Kybernetika

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Kybernetika, Vol. 23 (1987), No. 1, 67--81

Persistent URL: http://dml.cz/dmlcz/124770

KYBERNETIKA — VOLUME 23 (1987), NUMBER 1

PERTURBATION THEORY OF DUALITY IN VECTOR OPTIMIZATION VIA THE ABSTRACT DUALITY SCHEME

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The perturbation theory of duality has been usually constructed with help of the theory of conjugate functions (see [1, 2, 3, 4]). In this paper, unlike the traditional method, two kinds of perturbation duality in vector optimization are suggested on the basis of the abstract duality scheme which has been introduced in Tran Quoc Chien [5, 7] and further generalized in Tran Quoc Chien [8]. This approach is much simpler than the one using a generalization of conjugate functions in vector case (see Azimov [4]). It gives, however, almost the same or, in some cases, stronger results. The classical Fenchel duality is also generalized for maximizing a sum of n (instead of $n = 2$ in other works) concave functions. The only mathematical tool is separation of n convex sets introduced in [8].

0. INTRODUCTION

In this work X, Y and Q are real linear spaces if other requirements are not added, Y is ordered by a positive cone Y_+. All notations, concepts concerning linear spaces, ordering and optimality used in this paper are referred to Tran Quoc Chien [8].

Now given a function $f: X \to \overline{Y}$, $\overline{Y} = Y \cup \{-\infty, +\infty\}$, we shall be concerned with the probem

(P) $$\text{Max-Sup}\, f(X)\,.$$

We will consider a suitable function

$$\Phi: X \times Q \to \overline{Y}$$

termed the *perturbed essential objective* such that

0.1. $$\Phi(x, 0) = f(x) \quad \forall x \in X\,,$$

where 0 is the zero element of the space Q.

For every $q \in Q$ the problem

(P_q) $$\text{Max-Sup}\ \Phi(X, q)$$

is termed the *perturbed primal problem.*

In [1, 2, 3, 4] the perturbed primal problem and the theory of conjugate functions play a crucial role in establishing a dual to problem (P), although they have only an auxiliary character. The approach using the abstract duality scheme, needs no knowledge of conjugate functions and gives stronger results (note that in [4] the space Y is of finite dimension and the function Φ is convex).

Following [8] we put

$$\mathscr{P} = Q \times Y, \quad P(y) = E \cap E_y$$

where

$$E = \bigcup_{x \in X} \{(q, y) \in \mathscr{P}: y \leqq \Phi(x, q)\}$$

and

$$E_y = \{0\} \times (y + Y_+) .$$

It is evident that the multivalued function $P: Y \to \mathscr{P}$ satisfies the primal availability (see [8], Section 2). We have then

$$\mathscr{P}_0 = \{(q, y) \in \mathscr{P} \mid \exists y' \in Y: (q, y) \in P(y')\} =$$
$$= \{(0, y) \in \mathscr{P} \mid \exists x \in X: y \leqq \Phi(x, 0) = f(x)\}$$

and

$$\mu(0, y) = \{y' \in Y: (0, y) \in P(y')\} = y - Y_+$$

for every $(0, y) \in \mathscr{P}_0$.

Consequently

$$\mu(\mathscr{P}_0) = f(X) - Y_+$$

So from definition it follows

0.2. Proposition. Problem (P) is equivalent to the following abstract primal

(AP) $$\text{Max-Sup}\ \mu(\mathscr{P}_0)$$

in the sense that

$$\text{Max} f(X) = \text{Max}\ \mu(\mathscr{P}_0)$$

and

$$\text{Sup} f(X) = \text{Sup}\ \mu(\mathscr{P}_0) .$$

Now in order to be able to construct a dual to problem (AP) or, what is the same because of Proposition 0.2, to problem (P), we have to choose a suitable set $\mathscr{D}$ and a multivalued function $D: Y \to \mathscr{D}$ satisfying the dual availability (see [8], Section 2). Because of the complicatedness and difficulties joined with the vector structure we suggest the two following kinds of duality which coincide with each other in the scalar case. Each duality has its advantage and disadvantage and they complete each other.

The corresponding sets to these dualities are the following

$$\mathscr{D} = Q^* \times Y^*_{++}$$

where Y^*_{++} is the set of all positive linear functionals on Y ($y^* \in Y^*$ is called positive if $\langle y^*, y\rangle > 0\ \forall y \in Y_{++}$) and

$$\mathscr{L} \subset \mathscr{L}(Q, Y)$$

where $\mathscr{L}(Q, Y)$ is the space of linear operators from Q to Y.

1. FUNCTIONAL DUALITY

1.1. Definition. Given $\mathscr{D} = Q^* \times Y^*_{+++}$ we define

$$D(y) = \{(q^*, y^*) \in \mathscr{D}: \langle y^*, y\rangle \geqq \sup\{\langle q^*, q\rangle + \langle y^*, \Phi(x, q)\rangle: (x, q) \in X \times Q\}\}$$

and

$$v(q^*, y^*) = \{y \in Y: (q^*, y^*) \in D(y)\}.$$

Obviously function $D(y)$ satisfies the dual availability. Putting

$$\mathscr{D}_0 = \{(q^*, y^*) \in \mathscr{D}: v(q^*, y^*) \neq \emptyset\},$$

the problem

(D 1) $$\text{Min-Inf } v(\mathscr{D}_0)$$

is called the *functional Φ-dual* to program (P).

1.2. Theorem. (Weak Duality.) The weak duality condition is satisfied and hence

$$f(X) \gtreqqless v(\mathscr{D}_0).$$

1.3. Weak Optimality Condition:

$$Y_{++} = \text{cor } Y_+ \neq \emptyset.$$

1.4. Normality Condition:

$$E \cap E_y = \emptyset \Rightarrow E_{y'} \cap \text{lin } E = \emptyset \quad \forall y' > y.$$

1.5. Convexity Condition:

$$\text{icr } E \quad \text{is nonempty and convex.}$$

1.6. Lemma. If the weak optimality condition, the normality condition and the convexity condition hold then the Sup-Inf strong duality condition (see [8], Section 2) is fulfilled.

Proof. Let $y_0 \in Y$ be such that $P(y) = \emptyset\ \forall y > y_0$. Then, by the normality condition, $E_y \cap \text{lin } E = \emptyset\ \forall y > y_0$. Fix a $y_1 > y_0$ and $(q_*, y_*) \in \text{icr } E$. Consider the segment

$$[(q_*, y_*), (0, y_1)] = \{t(q_*, y_*) + (1 - t) \cdot (0, y_1): 0 \leqq t \leqq 1\}.$$

Since $(0, y_1) \notin \text{lin } E$, there exists $\varepsilon > 0$ such that

$$M = \{t \cdot (q_*, y_*) + (1 - t) \cdot (0, y_1): 0 \leqq t \leqq \varepsilon\} \cap E = \emptyset.$$

Hence, by Theorem 3.4.2 of [8], there exists a nonzero $(q^*, y^*) \in Q^* \times Y^*$ such that

(1.6.1)
$$a \leqq b$$

where
$$a = \sup\{\langle q^*, q\rangle + \langle y^*, y\rangle : (q, y) \in E\}$$

and
$$b = \inf\{t \,.\, \langle q^*, q\rangle + \langle y^*, ty_* + (1 - t)\, y_1\rangle : 0 \leqq t \leqq \varepsilon\}\,.$$

Obviously $y^* \in Y_+^*$. Suppose, on the contrary, that $(q^*, y^*) \notin \mathcal{D}$ which means $y^* = 0$. Then $q^* \neq 0$ for $(q^*, y^*) \neq 0$.

Since $(0, f(x)) \in E$ we have

(1.6.2)
$$a \geqq 0\,.$$

From (1.6.1) and (1.6.2) it follows that
$$\langle q^*, q_*\rangle \geqq 0\,.$$

Since $(q_*, y_*) \in \operatorname{icr} E$ there exists a pair $(q', y') \in E$ such that
$$\langle q^*, q'\rangle + \langle y^*, y'\rangle > 0$$

(for $\langle q^*, q'\rangle + \langle y^*, y'\rangle$ is not constant on E), which contradicts $a \leqq b = 0$. The proof is thus complete.

From Theorem 2.5 of [8] and Lemma 1.6 it follows

1.7. Theorem. (Strong Duality.) If the weak optimality condition, the normality condition and the convexity condition are satisfied, then
$$\operatorname{Sup} f(x) = \operatorname{Inf} v(\mathcal{D}_0)\,.$$

We formulate now another sufficient condition for the Sup-Inf strong duality to be valid.

1.8. Slater's Condition:
$$0 \in \operatorname{icr} E_Q$$

where
$$E_Q = \{q \in Q \mid \exists y \in Y : (q, y) \in E\}$$
$$= \{q \in Q \mid \exists x \in X : \Phi(x, q) \in Y\}\,.$$

1.9. Lemma. If E is convex then the Slater condition implies the normality condition.

Proof. Let $E \cap E_{y_0} = \emptyset$. Suppose, on the contrary, that there exists a $y_1 > y_0$ such that $E_{y_1} \cap \operatorname{lin} E \neq \emptyset$, which means
$$(0, y_1) \in \operatorname{lin} E\,.$$

There exists, by definition, a $(q_e, y_e) \in E$ such that
$$[(q_e, y_e), (0, y_1)[\subset E\,.$$

By the Slater condition there exists a $k > 0$ such that $-kq_e \in E_Q$ that means there exists an $y \in Y$ with $(-kq_e, y) \in E$.

Now let δ, ε be sufficiently small positive numbers. Obviously

$$\delta(q_e, y_e) + (1 - \delta) . (0, y_1) = (\delta q_e, \delta y_e + (1 - \delta) y_1) \in E$$

and

$$(q_2, y_2) = (1 - \varepsilon) . (\delta q_e, \delta y_e + (1 - \delta) y_1) + \varepsilon(-kq_e, y) =$$
$$= (((1 - \varepsilon) \delta - \varepsilon k) q_e, (1 - \varepsilon) \delta y_e + (1 - \varepsilon)(1 - \delta) y_1 + \varepsilon y) \in E .$$

Now it suffices to choose ε and δ so small that

$$q_2 = ((1 - \varepsilon) \delta - \varepsilon k) q_e = 0$$

and

$$y_2 = (1 - \varepsilon) \delta y_e + (1 - \varepsilon)(1 - \delta) y_1 + \varepsilon y > y_0 .$$

We obtain then $E \cap E_{y_2} \neq \emptyset$ that contradicts

$$E \cap E_{y_0} = \emptyset .$$

The proof is thus complete.

From Lemma 1.9 and Theorem 1.7 it follows

1.10. Corollary. If the weak optimality condition, the convexity condition and the Slater condition hold then

$$\operatorname{Sup} f(X) = \operatorname{Inf} v(\mathscr{D}_0) .$$

However, since the Slater condition is stronger than the normality one, it gives a stronger result.

1.11. Theorem. If the weak optimality condition, the convexity condition and the Slater condition hold then

$$\operatorname{Sup} f(X) = \operatorname{Inf} v(\mathscr{D}_0) = \operatorname{Min} v(\mathscr{D}_0) .$$

Proof. It suffices, in view of Corollary 1.10 and the fact that

$$\operatorname{Min} v(\mathscr{D}_0) \subset \operatorname{Inf} v(\mathscr{D}_0) ,$$

to prove

$$\operatorname{Sup} f(X) \subset \operatorname{Min} v(\mathscr{D}_0) .$$

Let $y_* \in \operatorname{Sup} f(x)$. Obviously $(\operatorname{icr} E) \cap \operatorname{icr} E_{y_*} = \emptyset$. So there exists a nonzero $(q^*, y^*) \in Q^* \times Y^*_+$ such that

$$\langle y^*, y_* \rangle \geqq \langle q^*, q \rangle + \langle y^*, \Phi(x, q) \rangle \quad \forall (x, q) \in X \times Q .$$

Since $0 \in \operatorname{icr} E_Q$, y^* cannot be zero. Consequently,

$$y_* \in v(q^*, y^*) \subset v(\mathscr{D}_0) .$$

Now, by the weak duality, we have $y_* \in \operatorname{Min} v(\mathscr{D}_0)$.

The weak optimality condition in Theorem 1.11 cannot be omitted, because y^*

is not generally positive. However, if the weak optimality condition is not supposed we have the following weaker assertion.

1.12. Theorem. (Direct Duality.) Suppose that the convexity and the Slater conditions are satisfied. If $y_* \in \operatorname{Max} f(X)$ is such that there exists a $y^* \in Y^*_{++}$ with

$$\langle y^*, y_* \rangle \geqq \langle y^*, f(x) \rangle \quad \forall x \in X$$

then

$$y_* \in \operatorname{Min} v(\mathscr{D}_0) .$$

Proof. Let $y_* \in f(X)$ and $y^* \in Y^*_{++}$ be such that

$$\langle y^*, y_* \rangle \geqq \langle y^*, f(x) \rangle \quad \forall x \in X .$$

By Lemma 1.13 below there exists a $q^* \in Q^*$ such that

$$\langle y^*, y_* \rangle \geqq \langle q^*, q \rangle + \langle y^*, \Phi(x, q) \rangle \quad \forall (x, q) \in X \times Q .$$

Hence $y_* \in v(\mathscr{D}_0)$ and, by the weak duality, $y_* \in \operatorname{Min} v(\mathscr{D}_0)$.

1.13. Lemma. Suppose that the convexity and the Slater conditions hold. Let $y_* \in f(X) - Y_+$ and $y^* \in Y^*_{++}$ be such that

$$\langle y^*, y_* \rangle \geqq \langle y^*, f(x) \rangle \quad \forall x \in X .$$

Then there exists a $q^* \in Q^*$ such that

$$\langle y^*, y_* \rangle \geqq \langle q^*, q \rangle + \langle y^*, \Phi(x, q) \rangle \quad \forall (x, q) \in X \times Q .$$

Proof. Put

$$M = \{(0, y) \in Q \times Y : \langle y^*, y \rangle = \langle y^*, y_* \rangle\} .$$

It is evident that $M \cap \operatorname{icr} E = \emptyset$. Hence, in view of Theorem 3.4.2 of [8], there exists a nonzero pair $(\bar{q}, \bar{y}) \in Q^* \times Y^*_+$ such that

$$\langle \bar{q}, q' \rangle + \langle \bar{y}, y' \rangle \leqq \langle \bar{y}, y \rangle \quad \forall (q', y') \in E \quad \forall (0, y) \in M .$$

We have, in particular,

$$\langle \bar{y}, y_* \rangle \leqq \langle \bar{y}, y \rangle \quad \forall (0, y) \in M$$

that means

$$\ker y^* \subset \{y \in Y : \langle \bar{y}, y \rangle \geqq 0\} .$$

Hence, by Corollary of Theorem 1.5 in [6], there exists a real k such that $\bar{y} = k \,.\, y^*$. Because of the Slater condition $\bar{y}$ cannot be zero and consequently $k \neq 0$. Now it suffices to put $q^* = \bar{q}/k$ and we obtain the required functional.

2. OPERATOR DUALITY

2.1. Definition. Given $\mathscr{L} \subset \mathscr{L}(Q, Y)$ we define

$$L(y) = \{l \in \mathscr{L} : y \not\leq l(q) + \Phi(x, q) \ \forall (x, q) \in X \times Q\} .$$

Obviously the point-set function $L: Y \to \mathscr{L}$ satisfies the dual availability. Putting

$$\nu(l) = \{y \in Y: l \in L(y)\}$$

and

$$\mathscr{L}_0 = \{l \in \mathscr{L}: \nu(l) \neq \emptyset\},$$

the problem

(D 2) $$\text{Min-Inf } \nu(\mathscr{L}_0)$$

is called the *operator Φ-dual* to problem (P).

Obviously

2.2. Theorem. (Weak Duality.) The weak duality condition is satisfied and hence

$$f(X) \gtreqless \nu(\mathscr{L}_0).$$

2.3. Lemma. It the weak optimality condition is satisfied and the function $\Phi(x, q)$ is concave then

$$\nu(\mathscr{L}_0) \subset \nu(\mathscr{D}_0).$$

Proof. Let $y' \in \nu(\mathscr{L}_0)$ then there exists an $l \in \mathscr{L}_0$ such that

$$y' \mathrel{\overline{\geq}} l(q) + \Phi(x, q) \quad \forall (x, q) \in X \times Q.$$

Put

$$M = \{y \in Y \mid \exists (x, q) \in X \times Q: y \leqq l(q) + \Phi(x, q)\}.$$

Obviously M is convex, cor $M \neq \emptyset$ and $y' \notin$ cor M. Hence there exists a $y^* \in Y^*_{++}$, and because of the weak optimality condition y^* is positive, such that

$$\langle y^*, y' \rangle \geqq \langle y^* \circ l, q \rangle + \langle y^*, \Phi(x, q) \rangle \quad \forall (x, q) \in X \times Q.$$

Putting $q^* = y^* \circ l$, we have $(q^*, y^*) \in \mathscr{D}_0$ and $y' \in \nu(q^*, y^*)$.

2.4. Completeness condition:

$$\forall (q^*, y^*) \in \mathscr{D}_0 \quad \exists l \in \mathscr{L}: q^* = y^* \circ l.$$

2.5. Lemma. If the completeness condition holds then

$$\nu(\mathscr{D}_0) \subset \nu(\mathscr{L}_0).$$

Proof. Let $y \in \nu(\mathscr{D}_0)$ then there exists a $(q^*, y^*) \in \mathscr{D}$ such that

(2.5.1) $$\langle y^*, y \rangle \geqq \langle q^*, q \rangle + \langle y^*, \Phi(x, q) \rangle \quad \forall (x, q) \in X \times Q.$$

In view of the completeness condition there exists an $l \in \mathscr{L}$ such that $q^* = y^* \circ l$. Now, if there is an $x' \in X$ and $q' \in Q$ such that

$$y < l(q') + \Phi(x', q'),$$

one has then, for y^* is positive,

$$\langle y^*, y \rangle < \langle y^* \circ l, q' \rangle + \langle y^*, \Phi(x', q') \rangle$$

that contradicts (2.5.1). Hence $y \in \nu(l)$.

Summarizing Lemmas 2.3 and 2.5 we obtain

2.6. Proposition. Suppose that $\Phi(x, q)$ is concave, the weak optimality and the completeness conditions hold. Then

$$v(\mathscr{D}_0) = v(\mathscr{L}_0).$$

Obviously, the concavity of function $\Phi(x, q)$ guarantees the convexity of the set E. Consequently from Theorem 1.7 and Proposition 2.6 it follows

2.7. Theorem. (Strong Duality.) If the function $\Phi(x, q)$ is concave, $\operatorname{icr} E \neq \emptyset$ and the weak optimality, the normality and the completeness conditions hold then

$$\operatorname{Sup} f(X) = \operatorname{Inf} v(\mathscr{L}_0).$$

Similarly, using Theorem 1.11 and Proposition 2.6 we obtain

2.8. Theorem. If $\Phi(x, q)$ is concave, $\operatorname{icr} E \neq \emptyset$ and the weak optimality, the Slater and the completeness conditions hold then

$$\operatorname{Sup} f(x) = \operatorname{Inf} v(\mathscr{L}_0) = \operatorname{Min} v(\mathscr{L}_0).$$

2.9. Theorem. (Direct Duality.) Suppose that the convexity, the Slater and the completeness conditions hold. If $y_* \in \operatorname{Max} f(X)$ is such that there exists a $y^* \in Y^*_{+++}$ with

$$\langle y^*, y_* \rangle \geqq \langle y^*, f(x) \rangle \quad \forall x \in X,$$

then

$$y_* \in \operatorname{Min} v(\mathscr{L}_0).$$

Proof. The proof of this statement is similar to that of Theorem 1.12

2.10. Strong Completeness Condition:

$$\forall (q^*, y^*) \in Q_0^* \times Y^*_{++} \quad \exists l \in \mathscr{L}: q^* = y^* \circ l,$$

where

$$Q_0^* = \{q^* \in Q^* \mid \exists y^* \in Y^*_{++}: \sup \{\langle q^*, q \rangle + \langle y^*, \Phi(x, q) \rangle : (x, q) \in X \times Q\} < +\infty\}.$$

2.11. Theorem. (Direct Duality.) Suppose that $\operatorname{icr} E \neq \emptyset$, the function $\Phi(x, q)$ is strictly concave and the Slater and the strong completeness conditions hold. Then

$$\operatorname{Max} f(X) \subset \operatorname{Min} v(\mathscr{L}_0).$$

Proof. Let $y_* = f(x_*) \in \operatorname{Max} f(X)$. Obviously $(\operatorname{icr} E) \cap E_{y_*} = \emptyset$. There exists a nonzero $(q^*, y^*) \in Q^* \times Y^*$ such that

$$\text{(2.11.1)} \qquad \langle y^*, y_* \rangle \geqq \langle q^*, q \rangle + \langle y^*, \Phi(x, a) \rangle \quad \forall (x, q) \in X \times Q.$$

In view of the Slater condition we have $y^* \in Y^*_{++}$. Then by the strong completeness condition there exists an $l \in \mathscr{L}$ such that $q^* = y^* \circ l$. If there exists an $(x', q') \in X \times Q$ such that

$$y_* < l(q') + \Phi(x', q'),$$

then by (2.11.1) we have

$$\langle y^*, y_* \rangle = \langle q^*, q' \rangle + \langle y^*, \Phi(x', q') \rangle .$$

Now since $\Phi(x, q)$ is strictly concave, for any

$$(x, q) \in \{t(x_*, 0) + (1 - t) . (x', q') : 0 < t < 1\}$$

we have

$$\langle y^*, y_* \rangle < \langle q^*, q \rangle + \langle y^*, \Phi(x, q) \rangle$$

which contradicts (2.11.1). Consequently $y_* \in v(l)$ and by the weak duality $y_* \in \in \text{Min}\, v(\mathscr{L}_0)$. The proof is complete.

2.12. Theorem. (Converse Duality.) Suppose that the Slater and the strong completeness conditions are satisfied, $\Phi(x, q)$ is concave and the set $f(x) - Y_+$ is closed. If $l^* \in \mathscr{L}_0$ is an optimal solution of problem

$$\text{Min}\, v(\mathscr{L}_0) ,$$

then there exists an optimal solution x^* of problem

$$\text{Max}\, f(X)$$

such that

$$f(x^*) \in v(l^*) .$$

Proof. By definition there exists a $y_* \in v(l^*) \cap \text{Min}\, v(\mathscr{L}_0)$. If $y_* \in f(X) - Y_+$ there exists an $x^* \in X$ with

$$f(x^*) \geqq y_*$$

and the weak duality guarantees that $y_* = f(x^*) \in v(l^*)$.

Suppose, on the contrary, that $y_* \notin f(X) - Y_+$. Then there exists a $y_0 \ll y$ such that

$$y_0 \in f(X) - Y_+ \quad \text{and} \quad]y_0, y_*] \cap (f(X) - Y_+) = \emptyset .$$

Consequently there exists a $\bar{y} \in Y^*_{++}$ such that

$$\langle \bar{y}, y_0 \rangle \geqq \langle \bar{y}, f(x) \rangle \quad \forall x \in X .$$

By Lemma 1.13 there exists a $\bar{q} \in Q^*$ such that

$$\langle \bar{y}, y_0 \rangle \geqq \langle \bar{q}, q \rangle + \langle \bar{y}, \Phi(x, q) \rangle \quad \forall (x, q) \in X \times Q .$$

By the strong completeness condition there exists an $l \in \mathscr{L}$ such that $\bar{q} = \bar{y} \circ l$. Now choose a y such that $y_0 \ll y \ll y_*$ one has

$$y \mathrel{\overline{\gtrless}} l(q) + \Phi(x, q) \quad \forall (x, q) \in X \times Q ,$$

which means $y \in v(\mathscr{L}_0)$, a contradiction to $y_* \in \text{Min}\, v(\mathscr{L}_0)$.

2.13. Theorem. (Inf-Sup formulation.) If Y_+ is reproducing and the weak optimality condition holds, then

$$\text{Inf}\, v(\mathscr{L}_0) = \underset{\mathscr{L}_0}{\text{Inf}}\, \text{Sup}\, \{l(q) + \Phi(x, q) : (x, q) \in X \times Q\} .$$

Proof. We have

$$\operatorname{Inf} v(\mathcal{L}_0) = \operatorname{Inf}_{\mathcal{L}_0} \operatorname{Inf} v(l) \quad \text{(by Propositions 1.13 and 1.17 of [8])}$$

$$= \operatorname{Inf}_{\mathcal{L}_0} \operatorname{Sup} \{l(q) + \Phi(x, q): (x, q) \in X \times Q\} \quad \text{(by Corollary 1.15 of [8])} .$$

2.14. Remark. The operator duality, thanks to its Inf-Sup formulation is more analytical and applicable. On the other hand, the (strong) completeness condition makes it less general than the functional duality. However, this condition is easily satisfied if $\dim Y < +\infty$ as it is shown in the following theorem.

2.15. Theorem. Suppose that $Y = R^n$, $Y_{++} = R^n_{++}$ and $\mathcal{L} \subset \mathcal{L}(Q, Y)$ is such that

$$\{t(\underbrace{q^*, q^*, \ldots, q^*}_{n\text{-times}}): q^* \in Q_0, t > 0\} \subset \mathcal{L} . \tag{2.15.1}$$

Then the strong completeness condition is fulfilled.

Proof. Let $q^* \in Q_0^*$ and $y = (y_1, y_2, \ldots, y_n) \in R^n_{++}$. Put

$$t = 1/(y_1 + y_2 + \ldots + y_n) \quad \text{and} \quad l = t \,.\, (\underbrace{q^*, q^*, \ldots, q^*}_{n\text{-times}})$$

Then, by condition (2.15.1), $l \in \mathcal{L}$ and $q^* = y \circ l$. The proof is complete.

2.16. Example 1. Lagrange dualization.

Suppose that A is a nonempty subset in X, Z is an ordered linear space with $\operatorname{cor} Z_+ \neq \emptyset$ and $f(x)$ and $g(x)$ are functions mapping X to Y and Z respectively. Consider the problem

$$\text{Max-Sup}\,\{f(x): x \in A \text{ and } g(x) \in Z_+\} . \tag{P_1}$$

Put

$$Q = Z$$

and

$$\Phi(x, q) = f(x) + \delta_A^-(x) + \delta_{Z_+}^-(g(x) - q) ,$$

where

$$\delta_A^-(x) = \begin{cases} 0 & \text{if } x \in A \\ -\infty & \text{if } x \notin A . \end{cases}$$

Given a set $\mathcal{L} \subset \mathcal{L}(Q, Y)$ we have, for $l \in \mathcal{L}$,

$$v(l) = \{y \in Y: y \geqq l(g(x)) + f(x) \ \forall x \in A\} .$$

Put

$$\mathcal{L}_0 = \{l \in \mathcal{L}: v(l) \neq \emptyset\} .$$

Problem

$$\text{Min-Inf}\, v(\mathcal{L}_0) \tag{D_1}$$

is then called the *Lagrange dual* to problem (P_1).

If the weak optimality condition holds and Y_+ is reproducing, then the dual

Inf-problem can be written as follows

$$\operatorname*{Inf}_{\mathscr{L}_0} \operatorname{Sup} \{f(x) + l(g(x)) : x \in A\} .$$

From definitions it follows immediately

2.16.1. Theorem. If the set A is convex and the functions $f(x)$ and $g(x)$ are concave then the convexity condition is fulfilled.

2.16.2. Theorem. If there exists a point $x_0 \in A$ such that

$$g(x_0) \in \operatorname{cor} Z_+$$

then the Slater condition is fulfilled.

2.16.3. Theorem. If $Y = R^n$, $Y_+ = R^n_+$ and

$$\{(\underbrace{q^*, q^*, \ldots, q^*}_{n\text{-times}}) : q^* \in Q^*_+\} \subset \mathscr{L}$$

then the strong completeness condition is fulfilled.

Proof. It is easily seen that $Q^*_0 \subset Q^*_+$. Consequently the statement follows from Theorem 2.15.

2.17. Example 2. Fenchel dualization.

Suppose that $S_1, S_2, \ldots, S_n$ are nonempty subsets in X, $f_1(y)\ f_2(x). \ldots, f_n(x)$ are functions from X to Y. We will be concerned with the problem

$$(\mathrm{P}_2) \qquad \text{Max-Sup} \{f_1(x) + f_2(x) + \ldots + f_n(x) : x \in \bigcap_{i=1}^{n} S_i\} .$$

Put

$$Q = \prod_{n\text{-times}} X$$

and

$$\Phi(x, q) = f_1(x + q_1) + f_2(x + q_2) + \ldots + f_n(x + q_n) + \sum_{i=1}^{n} \delta^-_{S_i}(x + q_i) ,$$

where $q = (q_1, q_2, \ldots, q_n) \in Q$.

Given a set $\mathscr{L} \subset \mathscr{L}(Q, Y) = \prod_{n\text{-times}} \mathscr{L}(X, Y)$ such that

$$\forall l = (l_1, \ldots, l_n) \in \mathscr{L} : \sum_{i=1}^{n} l_i = 0$$

we have, for $l = (l_1, \ldots, l_n) \in \mathscr{L}$,

$$v(l) = \{y \in Y : y \not\leqq \sum_{i=1}^{n} (l_i(q_i) + f_i(x + q_i) + \delta^-_{S_i}(x + q_i))$$

$$\forall (x, q) \in X \times Q\} =$$

$$= \{y \in Y : y \not\leqq \sum_{i=1}^{n} (l_i(x_i) + f_i(x_i))\ \forall (x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n\} .$$

Put

$$\mathscr{L}_0 = \{l \in \mathscr{L} : v(l) \neq \emptyset\} .$$

Problem

$$(\mathrm{D}_2) \qquad \text{Min-Inf}\, v(\mathscr{L}_0)$$

is then called the *Fenchel dual* to problem (P_2).

If the weak optimality condition holds and Y_+ is reproducing, then the dual Inf-problem can be written in the following form

$$\operatorname*{Inf}_{\mathscr{L}_0} \operatorname{Sup} \{ \sum_{i=1}^{n} (l_i(x_i) + f_i(x_i)) : (x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n \} .$$

From definitions it follows immediately

2.17.1. Theorem. If the sets $S_1, S_2, \ldots, S_n$ are convex and the functions $f_1(x), f_2(x), \ldots, f_n(x)$ are concave, then the convexity condition is fulfilled.

2.17.2. Theorem. If

$$\bigcap_{i=1}^{n} \operatorname{cor} S_i \neq \emptyset ,$$

then the Slater condition is satisfied.

2.17.3. Theorem. If $\mathscr{L}$ is the set of all $l = (l_1, \ldots, l_n)$ from $\prod_{n\text{-times}} \mathscr{L}(X, Y)$ such that $\sum_{i=1}^{n} l_i = 0$ and $Y_+ = R^m_+$, then the strong completeness holds.

Proof. Let $q^* = (q_1^*, \ldots, q_n^*) \in Q_0^* \subset \underbrace{X^* \times \ldots \times X^*}_{n\text{-times}}$. There exists a point $y^* \in R^m_{++}$ such that

$$\sup \{ \sum_{i=1}^{n} (\langle q_i^*, q_i \rangle + \langle y^*, f_i(x + q_i) \rangle + \delta_{S_i}^-(x + q_i)) : (x, q) \in X \times Q \}$$

$$= \sup \{ \sum_{i=1}^{n} (\langle q_i^*, x_i \rangle + \langle y^*, f_i(x_i) \rangle) - \langle \sum_{i=1}^{n} q_i^*, x \rangle : (x, x_1, \ldots, x_n) \in$$

$$\in X \times S_1 \times \ldots \times S_n \} < +\infty .$$

So we have $\sum_{i=1}^{n} q_i = 0$.

Now for any $y \in R^m_{++}$ we define

$$l_i = t \,.\, \underbrace{(q_i^*, \ldots, q_i^*)}_{m\text{-times}} \quad i = 1, \ldots, n$$

$$l = (l_1, \ldots, l_n)$$

where

$$t = 1/(y_1 + \ldots + y_m) .$$

It is easy to verify that

$$\sum_{i=1}^{n} l_i = 0$$

and

$$q^* = y \circ l .$$

The proof is thus complete.

3. EXTREMAL-VALUE FUNCTION

In the literature duality criteria have usually established with help of the so-called extremal-value function. Following this tradition in this section we derive, via the extremal-value function, some sufficient criteria for the convexity and the normality conditions to be valid.

3.1. Definition. Consider the perturbed primal problem (P_q) in Section 0. The function $h: Q \to 2^Y \cup \{-\infty, +\infty\}$ defined as follows

$$h(q) = \begin{cases} -\infty & \text{if } \Phi(X, q) = \emptyset \\ \operatorname{Sup} \Phi(X, q) & \text{if } \operatorname{Sup}(X, q) \neq \emptyset \\ +\infty & \text{if } \Phi(X, q) \text{ is not weakly bounded from above by any element from } Y \end{cases}$$

is called the extremal-value function of problem (P_q).

3.2. Definition. The multivalued function $h: Q \to 2^Y \cup \{-\infty, +\infty\}$ is said to be *concave* if for any $q_1, q_2 \in Q$ and $\alpha, \beta > 0: \alpha + \beta = 1$ we have

$$\alpha\, h(q_1) + \beta\, h(q_2) \subset h(\alpha q_1 + \beta q_2) = Y_+ .$$

Supposing that Q is a topological space we say that function h is *weakly upper semicontinuous* at $q_0 \in Q$ if $h(q_0) \subset Y$ and

$$\forall y_+ \in \operatorname{cor} Y_+ \quad \exists \text{ neighbourhood } U \text{ of } q_0 \quad \forall q \in U: h(q) \not> h(q_0) + y_+ .$$

Analogously are defined *convex* and *weakly lower semicontinuous functions.* Evidently the just defined notions are a generalization of the corresponding ones of the scalar functions.

We denote the *hypograph* of function $h(q)$ by H:

$$H = \{(q, y) \in Q \times Y: y \in h(q) - Y_+\} .$$

In the sequel we suppose that the weak optimality is fulfilled and Y_+ is reproducing.

Obviously

3.3. Lemma. If the function $h(q)$ is concave then the hypograph H is convex.

3.4. Lemma. We have

$$H \subset \operatorname{lin} E$$

where the set E has been defined in Section 0.

Proof. Let $(q_0, y_0) \in H$. By definition

$$y_0 \in \operatorname{lin} [\Phi(X, q_0) - Y_+] .$$

Then since

$$\{(q_0, y): y \in \Phi(X, q_0) - Y_+\} \subset E ,$$

we have

$$(q_0, y_0) \in \operatorname{lin} E .$$

3.5. Lemma. We have

$$E \subset H .$$

Proof. Given $(q_0, y_0) \in E$ then there exists an $x_0 \in X$ with

$$y_0 \leqq \Phi(x_0, q_0) .$$

If $h(q_0) = +\infty$ then obviously $(q_0, y_0) \in H$.

If $h(q_0) \neq +\infty$ then $\Phi(X, q_0)$ is weakly bounded from above and, by Proposition 1.12 of [8], $h(q_0)$ is sup-stable with respect to $\Phi(X, q_0)$. So there exists a $y \in h(q_0)$ with

$$y \geqq \Phi(x_0, q_0) \geqq y_0 .$$

Hence $(q_0, y_0) \in H$.

Summarizing Lemmas 3.3, 3.4 and 3.5 we obtain

3.6. Proposition. If function $h(q)$ is concave then the convexity condition is fulfilled.

3.7. Definition. Suppose that Q is a linear topological space then the problem (P) is said to be *normal* with respect to Φ if $h(q)$ is weakly upper semicontinuous at 0.

3.8. Proposition. If problem (P) is normal then the normality condition is satisfied.

Proof. By contradiction.

Let $y_0 \in Y$ be such that $E \cap E_{y_0} = \emptyset$ and $\bar{E} \cap E_{y_1} \neq \emptyset$ for some $y_1 \gg y_0$. Choose a point $y_+ \in \operatorname{cor} Y_+$, then on the segment

$$\{y_0 - t \,.\, y_+ : t \geqq 0\}$$

we can find an $y_* \in h(0) = \operatorname{Sup} f(X) \neq \emptyset$. Put

$$y_2 = \tfrac{1}{2}(y_1 + y_0) .$$

Then obviously $y_2 > y_*$ and hence, considering the upper semicontinuity of function $h(q)$ at 0, there exists a neighbourhood U of 0 such that

$$h(q) \not\geqq h(0) + y_2 - y_* \quad \forall q \in U . \tag{3.8.1}$$

But since $\bar{E} \cap E_{y_1} \neq \emptyset$ there exist a $q \in U$ and an $x \in X$ such that

$$\Phi(x, q) \gg y_2 .$$

Hence there exists, for $h(q)$ is sup-stable with respect to $\Phi(X, q)$, a $y \in h(q)$ such that

$$y \geqq \Phi(x, q) \gg y_2 ,$$

which contradicts (3.8.1). The proof is thus complete.

The following statement is evident from definition.

3.9. Lemma. Suppose that Q is a linear topological space. If there exist a neighbourhood U of 0 and a map $x: U \to X$ such that

$$\Phi(q, x(q)) \in Y \quad \forall q \in U ,$$

then the Slater condition is fulfilled.

From Lemma 3.9 it follows immediately

3.10. Proposition. If Q is a linear topological space and

$$0 \in \operatorname{int}\{q \in Q\colon h(q) \neq -\infty\},$$

then the Slater condition is satisfied.

Now summarizing Propositions 3.6, 3.8, 3.10 and Theorems 1.7 and 1.11 we obtain:

3.11. Theorem. Suppose that Q is a linear topological space and the extremal-value function $h(q)$ is concave. Then

(a) If problem (P) is normal with respect to Φ, then

$$\operatorname{Sup} f(x) = \operatorname{Inf} v(\mathscr{D}_0)$$

(b) If $0 \in \operatorname{int}\{q \in Q\colon h(q) \neq -\infty\}$, then

$$\operatorname{Sup} f(X) = \operatorname{Min} v(\mathscr{D}_0)\,.$$

(Received September 10, 1985.)

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