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ON ASYMPTOTIC BEHAVIOUR OF EMPIRICAL PROCESSES

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The paper studies asymptotic behaviour of special type of empirical processes, inspired by specific properties of empirical distribution functions. The problem of convergence in distribution is discussed in Theorem 1. The relevant and more useful result of Theorem 2 establishes convergence in the space D. Section 2 discusses possible consequences if empirical distribution functions in a regression model are considered. An example indicates that the space D is not sufficient to study the convergence even if empirical distributions of simple processes are considered.

1. INTRODUCTION

Asymptotic properties of empirical distribution function based on i. i. d. random variables belongs to the classical results of the probability theory. A slight violation of independence as well as different distributions of random variables bring raise a lot of difficulties. Therefore, considerable interest is given to empirical distribution function based on linear regression residuals. Portnoy [7] discusses an approach based on regression quantiles and Miller [5] investigates methods using a $n^{\frac{1}{2}}$-consistent estimator under Gaussian errors. Nevertheless, convergence in the space $D(0,1)$ is a frequently used tool when properties of empirical distribution are studied. The aim of the present paper is a generalization of the above approaches. Connection between the empirical process (1) and an empirical distribution function in linear regression is discussed in Section 2.

Throughout the paper we shall use the following notations:

$\mathbb{R}$ denotes the set of real numbers.

$\mathbb{N}$ denotes the set of natural numbers.

$X_n \xrightarrow{d} X$ denotes convergence in distribution of processes X_n to the process X; i. e. all finite-dimensional distributions of the process X_n converge to the corresponding finite-dimensional distribution of the process X.

$X_n \to X$ in $D_d(0,1)$ denotes weak convergence of processes X_n to the process X in the $D_d(0,1)$ topology (cf. [6], [8] for definition and [1], [3] for verifying criteria).

Moreover, we will accept a convention that every convergence will be considered for n tending to infinity. Further, a distribution function and the corresponding measure will be denoted by the same symbol.

2. MAIN RESULTS

We shall consider empirical processes in the following form:

$$X_n(a,\varphi,F) = \sum_{i=1}^{n}\sum_{j=1}^{k(i)} a_{ij}\left(I(X_i < \varphi_{ij}(t)) - F(\varphi_{ij}(t))\right), \qquad (1)$$

where $a_{ij}\in\mathbb{R}$, $k(i)\in\mathbb{N}$, $\varphi_{ij} : \langle 0,1)^d \to \mathbb{R}^q$, $t\in\langle 0,1)^d$ and $X_1,\dots,X_n$ are i.i.d. q-dimensional random vectors with a common d.f. F, moreover $a = (a_{ij})_{i=1,j=1}^{n\ \ k(i)}$, $\varphi = (\varphi_{ij})_{i=1,j=1}^{n\ \ k(i)}$ and d.f. F are considered as parameters of the process X_n.

We will investigate a sequence $X_n(a_n,\varphi_n,F_n)$ of such processes namely its asymptotic behaviour. To simplify notations we shall use additional shorthand notations for parameters like that

$$a_n = (a_{ijn})_{i=1,j=1}^{n\ \ k(i,n)}, \qquad \varphi_n = (\varphi_{ijn})_{i=1,j=1}^{n\ \ k(i,n)} \text{ and } F_n.$$

In Sections 1 and 2 two types of convergence of empirical processes (1) are investigated. The proofs are postponed to Section 3.

Theorem 1. Let $X_n(a_n,\varphi_n,F_n)$ be a sequence of random processes possessing the following properties:

(i)
$$\max_{i=1,\dots,n}\sum_{j=1}^{k(i,n)} |a_{ijn}| \longrightarrow 0; \qquad (2)$$

(ii) there exists a finite limit

$$\sum_{i=1}^{n}\sum_{j,p=1}^{k(i,n)} a_{ijn}\, a_{ipn}\left(F_n(\varphi_{ijn}(t)\wedge\varphi_{ipn}(s)) - F_n(\varphi_{ijn}(t))F_n(\varphi_{ipn}(s))\right) \longrightarrow H(t,s) \qquad (3)$$

for every $s,\ t\in\langle 0,1)^d$.

Then $X_n(a_n,\varphi_n,F_n) \xrightarrow{d} W$, where W is a Gaussian process with zero means and covariance function H.

This result is interesting but not sufficient for investigating properties of the whole trajectories of the process, e.g. supremum or some integral of the process. In such a case we need to use a stronger convergence properties as, for example, convergence in the space $D_d\langle 0,1)$. To begin with, we shall need some definitions.

Definition 1. The function $\varphi :\ \langle 0,1)^d \longrightarrow \mathbb{R}^d$ is said to be coordinatewise nondecreasing and continuous if $\varphi(t) = (\varphi^1(t^1),\dots,\varphi^d(t^d))$ and $\varphi^1,\dots,\varphi^d :\ \langle 0,1) \longrightarrow \mathbb{R}$ are nondecreasing and continuous.

Definition 2. For $d\in\mathbb{N}$, Ψ_d denotes the set of all permutations of coordinates in $\langle 0,1)^d$.

Definition 3. For $d \in \mathbb{N}$; $r = 0, 1, \ldots, d-1$; $p = 0, 1, \ldots, d-r-1$ and $\psi \in \Psi_d$, define a σ-field

$$\mathcal{B}_{d,r,p,\psi} = \psi\left((-\infty, 0)^r \times (-\infty, 1)^p \times \mathcal{B}(\langle 0, 1)^{d-r-p})\right),$$

where $\mathcal{B}(\langle 0, 1)^{d-r-p})$ denotes the Borel σ-field on $\langle 0, 1)^{d-r-p}$.

Definition 4. For $\varphi : \langle 0, 1)^d \longrightarrow \mathbb{R}^d$ coordinatewise nondecreasing and continuous and $A = \mathop{\times}\limits_{i=1}^{d} \langle a^i, b^i) \in \mathcal{B}_{d,r,p,\psi}$ denote

$$\varphi(A) = \mathop{\times}\limits_{i=1}^{d} \langle \varphi^i(a^i), \varphi^i(b^i)\rangle,$$

with $\varphi^i(-\infty) = -\infty$.

The definition is necessary because the function φ is not introduced outside the set $\langle 0, 1)^d$. Thus an image of the set A must be explained.

Now we are in a position to formulate the result concerning convergence in the space $D_d(0, 1)$.

Theorem 2. Let $X_n(a_n, \varphi_n, F_n)$ be a sequence of processes such that $q = d$ and all φ_{ijn} are coordinatewise nondecreasing and continuous. Moreover, assume that there exist numbers $c_{ijn} \in \mathbb{R}$ and finite measures $\mu_{r,p,\psi}$ on $\mathcal{B}_{d,r,p,\psi}$ with continuous marginals, i. e. the functions $f(t) = \mu_{r,p,\psi}\left(\psi((-\infty, 0)^r \times (-\infty, 1)^p \times \langle 0, t) \times \langle 0, 1)^{d-r-p-1})\right)$ are continuous; such that:

(i)

$$F_n(\varphi_{ijn}(A)) \le c_{ijn}\mu_{r,p,\psi}(A) \qquad \text{for every } A = \mathop{\times}\limits_{u=1}^{d} \langle a^u, b^u) \in \mathcal{B}_{d,r,p,\psi}. \tag{4}$$

(ii) There exist $Q \in \mathbb{R}$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{k(i,n)} a_{ijn}^2 c_{ijn} \le Q, \; \sum_{i=1}^{n} \left(\sum_{j=1}^{k(i,n)} |a_{ijn}|\, c_{ijn} \right)^2 \le Q \qquad \text{for every } n \in \mathbb{N}. \tag{5}$$

(iii) For arbitrary j, $p = 1, \ldots, k(i,n)$, $j \ne p$ and

$$A = \mathop{\times}\limits_{u=1}^{d} \langle a^u, b^u), \; B = \mathop{\times}\limits_{u=1}^{d} \langle g^u, h^u), \; A,\, B \in \mathcal{B}_{d,r,p,\psi},\; F_n(\varphi_{ijn}(A) \cap \varphi_{ipn}(B)) = 0. \tag{6}$$

(iv)

$$\max_{i=1,\ldots,n} \sum_{j=1}^{k(i,n)} |a_{ijn}| \longrightarrow 0. \tag{7}$$

(v) There exists a finite limit

$$\sum_{i=1}^{n} \sum_{j=1}^{k(i,n)} a_{ijn}\, a_{ipn} \left(F_n(\varphi_{ijn}(t) \wedge \varphi_{ipn}(s)) - F_n(\varphi_{ijn}(t))F_n(\varphi_{ipn}(s))\right) \longrightarrow H(t, s) \tag{8}$$

for every t, $s \in \langle 0, 1)^d$.

Then $X_n(a_n, \varphi_n, F_n) \longrightarrow W$ in $D_d(0, 1)$, where W is a Gaussian process with zero means and covariance function H.

3. APPLICATIONS

Consider a regression model $Y_i = X_i\beta + e_i$, $i = 1, 2, \ldots$, where

$$\begin{aligned}&Y_1, Y_2, \ldots \text{ are the observed } k\text{-dimensional vectors,}\\&X_1, X_2, \ldots \text{ are non-random } k \times d\text{-real matrices,}\\&\beta \text{ is an unknown } d\text{-dimensional vector,}\\&e_1, e_2, \ldots \text{ are i.i.d. random } k\text{-dimensional vectors with a common}\\&\text{distribution function } F.\end{aligned} \tag{9}$$

The problem is to find an estimator of F. One possibility is the following

$$\hat{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} I\left[Y_i - X_i\hat{\beta}_n < t\right] = \frac{1}{n}\sum_{i=1}^{n} I\left[e_i < t + X_i(\hat{\beta}_n - \beta)\right], \tag{10}$$

$t \in \mathbb{R}^k$, where $\hat{\beta}_n$ is a consistent estimator of β. A study of such a process results in an examination of the process

$$F_n^*(t, s) = \frac{1}{n}\sum_{i=1}^{n} I\left[e_i < t + \eta(n)X_i s\right], \qquad t \in \mathbb{R}^k,\ s \in \mathbb{R}^d. \tag{11}$$

Specifically, the function η could be the rate of consistency of $\hat{\beta}_n$. The relation between this two processes is given by $\hat{F}_n(t) = F_n^*\left(t, \frac{1}{\eta(n)}(\hat{\beta}_n - \beta)\right)$.

Let us denote $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, and let ∇F and $\nabla^2 F$ be the gradient of F and the matrix of second derivatives of F, respectively.

The processes F_n^* have the required structure (1). Thus the previous theorems can be employed to derive some asymptotic properties of F_n^*. Case 1 and Case 2 show a comparison with the theoretical error distribution function F. Case 3 and Case 4 study the difference between F_n^* and an empirical distribution function based on errors. Necessary definitions are before Case 3.

Case 1. Let $\sup_{n=1,2,\ldots} \|X_n\| < +\infty$, $\lim_{n\to+\infty} \sqrt{n}\,\eta(n) = \tilde{\eta} \in \mathbb{R}$, $A \subset \mathbb{R}^k$ such that F has the total differential at every point of A and F is continuous at every point $t_1 \wedge t_2$; $t_1, t_2 \in A$. Then

$$\left(\sqrt{n}(F_n^*(t, s) - F(t)) - \tilde{\eta}\nabla F(t)\bar{X}_n s; t \in A, s \in \mathbb{R}^d\right) \xrightarrow{d} W, \tag{12}$$

where W is a Gaussian process with zero mean and covariance function

$$H(t_1, s_1; t_2, s_2) = F(t_1 \wedge t_2) - F(t_1)F(t_2).$$

Proof. Take $t_1, t_2 \in A$, $s_1, s_2 \in \mathbb{R}^d$ and consider the sum

$$\frac{1}{n}\sum_{i=1}^{n} \{F\left((t_1 + \eta(n)X_i s_1) \wedge (t_2 + \eta(n)X_i s_2)\right)$$

$$
\begin{aligned}
&- \quad F(t_1+\eta(n)X_is_1)F(t_2+\eta(n)X_is_2)\} = \\
&= \quad F(t_1\wedge t_2) - F(t_1)F(t_2) + \\
&+ \quad \frac{1}{n}\sum_{i=1}^{n}\{F((t_1+\eta(n)X_is_1)\wedge(t_2+\eta(n)X_is_2)) - F(t_1\wedge t_2)\} - \\
&- \quad \frac{1}{n}\sum_{i=1}^{n}(F(t_1+\eta(n)X_is_1) - F(t_1))\,F(t_2+\eta(n)X_is_2) - \\
&- \quad F(t_1)\frac{1}{n}\sum_{i=1}^{n}(F(t_2+\eta(n)X_is_2) - F(t_2)) = \\
&= \quad F(t_1\wedge t_2) - F(t_1)\,F(t_2) + o(1).
\end{aligned}
$$

Then, by Theorem 1,

$$
\left(\sqrt{n}\left(F_n^*(t,s) - \frac{1}{n}\sum_{i=1}^{n}F(t+\eta(n)X_is)\right)\,;\ t\in A,\ s\in \mathbb{R}^d\right) \xrightarrow{d} W.
$$

Theorem 1 can be used since there always exists a surjective map $\langle 0,1\rangle^{k+d} \longrightarrow A\times\mathbb{R}^d$ and the convergence in distribution refers to finite distributions only.

The proof is completed if we note that

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}F(t+\eta(n)X_is) = \sqrt{n}\,F(t) + \sqrt{n}\,\eta(n)\nabla F(t)\bar{X}_ns + \sqrt{n}\,o(\eta(n)) = \\
&= \quad \sqrt{n}\,F(t) + \tilde{\eta}\nabla F(t)\bar{X}_n(s) + o(1). \qquad \square
\end{aligned}
$$

In what follows we shall need a generalization of the space $D_d(0,1)$.

Definition 5. Let $A_1,\ldots,A_d \subset \mathbb{R}$ be any intervals and $Z, Z_1, Z_2, \ldots$ be stochastic processes indexed by $A_1\times\ldots\times A_d$. Then

$$
Z_n \longrightarrow Z \qquad \text{in } D(A_1\times\ldots\times A_d)
$$

iff there exist increasing surjective functions $\varphi_i:\ \langle 0,1\rangle \to \bar{A}_i^*,\ i=1,\ldots,d$ $\left(\bar{A}_n^* = \left\langle \inf_{a\in A_i} a,\ \sup_{a\in A_i} a\right\rangle\right)$, and extensions $\bar{Z}$, $\bar{Z}_1$, $\bar{Z}_2,\ldots$ of Z, Z_1, $Z_2,\ldots$ indexed by $\bar{A}_n^*\times\ldots\times\bar{A}_d^*$ such that

$$
\bar{Z}_n\circ\varphi \longrightarrow \bar{Z}\circ\varphi \qquad \text{in } D_d(0,1),
$$

where $\varphi = \varphi_1\times\ldots\times\varphi_d$.

Observe that if the process $(F_n^*(t,s);\ t\in A_1\times\ldots\times A_k,\ s\in\langle -1,1\rangle^d)$ belongs to $D(A_1\times\ldots\times A_k\times\langle -1,1\rangle^d)$ then $d=k$ and X_i are diagonal matrices with non-negative elements. Hence an approximation of F_n^* must be considered.

Case 2. Let $\sup_{i=1,2,\dots} \|X_i\| < +\infty$, $\lim_{n\to+\infty} \sqrt{n}\,\eta(n) = \tilde{\eta} \in \mathbb{R}$, $A_1, \dots, A_n \subset \mathbb{R}$ be any intervals, $\bar{A} = \bar{A}_1^* \times \dots \times \bar{A}_k^*$, $A = \bar{A} \cap \mathbb{R}^k$. Moreover, let ∇F be continuous on A and $\nabla^2 F$ bounded on a neighbourhood of A and

$$\text{if} \qquad \sup_{a\in A} \|a\| = +\infty \qquad \text{then} \qquad \lim_{\substack{a\in A\\ \|a\|\to+\infty}} \|\nabla F(a)\| = 0.$$

There exist $\varepsilon > 0$ and a finite Borel measure μ on $\mathbb{R}^k$ with continuous marginals such that $F(B+t) \le \mu(B)$ for every $t \in \mathbb{R}^k$, $\|t\| < \varepsilon$, $B = \mathsf{X}_{i=1}^{d} (\alpha_i, \beta_i)$, $\alpha_i = -\infty$ or $\alpha_i \in \bar{A}_i^*$, $\beta_i \in \bar{A}_i^*$, $\alpha_i < \beta_i$. Put $Q_{i,n}^j = \sum_{p=1}^{d} \left|X_i^{j,p} - \bar{X}_n^{j,p}\right|$, $\bar{Q}_n = \frac{1}{n}\sum_{i=1}^{n} Q_{i,n}$. Then

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} I\left[e_i < t + \eta(n)\bar{X}_n s - \eta(n)Q_{i,n}\right] \le F_n^*(t,s) \le \\ \le\ &\frac{1}{n}\sum_{i=1}^{n} I\left[e_i < t + \eta(n)\bar{X}_n s + \eta(n)Q_{i,n}\right] \end{aligned} \tag{13}$$

for every $t \in \mathbb{R}^k$, $s \in \mathbb{R}^d$ and the following convergences take place

$$\begin{aligned} &\left(\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} I\left[e_i < t + \eta(n)Q_{i,n}\right] - F(t)\right) + \tilde{\eta}\nabla F(t)\bar{Q}_n;\ t\in A\right) \longrightarrow \\ \longrightarrow\ &(W(t),\ t \in A) \quad \text{in } D(A), \\ &\left(\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} I\left[e_i < t - \eta(n)Q_{i,n}\right] + F(t)\right) - \tilde{\eta}\nabla F(t)\bar{Q}_n;\ t\in A\right) \longrightarrow \\ \longrightarrow\ &(W(t),\ t \in A) \quad \text{in } D(A), \end{aligned} \tag{14}$$

where W is a Gaussian process with zero mean and covariance function $H(t_1,t_2) = F(t_1 \wedge t_2) - F(t_1)F(t_2)$.

Proof. The inequality (13) is evident. The assertion (14) is based on verifying the assumptions of Theorem 2 for the process

$$\left(\frac{1}{\sqrt{n}}\frac{1}{n}\sum_{i=1}^{n} \left\{I\left[e_i < \varphi(t) + \eta(n)Q_{i,n}\right] - F(\varphi(t) + \eta(n)Q_{i,n})\right\};\ t \in \langle 0,1\rangle^k\right),$$

where $\varphi = \varphi_1 \times \dots \times \varphi_n, \varphi_i : \langle 0,1\rangle \to A_n^*$ are increasing surjective functions.
(4) takes place since $\mu\circ\varphi$ has all marginals continuous and $F(\varphi(B)+\eta(n)Q_{i,n}) \le \mu\circ\varphi(B)$ for every set $B = \mathsf{X}_{i=1}^{d}\langle a_i, b_i\rangle \in \mathcal{B}_{d,r,p,\psi}$ if $2\eta(n)d \sup_{i=1,2,\dots} \|X_i\| < \varepsilon$.
(5), (6), (7) are evident.

(8) takes place since the same technique as in the proof of Case 1 implies

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\{F\left((\varphi(t_1)+\eta(n)Q_{i,n})\wedge(\varphi(t_2)+\eta(n)Q_{i,n})\right)-\\ -\quad&F(\varphi(t_1)+\eta(n)Q_{i,n})F(\varphi(t_2)+\eta(n)Q_{i,n})\}=\\ =\quad&F(\varphi(t_1)\wedge\varphi(t_2))-F(\varphi(t_1))F(\varphi(t_2))+o(1).\end{aligned}$$

Then in $D(A)$

$$\left(\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{I\left[e_i<t+\eta(n)Q_{i,n}\right]-F(t+\eta(n)Q_{i,n})\}\,;\ t\in A\right)\longrightarrow(W(t),t\in A).$$

Moreover,

$$\begin{aligned}g_n(t)\quad&=\quad\frac{1}{\sqrt{n}}\sum_{i=1}^{n}F(t+\eta(n)Q_{i,n})-\sqrt{n}\,F(t)-\tilde{\eta}\nabla F(t)\bar{Q}_n=\\ &=\quad\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\nabla F(t+\xi_{i,n})-\nabla F(t)\right)Q_{i,n}\eta(n)+\left(\sqrt{n}\,\eta(n)-\tilde{\eta}\right)\nabla F(t)\bar{Q}_n\longrightarrow 0\end{aligned}$$

since $\|\xi_{i,n}\|\le 2\eta(n)\sup\limits_{p=1,2,\dots}\|X_p\|$. Further,

$$\begin{aligned}g_n(t)-g_n(s)\quad&=\quad\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(F(t+\eta(n)Q_{i,n})-F(t)-F(s+\eta(n)Q_{in})+F(s)\right)-\\ &-\quad\tilde{\eta}\left(\nabla F(t)-\nabla F(s)\right)\bar{Q}_n=\\ &=\quad\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\nabla F(t+\xi^{*}_{i,n})-\nabla F(s+\xi^{*}_{i,n})\right\}Q_{i,n}\eta(n)-\\ &-\quad\tilde{\eta}\left(\nabla F(t)-\nabla F(s)\right)\bar{Q}_n=\\ &=\quad\frac{1}{\sqrt{n}}\eta(n)\sum_{i=1}^{n}(t-s)'\left(\frac{\partial^2 F}{\partial t_p\partial t_\ell}(t+\xi^{p}_{i,n})\right)_{p,\ell=1}^{k}Q_{in}-\\ &-\quad\tilde{\eta}\left(\nabla F(t)-\nabla F(s)\right)\bar{Q}_n,\end{aligned}$$

with $\|\xi^{*}_{i,n}\|$, $\|\xi^{p}_{i,n}\|\le 2\eta(n)\sup\limits_{p=1,2,\dots}\|X_p\|$, implies that the functions $g_n\circ\varphi$ are uniformly continuous on $\langle 0,1\rangle^k$, according to the assumption on the existence and the properties of ∇F and ∇^2F. The fact that

$$\begin{aligned}&\left(Z_n(t)+f_n(t),\ t\in\langle 0,1\rangle^k\right)\longrightarrow Z\text{ in }D_k(0,1)\text{ if}\\ &\left(Z_n(t),\ t\in\langle 0,1\rangle^k\right)\longrightarrow Z\text{ in }D_k(0,1)\text{ and }f_n\to 0\text{ uniformly on }\langle 0,1\rangle^k,\end{aligned}$$

completes the proof since the second part of (14) can be verified by a similar way. □

The second problem is to investigate the distance between $F^{*}_n(\cdot,\cdot)$ and the empirical distribution function of errors

$$\tilde{F}_n(t)=\frac{1}{n}\sum_{i=1}^{n}I[e_i<t].\tag{15}$$

Various properties of this distance are discussed in the following two cases.

Case 3. Let $\sup_{n=1,2,\ldots} \|X_n\| < +\infty$, $\lim_{n\to+\infty} \sqrt{n}\,\eta(n) = \tilde{\eta} \in \mathbb{R}$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i s_1 \wedge X_i s_2 - X_i s_1 \wedge 0 - X_i s_2 \wedge 0) = h(s_1, s_2) \in \mathbb{R}$$

for every $s_1,\ s_2 \in \mathbb{R}^d$, $A \subset \mathbb{R}^k$ such that the d.f. F has the bounded second derivatives on A and the total differential at every point $t_1 \wedge t_2,\ t_1,\ t_2 \in A$. Then

$$\left(n^{\frac{3}{4}} \left(F_n^*(t,s) - \tilde{F}_n(t)\right) - n^{\frac{3}{4}}\, \eta(n)\, \nabla F(t)\, \bar{X}_n s;\ t \in A,\ s \in \mathbb{R}^d\right) \xrightarrow{d} \tilde{W}, \tag{16}$$

where $\tilde{W}$ is a Gaussian process with zero mean and covariance function

$$H(t_1, s_1; t_2, s_2) = \tilde{\eta} \nabla F(t_1 \wedge t_2)\, h(t_1, s_1; t_2, s_2)$$

and

$$h(t_1, s_1; t_2, s_2)_j = \begin{cases} h(s_1, s_2)_j & \text{if } t_1^j = t_2^j \\ 0 & \text{if } t_1^j \neq t_2^j \end{cases}, \quad j = 1, \ldots, k$$

Proof. Consider that

$$n^{\frac{3}{4}} \left(F_n^*(t,s) - \tilde{F}_n(t)\right) = n^{-\frac{1}{4}} \sum_{i=1}^{n} (I[e_i < t + \eta(n) X_i s] - I[e_i < t]).$$

Hence the empirical process $X_n(a_n, \varphi_n, F_n)$ is such that

$$\begin{aligned} &k(i,n) = 2, \qquad a_{i1n} = n^{-\frac{1}{4}}, \qquad a_{i2n} = -n^{-\frac{1}{4}}, \\ &\varphi_{i1n}(t,s) = t + \eta(n) X_i s, \\ &\varphi_{i2n}(t,s) = t \qquad \text{and} \qquad F_n = F. \end{aligned}$$

Take $t_1,\ t_2 \in A,\ s_1,\ s_2 \in \mathbb{R}^d$ then

$$\begin{aligned} & n^{-\frac{1}{2}} \sum_{i=1}^{n} \{F((t_1 + \eta(n) X_i s_1) \wedge (t_2 + \eta(n) X_i s_2)) - F((t_1 + \eta(n) X_i s_1) \wedge t_2) - \\ - \ & F(t_1 \wedge (t_2 + \eta(n) X_i s_2)) + F(t_1 \wedge t_2)\} - \\ - \ & n^{-\frac{1}{2}} \sum_{i=1}^{n} \{F(t_1 + \eta(n) X_i s_1) F(t_2 + \eta(n) X_i s_2) - F(t_1 + \eta(n) X_i s_1) F(t_2) - \\ - \ & F(t_1) F(t_2 + \eta(n) X_i s_2) + F(t_1) F(t_2)\} = \\ = \ & n^{-\frac{1}{2}} \nabla F(t_1 \wedge t_2) \sum_{i=1}^{n} \{(t_1 + \eta(n) X_i s_1) \wedge (t_2 + \eta(n) X_i s_2) - (t_1 + \eta(n) X_i s_1) \wedge t_2 - \\ - \ & t_1 \wedge (t_2 + \eta(n) X_i s_2) + t_1 \wedge t_2\} + \sqrt{n}\, o(\eta(n)) - \\ - \ & n^{-\frac{1}{2}} \eta^2(n) \sum_{i=1}^{n} \nabla F(t_1) X_i s_1 \cdot \nabla F(t_2) X_i s_2 + \sqrt{n}\, \eta(n)\, o(\eta(n)) = \\ = \ & \sqrt{n}\, \eta(n) \nabla F(t_1 \wedge t_2)\, h(t_1, s_1; t_2, s_2) + o(1) = \\ = \ & \tilde{\eta} \nabla F(t_1 \wedge t_2)\, h(t_1, s_1; t_2, s_2) + o(1). \end{aligned}$$

The assumptions of Theorem 1 are fulfilled, therefore

$$\left(n^{\frac{3}{4}}\left(F_n^*(t,s)-\hat{F}_n(t)\right)-n^{-\frac{1}{4}}\sum_{i=1}^{n}(F(t+\eta(n)X_is)-F(t))\,;\ t\in A,\ s\in\mathbb{R}^d\right)\xrightarrow{d}\tilde{W}.$$

Theorem 1 can be employed because of there always exists a surjective map $\langle 0,1\rangle^{k+d}\longrightarrow A\times\mathbb{R}^d$ and convergence in distribution refers to finite distributions only.

The nonstochastic part could be further approximated

$$\begin{aligned} & n^{-\frac{1}{4}}\sum_{i=1}^{n}(F(t+\eta(n)\,X_is)-F(t)) = \\ = \ & n^{-\frac{1}{4}}\sum_{i=1}^{n}\eta(n)\,\nabla F(t)\,X_is+n^{\frac{3}{4}}O(\eta^2(n)) = \\ = \ & n^{\frac{3}{4}}\eta(n)\,\nabla F(t)\,\bar{X}_ns+O(n^{-1/4}), \end{aligned}$$

since the second derivatives of F are bounded on A. □

The convergence in the space D cannot be derived, because $\sup\limits_{t\in A}\|\tilde{W}(t,s)\|=+\infty$ if $h(s,s)=\lim\limits_{n\to+\infty}\frac{1}{n}\sum\limits_{i=1}^{n}|X_i|s>0$ and $\#A=+\infty$. The necessity leads us to fix a point $t\in A$ and to assume $X_i=\operatorname{diag}\rho_i$, $\rho_i\geq 0$, $k=d$, because this is the only one case where $(F_n^*(t,s),\ s\in\langle -1,1\rangle^k)\in D(\langle -1,1\rangle^k)$.

Case 4. Let $t\in\mathbb{R}^k$, $X_i=\operatorname{diag}\rho_i$, $\rho_i\geq 0$, $\sup\limits_{i=1,2,\dots}\|\rho_i\|<+\infty$, $\bar{\rho}_n=\frac{1}{n}\sum\limits_{i=1}^{n}\rho_i$, $\bar{\rho}_n\longrightarrow\bar{\rho}\in\mathbb{R}^n$, $\sqrt{n}\,\eta(n)\longrightarrow\tilde{\eta}\in\mathbb{R}$. Let ∇^2F be bounded in a neighbourhood of the point t and $\varepsilon>0$ be such that the function

$$F_{p,\xi}(x)=F\left(t+\xi\left((-\infty,0\rangle^p\times\langle-\varepsilon,x_1\rangle\times\ldots\times\langle-\varepsilon,x_{k-p}\rangle\right)\right)$$

is absolutely continuous with respect to the Lebesgue measure on $\langle-\varepsilon,\varepsilon\rangle^{k-p}$ with a bounded density, for every $p=0,1,\dots,k-1$ and for every permutation ξ of coordinates. Then

$$\begin{aligned} & \left(n^{\frac{3}{4}}\left(F_n^*(t,s)-\hat{F}_n(t)\right)-n^{\frac{3}{4}}\eta(n)\,\nabla F(t)\operatorname{diag}\bar{\rho}_ns;\ s\in\langle-1,1\rangle^k\right)\longrightarrow \qquad (17)\\ \longrightarrow\ & \tilde{W}\ \text{in}\ D(\langle-1,1\rangle^k), \end{aligned}$$

where $\tilde{W}$ is a Gaussian process with zero mean and covariance function

$$H(s_1,s_2)=\tilde{\eta}\nabla F(t)\operatorname{diag}\bar{\rho}\,(s_1\wedge s_2-s_1\wedge 0-s_2\wedge 0)\,.$$

Proof. Use the transformation $\varphi(t)=2t-1$ and verify the assumptions of Theorem 2 for empirical processes $X_n(a_n,\varphi_n,F_n)$ where

$$\begin{aligned} & k(i,n)=2, \qquad a_{i1n}=n^{-\frac{1}{4}}, \qquad a_{i2n}=-n^{-\frac{1}{4}},\\ & \varphi_{i1n}(s)=t+\eta(n)\,X_is,\\ & \varphi_{i2n}(s)=t \qquad \text{and} \qquad F_n=F. \end{aligned}$$

First, the assumption (4):

Take $p = 0, 1, \ldots, k-1, \xi$ a permutation of coordinates and $A \in \mathcal{B}(\langle 0,1)^{k-p})$

$$F(t+\eta(n)X_i\xi \circ \varphi((-\infty,1)^p \times A)) \leq M \int_{2\eta(n)\sup_{i=1,2,\ldots}\|\rho_i\|A} d\lambda_{k-p} = \tilde{M}\eta(n)\lambda_{k-p}(A).$$

The assumption (5):

$$\sum_{i=1}^{n} n^{-\frac{1}{2}}\tilde{M}\eta(n) = \tilde{M}\sqrt{n}\,\eta(n) \leq Q,$$
$$\sum_{i=1}^{n}\left(n^{-\frac{1}{2}}\tilde{M}\eta(n)\right)^2 = \tilde{M}\sqrt{n}\,\eta(n) \leq \tilde{Q}, \text{ since } \eta(n) \to \tilde{\eta} \in \mathbb{R}.$$

The assumption (6) is evident since $\varphi_{i2n} \equiv t$ and then $\varphi_{i2n}(B) = \emptyset$ everytime.

The assumption (7) is evident.

The assumption (8) was verified in the proof of Case 3.

Then, by Theorem 2,

$$\left(n^{\frac{3}{4}}\left(F_n(t,s) - \hat{F}_n(t)\right) - n^{-\frac{1}{4}}\sum_{i=1}^{n}(F(t+\eta(n)\,X_is) - F(t));\ s \in \langle -1,1)^k\right) \longrightarrow$$
$$\longrightarrow\ \tilde{W} \text{ in } D(\langle -1,1)^k).$$

Moreover,

$$g_n(s) = n^{-\frac{1}{4}}\sum_{i=1}^{n}(F(t+\eta(n)\,X_is) - F(t)) - n^{\frac{3}{4}}\eta(n)\,\nabla F(t)\bar{X}_ns \longrightarrow 0$$

and

$$\begin{aligned} g_n(s) - g_n(v) &= n^{-\frac{1}{4}}\sum_{i=1}^{n}(F(t+\eta(n)\,X_is) - \eta(n)\nabla F(t)X_is - \\ &- F(t+\eta(n)\,X_iv) + \eta(n)\nabla F(t)\,X_iv) = \\ &= n^{-\frac{1}{4}}\eta(n)\sum_{i=1}^{n}\{\nabla F(t+\eta(n)\xi_{i,n}) - \nabla F(t))\ X_i(s-v) = \\ &= n^{-\frac{1}{4}}\eta^2(n)\sum_{i=1}^{n}\xi'_{i=1}\left(\frac{\partial^2 F}{\partial t_\ell \partial t_p}(t+\eta(n)\xi^p_{i,n})\right)^k_{\ell,p=1} X_i(s-v), \end{aligned}$$

where

$$\|\xi_{i,n}\|,\ \left\|\xi^p_{i,n}\right\| \leq \sup_{p=1,2,\ldots}\|x_p\|.$$

Then the functions g_n are uniformly continuous on $\langle -1,1)^k$. This completes the proof in the same way as the proof of Case 2. □

Case 2 and Case 4 give immediately consequence for the empirical process $\hat{F}_n$.

Corollary. The condition $\hat{\beta}_n - \beta = O_p\left(n^{-\frac{1}{2}}\right)$ leads to $\sup_{t\in A}\left|\hat{F}_n(t) - F(t)\right| = O_p\left(n^{-\frac{1}{2}}\right)$ in Case 2 and

$$\left|\hat{F}_n(t) - \tilde{F}_n(t) - \nabla F(t)\operatorname{diag}\bar{\rho}_n(\hat{\beta}_n - \beta)\right| = O_p(n^{-\frac{3}{4}}) \qquad \text{in Case 4.}$$

4. PROOFS OF THEOREMS

We shall write $X_n(a_n, \varphi_n, F_n) = \sum_{i=1}^{n} X_{in}(a_n, \varphi_n, F_n)$ where $X_{in}(a_n, \varphi_n, F_n)(t) = \sum_{j=1}^{k(i,n)} a_{ijn}\left(I(X_i < \varphi_{ijn}(t)) - F_n(\varphi_{ijn}(t))\right)$, $i = 1, \ldots, n$. The covariance of $X_{in}(a_n, \varphi_n, F_n)(t)$ and $X_{in}(a_n, \varphi_n, F_n)(s)$ is expressed as

$$\begin{aligned} &\operatorname{cov}\left(X_{in}(a_n, \varphi_n, F_n)(t),\, X_{in}(a_n, \varphi_n, F_n)(s)\right) = \\ &= \sum_{j,p=1}^{k(i,n)} a_{ijn}a_{ipn}\left(F_n(\varphi_{ijn}(t) \wedge \varphi_{ipn}(s)) - F_n(\varphi_{ijn}(t))F_n(\varphi_{ipn}(s))\right). \end{aligned} \tag{18}$$

Convergence in distribution can be verified by McLeish's CLT for martingale differences. Let us recall that $Z_1, \ldots, Z_n$ are martingale differences iff

$$\mathsf{E}\, Z_1 = 0,\ \mathsf{E}\left[Z_j \mid Z_1, \ldots, Z_{j-1}\right] = 0 \qquad \text{for } j = 2, \ldots, n.$$

For example, independent variables with zero means are martingale differences.

Lemma 1 (McLeish CLT). Let $Y_{1n}, \ldots, Y_{nn}$ be martingale differences with the following properties:

$$\mathsf{E} \max_{j=1,\ldots,n} |Y_{jn}| \longrightarrow 0; \tag{19}$$

$$\sum_{j=1}^{n} Y_{jn}^2 \xrightarrow{d} \sigma^2, \qquad \text{where } \sigma^2 \text{ is a finite constant.} \tag{20}$$

Then $\sum_{j=1}^{n} Y_{jn} \xrightarrow{d} Y$, where Y is a Gaussian r. v. with zero mean and variance σ^2.

Proof. Cf. [2] or [4]. □

Proof of Theorem 1. By the Cramér–Wald Theorem (see, e. g., [9], III.4.6, p. 217), it is sufficient to verify the convergence

$$\sum_{k=1}^{K} \beta_k X_n(a_n, \varphi_n, F_n)(t_k) \xrightarrow{d} \sum_{k=1}^{K} \beta_k W(t_k)$$

for arbitrarily chosen $\beta_1, \ldots, \beta_K \in \mathbb{R}$; $t_1, \ldots, t_K \in \langle 0, 1\rangle^d$ and $K \in \mathbb{N}$.

Therefore, fix $K \in \mathbb{N}$; $\beta_1, \ldots, \beta_K \in \mathbb{R}$; $t_1, \ldots, t_K \in \langle 0, 1\rangle^d$ and put $Y_{in} = \sum_{k=1}^{K} \beta_k X_{in}(a_n, \varphi_n, F_n)(t_k)$.

a) Since $Y_{1n}, \dots, Y_{nn}$ are independent then they can be considered as martingale differences as well.

b) $|Y_{in}| \leq \sum_{k=1}^{K} |\beta_k| \sum_{j=1}^{k(i,n)} |a_{ijn}|$, hence
$\mathsf{E} \max\limits_{i=1,\dots,n} |Y_{in}| \leq \sum\limits_{k=1}^{K} |\beta_k| \max\limits_{i=1,\dots,n} \sum\limits_{j=1}^{k(i,n)} |a_{ijn}| \longrightarrow 0$. Now (19) follows immediately.

c) We will compute mean and variance of the sum $\sum_{i=1}^{n} Y_{in}^2$.

$$\begin{aligned}
&\mathsf{E} \sum_{i=1}^{n} Y_{in}^2 = \sum_{i=1}^{n} \operatorname{var} Y_{in} = \\
&= \sum_{i=1}^{n} \sum_{u,v=1}^{K} \beta_u \beta_v \operatorname{cov} \left(X_{in}(a_n, \varphi_n, F_n)(t_u), X_{in}(a_n, \varphi_n, F_n)(t_v)\right) = \\
&= \sum_{u,v=1}^{K} \beta_u \beta_v \sum_{i=1}^{n} \sum_{j,p=1}^{k(i,n)} a_{ijn} a_{ipn} \left(F_n(\varphi_{ijn}(t_u) \wedge \varphi_{ipn}(t_v)) - \right. \\
&- \left. F_n\left(\varphi_{ijn}(t_u)\right) F_n(\varphi_{ipn}(t_v))\right) \longrightarrow \sum_{u,v=1}^{K} \beta_u \beta_v H(t_u, t_v)
\end{aligned}$$

by (3).
Further,

$$\begin{aligned}
&\operatorname{var} \sum_{i=1}^{n} Y_{in}^2 = \sum_{i=1}^{n} \operatorname{var} Y_{in}^2 \leq \sum_{i=1}^{n} \mathsf{E} Y_{in}^4 \leq \\
&\leq \left(\sum_{u=1}^{k} |\beta_u|\right)^2 \cdot \left(\max_{i=1,\dots,n} \sum_{j=1}^{k(i,n)} |a_{ijn}|\right)^2 \cdot \sum_{i=1}^{n} \mathsf{E} Y_{in}^2 \longrightarrow 0.
\end{aligned}$$

Hence, $\sum\limits_{i=1}^{n} Y_{in}^2 \xrightarrow{d} \sum\limits_{u,v=1}^{n} \beta_u \beta_v H\left(t_u, t_v\right)$.
We have verified the assumptions of Lemma 1 therefore
$\sum\limits_{i=1}^{n} Y_{in}^2 = \sum\limits_{u=1}^{U} \beta_u X_n\left(a_n, \varphi_n, F_n\right)\left(t_u\right) \xrightarrow{d} \sum\limits_{u=1}^{U} \beta_u W(t_u)$ for arbitrarily chosen $U \in \mathbb{N}$, $\beta_1, \dots, \beta_U \in \mathbb{R}$, $t_1, \dots, t_U \in \langle 0, 1)^d$. The proof is complete. □

In order to prove Theorem 2, we introduce some lemmas and definitions.

Lemma 2. Let $f_1, \dots, f_n, g_1, \dots, g_n$ be r. v.'s with zero mean and $\mathsf{E} f_i^2 g_i^2 < +\infty$, $\mathsf{E} f_i^2 < +\infty$, $\mathsf{E} g_i^2 < +\infty$ for every $i = 1, \dots, n$. Let the pair (f_i, g_i) be independent of all the other r. v.'s for every $i = 1, \dots, n$. Then

$$\begin{aligned}
&P\left(\left|\sum_{i=1}^{n} f_i\right| \geq y, \left|\sum_{j=1}^{n} g_j\right| \geq y\right) \leq \qquad (21) \\
&\leq y^{-4} \left(\sum_{i=1}^{n} \mathsf{E} f_i^2 g_i^2 + \sum_{i=1}^{n} \mathsf{E} f_i^2 \sum_{j=1}^{n} \mathsf{E} g_j^2 + 4 \left(\sum_{i=1}^{n} \mathsf{E} f_i g_i\right)^2\right)
\end{aligned}$$

for arbitrary $y > 0$.

Proof.

$$P\left(\left|\sum_{i=1}^{n} f_i\right| \geq y, \left|\sum_{j=1}^{n} g_j\right| \geq y\right) y^{-4}\mathsf{E}\left[\left(\sum_{i=1}^{n} f_i\right)^2 \left(\sum_{j=1}^{n} g_j\right)^2\right] \leq$$

$$\leq y^{-4}\left(\sum_{i=1}^{n} \mathsf{E} f_i^2 g_i^2 + \sum_{i=1}^{n} \mathsf{E} f_i^2 \sum_{i_j=1}^{n} \mathsf{E} g_j^2 + 4\left(\sum_{i=1}^{n} \mathsf{E} f_i g_i\right)^2\right)$$

since the other terms vanish because of the independence and zero means of the considered r. v.'s. □

The concept of an increment of a stochastic process around a set is crucial in what follows.

Definition 6. Let $X = \left((X(t),\, t \in \langle 0,1\rangle^d\right)$ be a stochastic process and $A = \mathop{\mathsf{X}}_{i=1}^{d} \langle a_i, b_i\rangle \in \mathcal{B}_{d,r,p,\psi}$ (cf. Definition 3); then

$$\Delta X(A) = \sum_{\substack{\delta_i = a_i, b_i \\ \delta_i > -\infty}} (-1)^{\#\{i|\delta_i = a_i\}} X(\delta_1, \dots, \delta_d) \tag{22}$$

is called the increment of X around A.

The increment of an empirical process has a useful special form.

Lemma 3. Let $X_n(a,\varphi,F)$ be an empirical process and φ_{ij} be coordinatewise nondecreasing and continuous for arbitrary $i = 1,\dots,n$; $j = 1,\dots,k(i)$. If F possesses continuous marginals, it means that the functions

$$f(t) = F\left(\psi((-\infty,0\rangle^r \times (-\infty,1\rangle^p \times \langle 0,t\rangle \times \langle 0,1\rangle^{d-r-p-1})\right)$$

are continuous, then

$$X_n(a,\varphi,F) \in D_d(0,1) \text{ a.s.}; \tag{23}$$

$$\Delta X_n(a,\varphi,F)(A) = \sum_{i=1}^{n} \sum_{j=1}^{k(i)} a_{ij}\left(I(X_i \in \varphi_{ij}(A)) - F(\varphi_{ij}(A))\right) \tag{24}$$

$$\text{for every } A = \mathop{\mathsf{X}}_{i=1}^{d} \langle d_i, h_i\rangle \in \mathcal{B}_{d,r,p,\psi}.$$

Proof. The process has the desired limits at every point of $\langle 0,1\rangle^d$. A confusion may appear on the lower boundary of $\langle 0,1\rangle^d$, but the probability of such an event is zero since F is continuous. Therefore, $X_n(a,\varphi,F) \in D_d(0,1)$ a. s. The second part of Lemma follows immediately since φ_{ij} are coordinatewise nondecreasing. Recall that the set $\varphi_{ij}(A)$ was introduced in Definition 4. □

The convergence in $D_d(0,1)$ will be proved using the following lemma.

Lemma 4. Let $X_n = \big(X_n(t),\, t \in \langle 0,1\rangle^d\big) \in D_d(0,1)$ a.s. be stochastic processes satisfying the following conditions:

There is a stochastic process $X = \big(X(t),\, t \in \langle 0,1\rangle^d\big)$ such that $X_n \xrightarrow{d} X$. (25)

There exist $\alpha,\, \beta > 0$ and finite measures $\mu_{r,p,\psi}$ on $\mathcal{B}_{d,r,p,\psi}$ (26)
for $r = 0,1,\dots,d-1;\ p = 0,1,\dots,d-r-1;\ \psi \in \Psi_d$ such that

(i) every measure $\mu_{r,p,\psi}$has continuous marginals; it means that the functions $f(t) = \mu_{r,p,\psi}\left(\psi((-\infty,0)^r \times (-\infty,1)^p \times \langle 0,t\rangle \times \langle 0,1\rangle^{d-r-p-1})\right)$ are continuous;

(ii) $P\left(|\Delta X_n(A)| > y,\ |\Delta X_n(B)| > y\right) \le y^{-\alpha}\left(\mu_{r,p,\psi}(A \cup B)\right)^{1+\beta}$

for every $y > 0$, $A = \bigtimes_{i=1}^{d} (a_i, b_i\rangle$, $B = \bigtimes_{i=1}^{d} (g_i, h_i\rangle$,

$A,\, B \in \mathcal{B}_{d,r,p,\psi}$, $A \cap B = \emptyset$, $\operatorname{clo} A \cap \operatorname{clo} B \ne \emptyset$.

Then there exists a stochastic process $Y = (Y(t),\ t \in \langle 0,1\rangle^d)$ such that $X_n \to Y$ in $D_d(0,1)$.

Proof. Straf [6] or Neuhaus [5] have shown that if $X_n \xrightarrow{d} X$ and X_n satisfy the tightness condition for $D_d(0,1)$ then $X_n \longrightarrow Y$ in $D_d(0,1)$. (26) implies the tightness condition for X_n which is proved in Lachout [3]. □

The following moment inequalities will be used in the sequel.

Lemma 5. Let $\varphi_1,\dots,\varphi_k : \langle 0,1\rangle^d \longrightarrow \mathbb{R}^d$ be coordinatewise nondecreasing and continuous. Let

$$Y(A) = \sum_{j=1}^{k} a_j \left(I(X \in \varphi_j(A)) - F(\varphi_j(A))\right), \tag{27}$$

where X is a d-dimensional random vector with d.f. F, $A \in \mathcal{B}_{d,r,p,\psi}$ and $a_1,\dots,a_k \in \mathbb{R}$.

Let $A, B \in \mathcal{B}_{d,r,p,\psi}$ be given such that (28)
$F(\varphi_j(A) \cap \varphi_j(B)) = F(\varphi_j(A) \cap \varphi_\ell(A)) = F(\varphi_j(A) \cap \varphi_\ell(B)) = 0$

for arbitrary $j,\, \ell = 1,\dots,k, j \ne \ell$. Then

$$\mathsf{E}\, Y(A)Y(B) = -\sum_{j=1}^{k} a_j F(\varphi_j(A)) \cdot \sum_{p=1}^{k} a_p F(\varphi_p(B)); \tag{29}$$

$$\mathsf{E}\, Y^2(A) = \sum_{j=1}^{k} a_j^2 F(\varphi_j(A)) - \left(\sum_{j=1}^{k} a_j F(\varphi_j(A))\right)^2; \tag{30}$$

$$\mathsf{E}\, Y^2(A)Y^2(B) \le \left(\sum_{j=1}^{k} a_j F(\varphi_j(A))\right)^2 \cdot \sum_{p=1}^{k} a_p^2 F(\varphi_p(B)) + \tag{31}$$

$$+\left(\sum_{j=1}^{k} a_j F(\varphi_j(B))\right)^2 \cdot \sum_{p=1}^{k} a_p^2 F(\varphi_p(A)).$$

Proof. a)

$$\mathsf{E}\, Y(A)Y(B) = \sum_{j,p=1}^{k} a_j a_p \mathrm{cov}\,(I(x \in \varphi_j(A)),\, I(x \in \varphi_p(B))) =$$

$$= \sum_{j,p=1}^{k} a_j a_p \left(F(\varphi_j(A) \cap \varphi_p(B)) - F(\varphi_j(A))F(\varphi_p(B))\right) =$$

$$= -\sum_{j=1}^{k} a_j F(\varphi_j(A)) \cdot \sum_{p=1}^{k} a_p F(\varphi_p(B))$$

since $F(\varphi_j(A) \cap \varphi_p(B)) = 0$ by (28).

b)

$$\mathsf{E}\, Y^2(A) = \sum_{j,p=1}^{k} a_j a_p \left(F(\varphi_i(A) \cap \varphi_p(A)) - F(\varphi_j(A))F(\varphi_p(A))\right) =$$

$$= \sum_{j=1}^{k} a_j^2 F(\varphi_j(A)) - \left(\sum_{j=1}^{k} a_j F(\varphi_j(A))\right)^2$$

since the other members are vanishing by (28).

c)

$$\mathsf{E}\, Y^2(A)\, Y^2(B) = \sum_{j_1,j_2,j_3,j_4=1}^{k} a_{j_1} a_{j_2} a_{j_3} a_{j_4} \mathsf{E}\left\{(I(X \in \varphi_{j_1}(A)) - F(\varphi_{j_1}(A)))\cdot\right.$$
$$\cdot\ (I(X \in \varphi_{j_2}(A)) - F(\varphi_{j_2}(A)))\,(I(X \in \varphi_{j_3}(B)) - F(\varphi_{j_3}(B)))\cdot$$
$$\cdot\ \left.(I(X \in \varphi_{j_4}(B)) - F(\varphi_{j_4}(B)))\right\}.$$

Perform the multiplication. Most of the terms vanish according to (28). The other terms either contain no indicator or contain only one indicator or contain two indicators having the common set A (resp. B). Thus, we have

$$\mathsf{E}\, Y^2(A) Y^2(B) =$$
$$= \sum_{j_1,j_2,j_3,j_4=1}^{k} a_{j_1} a_{j_2} a_{j_3} a_{j_4} F(\varphi_{j_1}(A))\, F(\varphi_{j_2}(A))\, F(\varphi_{j_3}(B))\, F(\varphi_{j_4}(B)) -$$
$$- 4 \sum_{j_1,j_2,j_3,j_4=1}^{k} a_{j_1} a_{j_2} a_{j_3} a_{j_4} F(\varphi_{j_1}(A))\, F(\varphi_{j_2}(A))\, F(\varphi_{j_3}(B))\, F(\varphi_{j_4}(B)) +$$
$$+ \sum_{j_1,j_3,j_4=1}^{k} a_{j_1}^2 a_{j_3} a_{j_4} F(\varphi_{j_1}(A))\, F(\varphi_{j_3}(B))\, F(\varphi_{j_4}(B)) +$$
$$+ \sum_{j_1,j_2,j_3=1}^{k} a_{j_1} a_{j_2} a_{j_3}^2 F(\varphi_{j_1}(A))\, F(\varphi_{j_2}(A))\, F(\varphi_{j_3}(B)) \leq$$

$$\leq \left(\sum_{j=1}^{k} a_j F(\varphi_i(B))\right)^2 \sum_{p=1}^{k} a_p^2 F(\varphi_p(A)) + \left(\sum_{j=1}^{k} a_j F(\varphi_i(B))\right)^2 \sum_{p=1}^{k} a_p^2 F(\varphi_p(B)). \square$$

We are now ready to complete the proof of Theorem 2.

Proof of Theorem 2. We will verify the assumptions of Lemma 4.

i) (7) and (8) imply (25) by Theorem 1.

ii) Let $A = \mathop{\times}_{i=1}^{d} (a_i, b_i\rangle$, $B = (g_i, h_i\rangle$, $A, B \in \mathcal{B}_{d,r,p,\psi}$,

$A \cap B = \emptyset$ and $y > 0$. Using Lemmas 2 and 3 gets the estimation

$$\begin{aligned}
& P\left(|\Delta X_n(a_n,\varphi_n)(A)| > y,\ |\Delta X_n(a_n,\varphi_n)(B)| > y\right) = \\
= \ & P\left(\left|\sum_{i=1}^{n} Y_{in}(A)\right| > y,\ \left|\sum_{i=1}^{n} Y_{in}(B)\right| > y\right) \leq \\
\leq \ & y^{-4}\left(\sum_{i=1}^{n} \mathsf{E}\, Y_{in}^2(A) Y_{in}^2(B) + \sum_{i=1}^{n} \mathsf{E}\, Y_{in}^2(A) \sum_{i=1}^{n} \mathsf{E}\, Y_{in}^2(B) + \right. \\
+ \ & \left. 4\left(\sum_{i=1}^{n} \mathsf{E}\, Y_{in}(A) Y_{in}(B)\right)^2\right)
\end{aligned}$$

where $Y_{in}(A) = \sum_{j=1}^{k(i,n)} a_{ijn}\left(I(X_{in} \in \varphi_{ijn}(A)) - F_n(\varphi_{ijn}(A))\right)$.

$Y_{in}(A)$ has the form and properties required in Lemma 5. Therefore, by assumption (4),

$$\begin{aligned}
& P\left(|\Delta X_n(a_n,\varphi_n)(A)| > y,\ |\Delta X_n(a_n,\varphi_n)(B)| > y\right) \leq \\
\leq \ & y^{-4}\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{k(i,n)} a_{ijn} F_n(\varphi_{ijn}(A))\right) \cdot \sum_{p=1}^{k(i,n)} a_{ipn}^2 F_n(\varphi_{ipn}(B)) + \right. \\
+ \ & \sum_{i=1}^{n}\left(\sum_{j=1}^{k(i,n)} a_{ijn} F_n(\varphi_{ijn}(B))\right) \cdot \sum_{p=1}^{k(i,n)} a_{ipn}^2 F_n(\varphi_{ipn}(A)) + \\
+ \ & \left(\sum_{i=1}^{n}\sum_{j=1}^{k(i,n)} a_{ijn}^2 F_n(\varphi_{ijn}(A))\right) \cdot \left(\sum_{i=1}^{n}\sum_{p=1}^{k(i,n)} a_{ipn}^2 F_n(\varphi_{ipn}(B))\right) + \\
+ \ & \left. 4\left(\sum_{i=1}^{n}\sum_{j=1}^{k(i,n)} a_{ijn} F_n(\varphi_{ijn}(A)) \cdot \sum_{p=1}^{k(i,n)} a_{ipn} F_n(\varphi_{ipn}(B))\right)^2\right\} \leq \\
\leq \ & y^{-4}\left\{2\sum_{i=1}^{n}\left(\sum_{j=1}^{k(i,n)} |a_{ijn}|\, c_{ijn}\right)^2 \cdot \sum_{p=1}^{k(i,n)} a_{ipn}^2 c_{ipn} \cdot \mu_{r,p,\psi}^3(A \cup B) + \right.
\end{aligned}$$

$$+ \left(\sum_{i=1}^{n} \sum_{j=1}^{k(i,n)} a_{ijn}^2 c_{ijn} \right)^2 \cdot \mu_{r,p,\psi}^2 (A \cup B) +$$

$$\left. + \; 4 \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{k(i,n)} |a_{ijn}| \, c_{ijn} \right)^2 \right)^2 \cdot \mu_{r,p,\psi}^4 (A \cup B) \right\}.$$

The assumptions (5), (7) together with the fact that $\mu_{r,p,\psi}$ is a finite measure ensure that

$$P\left(|\Delta X_n(a_n, \varphi_n)(A)| > y, \; |\Delta X_n(a_n, \varphi_n)(B)| > y \right) \le W y^{-4} \mu_{r,p,\psi}^2 (A \cup B)$$

for some constant W, hence (26) holds.

We have verified all assumptions of Lemma 4 hence Theorem 2 holds. □

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