) \geqq \gamma(y)$; (ii) $x \geqq y \Rightarrow$ there is a state $z \in S$ such that $z \wedge y = 0$, $\gamma(y) \wedge \gamma(z) = 0$ and $\gamma(z) \vee \gamma(y) = \gamma(x)$; (iii) $\gamma(x \wedge y) \leqq \gamma(x) \wedge v(y)$; (iv) $\gamma(x) = 0 \Rightarrow x = 0$.

Not every assignment need be realizable by some state of S. It is common for practical networks to realize only a small fraction of the possible assignments.

A subset $X \subset S$ is said to have the intersection property if and only if for every $x \in X$ and every $a \in A$ there exists $y \in X$ such that $\gamma(y) = a$ and $\gamma(x) \wedge \gamma(y) = \gamma(x \wedge y)$. The following results derived by Beneš [2] illuminate the concepts of non blocking.

Lemma 2. A communication graph G is nonblocking in the wide sense if and only if some subset $X \subset S$ has the intersection property.

Lemma 3. A communication graph G is strictly nonblocking if and only if S has the intersection property.

If u is not a member of the inlets of a state x and v not in the outlets of x, the pair (u, v) is called an idle pair of x. G is called rearrangeable, if for any $x \in S$ of G and any idle pair (u, v) of x there is a state $y \in S$ such that $\gamma(y)$ realizes the assignment $\gamma(x) \vee \{(u, v)\}$, i.e. $\gamma(y) = \gamma(x) \vee \{(u, v)\}$.

3. A CHARACTERIZATION OF COMMUNICATION GRAPHS

We characterize first rearrangeable graphs by means of graph homomorphisms.

Theorem 1. Let G be a given channel-regular n-stage graph, S and A its semilattices of states and assignments, respectively, and γ the mapping $S \to A$. G is rearrangeable if and only if γ is a t-type homomorphism: $D_S \to D_A$ and $\gamma(D_S) = D_A$, where D_S and D_A are directed graphs obtained from the Hasse diagrams of S and A by directing their edges downward.

Proof. The proof is valid also for communication graphs, where the intersection of inlets and outlets is the empty set.

Let γ be t-type homomorphism: $D_S \to D_A$ and $\gamma(D_S) = D_A$. Assume that (u, v) is an idle pair of a state $x \in S$. Then $\gamma(x) \vee \{(u, v)\} \in A$, and as $\gamma(D_S) = D_A$, there is a state $y \in S$ such that $\gamma(y) = \gamma(x) \vee \{(u, v)\}$. Hence G is rearrangeable.

Let G be rearrangeable. If $\gamma(x) \in A$ and if (u, v) is an idle pair of x, then $\gamma(x) \vee \vee \{(u, v)\} \in A$, and there is a state $y \in S$ such that $\gamma(y) = \gamma(x) \vee \{(u, v)\}$, as G is rearrangeable. By putting $x = 0$ (the empty state) and by applying the step reported above, one can show that each assignment is the image of a state of G under the mapping γ, i.e. $\gamma(D_S) = D_A$.

Let us consider $\gamma(x) \in A$. Each element $y \in \Gamma x$ in D_S is obtained by removing a path from x and this means the removal of the corresponding pair from $\gamma(x)$. Also conversely, each vertex $b \in \Gamma\, \gamma(x)$ in D_A is obtained from $\gamma(x)$ by removing a pair from $\gamma(x)$, and as x realizes (γx), there is in x a path for any pair of $\gamma(x)$. Thus there is a one-to-one correspondence between the sets Γx and $\Gamma\, \gamma(x)$, and this holds for any state $z \in S$ with the property $\gamma(z) = \gamma(x)$. The classification $C_1, \ldots, C_n$ of the vertices of D_S, where $x, y \in C_i \Leftrightarrow \gamma(x) = \gamma(y)$, and $\gamma(x) \neq \gamma(y)$, if $x \in C_i$, $y \in C_j$ and $i \neq j$, is clearly a classification satisfying the demands of a t-type homorphism between D_S and D_A. In this homomorphism $C_i = \gamma(x)$, when $x \in C_i$. Hence γ is a t-type homorphism: $D_S \to D_A$.

Theorem 2. Let G be a given channel-regular n-stage graph, S and A its semi-lattices of states and assignments, respectively, and γ the mapping $S \to A$. G is strictly nonblocking if and only if G is rearrangeable and γ is also a t-type homomorphism: $U_S \to U_A$ and $\gamma(U_S) = U_A$, where U_S and U_A are directed graphs obtained from the Hasse diagrams of S and A by directing all their edges upward.

Proof. The proof is valid also for communication graphs, where the intersection of inlets and outlets is the empty set.

Let G be strictly nonblocking. As shown in [1, Section 2 : 10], if G is strictly nonblocking, it is also rearrangeable, and thus it remains to show that γ is a t-type homomorphism: $U_S \to U_A$. Because $V(D_S) = V(U_S)$, $V(D_A) = V(U_A)$, and $\gamma(D_S) = = D_A$, also $\gamma(U_S) = U_A$ holds.

Let us consider a state $x \in S$. All the vertices Γx in U_S are obtained from x by adding an idle pair (u, v) of x to x. This corresponds to the adding of a new pair $\{(u, v)\}$ to $\gamma(x)$, and all the vertices $\gamma(x) \vee \{(u, v)\}$ belong to the set $\Gamma\, \gamma(x)$ in U_A. Hence for any vertex of Γx in U_S there is a vertex of $\Gamma\, \gamma(x)$ in U_A. On the other hand, if $b \in \Gamma\, \gamma(x)$ in U_A, it means that there is a pair $\{(u, v)\}$ not present in $\gamma(x)$ and hence (u, v) is an idle pair of x. As G is strictly nonblocking, there is a state $y > x$ in S such that $\gamma(y) = \gamma(x) \vee \{(u, v)\}$ [1, Section 2 : 10], and as $y > x$, it is obtained from x by adding a path joining u and v in G and being disjoint from those in x. Thus $y \in \Gamma x$ in U_S. Hence there is a one-to-one correspondence between the vertices of the sets Γx and $\Gamma\, \gamma(x)$ in U_S and U_A. As above, the classification $C_1, \ldots, C_n$ of the vertices of U_S, where $x, y \in C_i \Leftrightarrow \gamma(x) = \gamma(y)$, and $\gamma(x) \neq \gamma(y)$, if $x \in C_i$, $y \in C_j$ and $i \neq j$, is a classification satisfying the demands of a t-type homomorphism between U_S and U_A. This homomorphism is γ.

Conversely, let G be rearrangeable and γ satisfy the properties of the theorem. Let $x \in S$ and $a \in A$. a determines a class $C_i = \{z \mid z \in S \text{ and } \gamma(z) = a\}$ of D_S and U_S being a class under t-type homomorphism $\gamma : D_S \to D_A$ and $U_S \to U_A$. As A is a meet-semilattice, $a \wedge \gamma(x)$ exists, and according to the properties of semilattices, there is a shortest path P_1 between $\gamma(x)$ and $\gamma(x) \wedge a$, and P_2 beetwen a and $\gamma(x) \wedge a$ (P_1 and P_2 are not necessarily unique) in the Hasse diagram of A. As γ is a t-type homomorphism $D_S \to D_A$, there is a downward directed path from x to vertex w in D_S such that γ maps the vertices on this path onto the vertices on P_1 in D_A and in particular, $\gamma(w) = \gamma(x) \wedge a$. As γ is a t-type homomorphism $U_S \to U_A$, there is an upward directed path from w to a vertex y in U_S such that γ maps the vertices on this path onto the path P_2 in U_A, and thus $\gamma(y) = a$. According to the definition of the t-type homomorphisms and as P_1 and P_2 are shortest possible paths in the Hasse diagram of A, $y \wedge x = z$ in S, and thus y is an element of S satisfying the demand $\gamma(y \wedge x) = \gamma(z) = \gamma(x) \wedge a = \gamma(x) \wedge \gamma(y)$. Hence S has the intersection property and consequently (Lemma 3), G is strictly nonblocking.

Theorem 3. Let G be a given channel-regular n-stage graph, S and A its semilattices of states and assignments, respectively, and γ the mapping $S \to A$. G is widely nonblocking if and only if G is rearrangeable and there is an induced subgraph H of U_S such that γ is a t-type homomorphism $H \to U_A$ and $\gamma(H) = U_A$, where U_S and U_A are defined like in Theorem 2.

Proof. The proof is valid also for communication graphs, where the intersection of inlets and outlets is the empty set.

Assume that there is an induced subgraph H of U_S such that $\gamma(H) = U_A$ and $\gamma : H \to U_A$ is a t-type homomorphism. As $\gamma(H) = U_A$, H is an induced subgraph of U_S, and as $\gamma : D_S \to D_A$ is a t-type homomorphism, the vertices $V(H)$ induce in D_S a subgraph D_H such that $\gamma(D_H) = D_A$ and γ is a t-type homomorphism $D_H \to D_A$. According to Theorem 2, $V(H) \subseteq S$ is then a set having the intersection property, and thus G is widely nonblocking.

Conversely, let G be widely nonblocking. Beneš' results (Lemma 2) imply the existence of a set $X \subseteq S$ having the intersection property. Moreover, as shown in [2, Lemma 1], when X has the intersection property, then also the set $\bar{X} = \{y \mid y \in S \text{ and } y \leqq x \text{ for some } x \in X\}$ has the intersection property. According to [2, Theorems 4, 5 and 6], there is a strictly nonblocking graph having $\bar{X}$ as its state semilattice and realizing the assignments A of G. This shows that the elements of $\bar{X}$ induce a subgraph of U_S having the properties of the theorem.

In the undirected case Zelinka [8] has derived a matrix criterion for tolerances and the corresponding homomorphisms [5]. We hope that the directed version of this matrix criterion would offer a design tool when designing communication graphs which are rearrangeable or nonblocking.

Let $T = (V(T), E(T))$ and $C = (V(C), E(C))$ be the given terminal and channel graphs of an n-stage channel-regular communication graph, respectively, and let the corresponding sets of states be S_T and S_c. In S_T the empty state is the least state of T and the other states consist of pairwise disjoint paths of T having the length n between an inlet and an outlet; two paths are disjoint if they have no common vertices. If $x, y \in S_T$, $x \wedge y$ consists of paths contained in x and y. We define S_C so that there is no least element and the states of C consists of pairwise limitedly disjoint paths of C having the length n and joining the inlet and the outlet of C. Two paths of C are limitedly disjoint if they have no common vertices except the inlet and the outlet. If $x, y \in S_C$ and $x \wedge y$ exists (i.e. $x \cap y \neq \emptyset$), $x \wedge y$ consists of paths contained in x and y.

Lemma 4. Let T and C be the terminal and channel graphs of a given channel-regular n-stage graph G and S_T, S_C and S_G the corresponding sets of states ordered by the set inclusion. Then $G = T \circ C$ and each path P^G of G is the fibred product of corresponding paths P^T and P^C in T and C : $P^G = P^T \circ P^C$. Moreover, two paths $P_1^G = P_1^T \circ P_1^C$ and $P_2^G = P_2^T \circ P_2^C$ belong to the same state of G if and only if either (i) or (ii) holds:

(i) P_1^T and P_2^T are disjoint,

(ii) P_1^C and P_2^C are limitedly disjoint, $v_{i1} \neq v_{i2}$ and $v_{n1} \neq v_{n2}$, where v_{i1} and v_{n1} are the inlet and the outlet of P_1^T, and v_{i2} and v_{n2} those of P_2^T.

The proof follows directly from the definition of the fibred product and the fact that $G = T \circ C$.

It is now possible to determine also S_G as a product of S_T and S_C; the least element of S_G is the empty state.

Theorem 4. Let T and C be the terminal and channel graphs of a given channel-regular n-stage graph G. $x \in S_G$ if and only if there are different states $t_1. \ldots, t_m \in S_T$, $y_1, \ldots y_k \in S_C$, and an ordering of the paths in the set $\mathscr{P} = \{P_1^C, \ldots, P_m^C\}$ of all the paths of the states $y_1, \ldots, y_k$ such that $\{t_1 \circ P_1^C\} \cup \{t_2 \circ P_2^C\} \cup \ldots \cup \{t_m \circ P_m^C\}$ constitutes a set of pairwise disjoint paths of G, where $\{t_j P_{\circ j}^C\} = \{P_{jr}^T \circ P_j^C \mid P_{jr}^T$ is a path of the state t_j of $T\}, j = 1, \ldots, m$.

Proof. If $\{t_1 \circ P_1^C\} \cup \ldots \cup \{t_m \circ P_m^C\}$ contains only pairwise disjoint paths of G, it is a state of G. Let, conversely, x be a state of G, i.e. a set of pairwise disjoint paths of G of length n from an inlet to an outlet. As $G = T \circ C$, each path in x is a fibred product of the type $P^T \circ P^C$. Thus the paths in x can be divided into classes according to the path P^C in the product $P^T \circ P^C$: a class has the form $t \circ P^C$, where t consists of paths in T. As x is a state, the condition (i) in Lemma 4 implies that t is a state of T. Clearly the paths P^C can now be grouped into states of C, and the theorem follows.

Let $a_1, \ldots, a_m$ be all the maximal assignments of G in A_G and $\mathscr{P}^T$ and $\mathscr{P}^C$ the least families of paths in T and C, respectively, needed to realize the assignments $a_1, \ldots, a_m$; the states of S_G realizing $a_1, \ldots, a_m$ are made up by the paths in $\mathscr{P}^T$ and $\mathscr{P}^C$ as reported in Theorem 4. Remove all the edges not belonging to the paths of $\mathscr{P}^C$ from C and denote the resulting reduced graph by C_r; the reduced graph T_r is obtained analogously. Then already the communication graph $G_r = T_r \circ C_r$ has a set S of states realizing all the assignments $a_1, \ldots, a_m \in A_G$ as T_r and C_r contain all the paths needed to realize $a_1, \ldots, a_m$. As a set of states in S realizes all the maximal assignments of A_G, there is consequently at least one state $s \in S$ realizing an arbitrary assignment a, $a < a_i$ for some a_i, $i = 1, \ldots, m$. Thus Theorem 4 offers a way of reducing the number of edges in C, T and consequently in G.

If there are at least m states, $s_1, \ldots, s_m$ in S of a communication graph G realizing all the maximal elements $a_1, \ldots, a_m \in A$ of G, then, as $s \in S$ is a set of paths and $a \in A$ is a set of pairs, γ maps each path from s_i to 0 in D_S onto a path from a_i to 0 in D_A, $i = 1, \ldots, m$. Hence γ is a t-type homomorphism $D_S \to D_A$, $\gamma(D_S) = D_A$ and G is rearrangeable. Thus each graph G_r obtained by reducing T and C is rearrangeable.

Let S be the state semilattice of the reduced G_r, $\gamma(D_S) = D_A$, γ a t-type homomorphism $D_S \to D_A$, and let $s_1, \ldots, s_k \in S$ be all the maximal states of S. If γ maps each maximal element s_j of S to a maximal element a_i of A, then G_r is strictly nonblocking. According to Theorem 2, it suffices to show that γ is a t-type homomorphism $U_S \to U_A$. This property of γ follows from the facts that $\gamma(D_S) = D_A$, $\gamma : D_S \to D_A$ is a t-type homomorphism, and γ maps each maximal element s_j of S to a maximal element a_i of A. Thus the reduction offers also a way of obtaining strictly nonblocking graphs, if such exists.

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