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ALGEBRAIC EQUIVALENCES OVER FUZZY QUANTITIES[1]

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Fuzzy numbers and fuzzy quantities do not generally fulfil some fundamental algebraic properties valid for crisp numbers, as shown e.g. in [1]. But it is possible to avoid this discrepancy if the strict equality between fuzzy quantities is substituted by rather weaker equivalence relations (cf. [5, 8, 9]) more respecting the natural vagueness of fuzzy phenomena.

The equivalence relations suggested in the referred papers are based on analogous principles, however they are modified for the specific cases of addition and multiplication relations. Here we suggest a generalized equivalence model covering both previous equivalences (additive and multiplicative) as its special cases, and show its applicability to adequate description of certain class of algebraic treatments of fuzzy numbers and fuzzy quantities.

1. INTRODUCTION

It is well known that even the simplest arithmetical operations like addition and multiplication over fuzzy quantities (see e.g. [1, 5, 9]) are connected with serious formal difficulties. They do not fulfil the important group property of the existence of opposite (inverse) element, are not distributive, etc. This fact, which is a natural consequence of the vagueness of fuzzy quantities and subsequent uncertainty in mutual relations between them, does seriously complicate the routine fuzzy data processing.

As shown in some of the referred works, e.g. in [5, 7, 9], these difficulties are deeply connected with the inadequacy of the strict equality to the vague nature of fuzziness. The equivalence relations suggested in [5] and [9] and based on the equality "up to fuzzy zero" (in the additive case [5]) or "up to multiplicative fuzzy 1" (in the multiplicative case [9]) much better reflect the essential structure of fuzzy numbers and fuzzy quantities. They can guarantee the validity of the group properties (for addition or multiplication, respectively), and also the distributivity of the multiplication of fuzzy quantity by crisp numbers for a useful class of symmetric fuzzy quantities (cf. [8]).

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Those two equivalences, additive and multiplicative, however analogous in their formal structure, essentially differ regarding the form of similarity between the equivalent elements. It means that they cannot be arbitrarily substituted by each other.

It would be evidently very useful to construct a combination or generalization of those particular equivalences which could be applied to the addition as well as multiplication and which would include both equivalences mentioned above as its special cases. In this paper we attempt to suggest such generalized algebraical equivalence and to derive some of its properties.

Before doing so we briefly remember the necessary basic notions as well as the formal definitions of the additive and multiplicative equivalence. Then, in the main section of the paper, we deal with their generalization.

2. NORMAL FUZZY QUANTITIES

In the whole paper we denote by R the set of all real numbers and by R_0 the set of all non-zero numbers from R, i.e. $R_0 = R - \{0\}$.

A *normal fuzzy quantity* over R, is a fuzzy subset a of R with membership function $f_a : R \to [0,1]$ such that

$$\sup\{f_a(x) : x \in R\} = 1. \tag{1}$$

The set of all normal fuzzy quantities fulfilling (1) is denoted by $\mathbb{R}$.

In some cases, namely, when some additive properties are investigated, it is comfortable to suppose also that the set

$$\{x \in R : f_a(x) > 0\} \tag{2}$$

is bounded. This assumption is too limiting in the multiplicative case and it is not generally requested below.

The set of all normal fuzzy quantities a from $\mathbb{R}$ such that

$$f_a(0) = 0 \tag{3}$$

is in the following explanation denoted by $\mathbb{R}_0$.

If $a, b \in \mathbb{R}$ are normal fuzzy quantitites then the equality symbol $a = b$ means that $f_a(x) = f_b(x)$ for all $x \in R$.

Due to [2, 5, 9] and other papers we introduce the arithmetical operations over normal fuzzy quantity in the following way. Let $a, b \in \mathbb{R}$ be normal fuzzy quantities with membership functions f_a, f_b, respectively. Then the normal fuzzy quantity $a \oplus b \in \mathbb{R}$ with membership function

$$f_{a\oplus b}(x) = \sup_{y\in R}\left(\min\left(f_a(y),\, f_b(x-y)\right)\right), \qquad x \in R, \tag{4}$$

is called the *sum* of a and b. If, moreover, $a, b \in \mathbb{R}_0$ then the normal fuzzy quantity $a \odot b \in \mathbb{R}$ such that

$$f_{a\odot b}(x) = \sup_{y\in R_0}\left(\min\left(f_a(y),\, f_b(x/y)\right)\right), \qquad x \in R \tag{5}$$

is called the *product* of a and b.

The algebraic properties of the operations $\oplus$ and $\odot$ are presented e. g. in [2, 5, 9]. It is very easy to verify that for $a, b, c \in \mathbb{R}$

$$a \oplus b = b \oplus a \qquad \text{and} \qquad a \oplus (b \oplus c) = (a \oplus b) \oplus c. \tag{6}$$

If we denote for $x \in R$ by $\langle x \rangle$ the normal fuzzy quantity for which

$$f_{\langle x \rangle}(x) = 1, \qquad f_{\langle x \rangle}(y) = 0 \qquad \text{if } y \neq x \tag{7}$$

then also

$$a + \langle 0 \rangle = a. \tag{8}$$

Nevertheless, if $a \in \mathbb{R}$ and if we denote by $-a$ the normal fuzzy quantity for which

$$f_{-a}(x) = f_a(-x) \qquad \text{for all } x \in R, \tag{9}$$

then the equality $a \oplus (-a) = \langle 0 \rangle$ does not generally hold.

Analogously for the multiplication, if $a, b, c \in \mathbb{R}_0$ then

$$a \odot b = b \odot a, \qquad a \odot (b \odot c) = (a \odot b) \odot c, \qquad a \odot \langle 1 \rangle = a, \tag{10}$$

but for $1/a \in \mathbb{R}_0$ such that

$$f_{1/a}(x) = f_a(1/x), \qquad x \neq 0, \tag{11}$$

the last one of the desired group relations, namely $a \odot (1/a) = \langle 1 \rangle$ is not generally fulfilled. It is also evident that the limitedness of the support set (2), mentioned above, implies that $1/a$ can have limited support iff there exists a closed neighborhood of 0, let us denote it by $\mathcal{U}_0$, such that $f_a(x) = 0$ for all $x \in \mathcal{U}_0$. This essentially limits the validity of results presented in Section 5 not offering qualitatively new ones. It is the main reason why the limitedness assumption was omitted.

Moreover, if $r, t \in R$, $a, b \in \mathbb{R}$, and if we denote by $r \cdot a$ the multiplication $\langle r \rangle \odot a$, i.e. for $x \in R$

$$\begin{aligned} f_{r \cdot a}(x) &= f_a(x/r) = f_{\langle r \rangle \odot a}(x) \quad \text{for } r \neq 0 \\ &= f_{\langle 0 \rangle}(x) \qquad\qquad\qquad \text{for } r = 0 \end{aligned} \tag{12}$$

then

$$r \cdot (a \oplus b) = r \cdot a \oplus r \cdot b \tag{13}$$

but the complementary distribution law $(r + t) \cdot a = r \cdot a \oplus t \cdot a$ is not generally fulfilled.

However, it is possible to ensure the validity of all group properties at least in a weakened form. The method of doing it was suggested in [5, 6, 7] for additive case, and in [9] for the multiplicative variant. Also the distributivity of the crisp product (12) can be assured in the same way for an interesting subclass of fuzzy quantities as shown in [8]. The principles of that approach are briefly remembered in the following two sections.

3. ADDITIVE EQUIVALENCE

The strict equality between n. f. q. s introduced above is too strong to be adequate to the vague nature of fuzzy phenomena like fuzzy quantities. This is the main essential cause of the invalidity of some algebraic properties which are self-evident for crisp numbers. If the strict equality is substituted by a properly weaker relation, the group (and some other) properties keep valid. In the case of addition, considered here, the equivalence defined as "equality up to fuzzy zero" proved to be adequate.

Let $y \in R$ and $s \in \mathbb{R}$ be such that for any $x \in R$

$$f_s(y+x) = f_s(y-x). \tag{14}$$

Then we say that s is y-symmetric, and the set of all y-symmetric n. f. q. s will be denoted by $\mathbb{S}_y$. By $\mathbb{S}$ we denote the union

$$\mathbb{S} = \bigcup_{y \in R} \mathbb{S}_y \subset \mathbb{R}.$$

If a, $b \in \mathbb{R}$ then we say that *a is additively equivalent to b* and write $a \sim_\oplus b$ iff there exist s_1, $s_2 \in \mathbb{S}_0$ such that

$$a \oplus s_1 = b \oplus s_2. \tag{15}$$

It is easy to verify (cf. for example [6, 7, 8]) that the equality $a = b$ for a, $b \in \mathbb{R}$ evidently implies $a \sim_\oplus b$, and that

$$a \oplus (-a) \sim_\oplus \langle 0 \rangle, \tag{16}$$

so that all group properties are fulfilled up to the additive equivalence ([6], Theorem 4).

Moreover, if $a \in \mathbb{S}$ and r, $t \in R$ then

$$(r+t) \cdot a \sim_\oplus r \cdot a \oplus t \cdot a \tag{17}$$

(cf. [8], Lemma 4), and evidently (see also [8], Lemma 1) for every $a \in \mathbb{S}$ there exist $y \in R$ and $s \in \mathbb{S}_0$ such that $a = \langle y \rangle \oplus s$.

4. MULTIPLICATIVE EQUIVALENCE

An analogous method can be used to reach the weakened fulfillment of the multiplicative group properties. Due to [9] we limit its application to the normal fuzzy quantities which are either positive or negative.

Let $a \in \mathbb{R}_0$. We say that a is positive iff $f_a(x) = 0$ for all $x \le 0$, and a is negative iff $f_a(x) = 0$ for all $x \ge 0$. The sets of all positive or negative n. f. q. s will be denoted by $\mathbb{R}^+$ or $\mathbb{R}^-$, respectively. By $\mathbb{R}^\star$ we denote the union

$$\mathbb{R}^\star = \mathbb{R}^+ \cup \mathbb{R}^-, \tag{18}$$

and the normal fuzzy quantities from $\mathbb{R}^\star$ will be called polarized.

Let $y \in R_0$ and $a \in \mathbb{R}^\star$. We say that a is y-transversible iff

$$f_a(y \cdot x) = f_a(y/x) \quad \text{for } x > 0, \qquad f_a(y \cdot x) = 0 \quad \text{for } x \leq 0. \tag{19}$$

The set of all y-transversible polarized normal fuzzy quantity will be denoted by $\mathbb{T}_y$, and by $\mathbb{T}$ we denote the union

$$\mathbb{T} = \bigcup_{y \in R_0} \mathbb{T}_y. \tag{20}$$

Evidently (cf. [9] Lemma 9) for any $a \in \mathbb{T}_y$ there exists $t \in \mathbb{T}_1$ such that $a = t \odot \langle y \rangle$. If a, $b \in \mathbb{R}^\star$ then we say that they are *multiplicatively equivalent* and write $a \sim_\odot b$ iff there exist 1-transversible n.f.q.s t_1, $t_2 \in \mathbb{T}_1$ such that

$$a \odot t_1 = b \odot t_2. \tag{21}$$

It can be easily seen (cf. [9] Theorem 7) that

$$a \odot (1/a) \sim_\odot \langle 1 \rangle, \tag{22}$$

which is the weakened form of the remaining group property, and consequently $\mathbb{R}^\star$ is a multiplicative group up to the equivalence relation $\sim_\odot$.

5. COMBINED EQUIVALENCE

Each of the equivalences, the additive $\sim_\oplus$ and the multiplicative $\sim_\odot$ one, are adapted to the specific problems of the respective arithmetic operations over normal fuzzy quantities. However, it could be at least interesting to find a common generalization of both of them. In this section we suggest such a generalized equivalence and derive some of its properties.

It is based on the notions of symmetric and transversible normal fuzzy quantities and combines their properties. Combining the definitions of $\sim_\oplus$ and $\sim_\odot$, it is evident that we have to sum and multiply the normal fuzzy quantities in question. But in this case the definition of multiplication given in (5) (and also in [9]) cannot satisfy the demands of the new generalized equivalence concept as it is defined for normal fuzzy quantities from $\mathbb{R}_0$ only.

If $a \in \mathbb{R}_0$ and $b \neq \langle 0 \rangle$ then (4) implies that $a \oplus b$ need not belong to $\mathbb{R}_0$, which fact essentially limits the possibilities of combined use of the addition $\oplus$ and multiplication $\odot$. Hence, the first task of this section is to extend the operation $\odot$ from $\mathbb{R}_0$ to the whole set $\mathbb{R}$.

5.1. Extended multiplication

Evidently, the crucial problem of the intended extension of the product operation is its behaviour in the zero-point.

Definition 1. Let a, $b \in \mathbb{R}$ be normal fuzzy quantities Then the normal fuzzy quantity $a \odot b$ with membership function $f_{a\odot b}$ defined by (5), i.e.

$$f_{a\odot b}(x) = \sup_{y\in R_0} (\min(f_a(y),\, f_b(x/y)))$$

for $x \in R_0$ and

$$f_{a\odot b}(0) = \max(f_a(0),\, f_b(0)), \tag{23}$$

is called the product of a and b.

Remark 1. If a, $b \in \mathbb{R}_0$ then Definition 1 coincides with (5).

Remark 2. If $a \in \mathbb{R}$, $b \in \mathbb{R}_0$ then $f_{a\odot b}(0) = f_a(0)$. Especially for $t \in \mathbb{T} \subset \mathbb{R}_0$ $f_{a\odot t}(0) = f_a(0)$ (which could not be reached if minimum is considered instead of maximum in (23)).

Remark 3. If a, b, $c \in \mathbb{R}$ then

$$f_{a\odot b}(x) = f_{b\odot a}(x), \qquad f_{(a\odot b)\odot c}(x) = f_{a\odot(b\cdot c)}(x), \qquad f_{a\odot\langle 1\rangle}(x) = f_a(x) \tag{24}$$

for all $x \in R$ as follows from (5) (i.e. from [9], Theorem 1), from (23) and from the previous remarks.

The inverse value $1/a$ for $a \in \mathbb{R}$ is defined only for $a \in \mathbb{R}_0$. Fuzzy quantities from $\mathbb{R} - \mathbb{R}_0$ represent certain form of zero elements in $\mathbb{R}$, and they cannot fulfill the last group condition. Moreover, as shown in [9], Section 4.4, there exist serious difficulties concerning the last group property (for any a there exists $1/a$ such that $a \odot (1/a) \sim_\odot \langle 1\rangle$) if $a \in \mathbb{R}_0 - \mathbb{R}^*$. These difficulties can be caused by n.f.q.s from $\mathbb{R}_0 - \mathbb{R}^*$ for which (19) is fulfilled and which are not polarized. Eventual attempt to extend the sets $\mathbb{T}_y$, $y \in R_0$ also to non-polarized case leads to unpleasant paradoxes, and without this extension the last group property cannot be fulfilled outside $\mathbb{R}^*$.

It means that $\mathbb{R}$ is a monoid regarding the multiplication (5), (23), and its subset $\mathbb{R}^*$ forms a group up to the equivalence relation $\sim_\odot$.

5.2. Weak equivalence

The attempt to define some kind of weaker equivalence is motivated by the endeavour to find a more general relation combining the advantages of additive equivalence $\sim_\oplus$ and multiplicative equivalence $\sim_\odot$ applicable in one computational procedure. Such relation should be an equivalence (reflexive, symmetric and transitive), it should include both former equivalences as its special cases, and it should preserve the useful properties (e.g. guaranteeing the group conditions) parallelly for both, addition and multiplication. Here we suggest one possible version of such relation.

Definition 2. If $a, b \in \mathbb{R}$ then we say that a and b are related and write $a \sim b$ iff $a \sim_\oplus b$ or $a \sim_\odot b$.

Evidently the similarity relation $\sim$ is not an equivalence as it is not transitive. However it can be an elementary part of the weak equivalence.

Definition 3. Let $a, b \in \mathbb{R}$ be normal fuzzy quantities. We say that they are weakly equivalent and write $a \approx b$ iff there exists a finite sequence of n.f.q.s $\{c_1, \ldots, c_n\} \subset \mathbb{R}$ such that

$$c_1 = a, \qquad c_n = b \qquad c_i \sim c_{i+1} \qquad \text{for } i = 1, \ldots, n-1. \tag{25}$$

Remark 4. If $a \sim_\oplus b$ or $a \sim_\odot b$ then evidently $a \approx b$.

Theorem 1. The weak equivalence relation $\approx$ is reflexive, symmetric and transitive.

Proof. The validity of this theorem obviously follows from Definition 3. If $a \in \mathbb{R}$ then $a \approx a$ as $a = a$, $a \sim_\oplus a$ and $a \sim_\odot a$. Analogously the symmetry follows from (25) and from the symmetry of $\sim_\oplus$ and $\sim_\odot$. If $a \approx b$ and $b \approx c$, $a, b, c \in \mathbb{R}$, then there exist sequences

$$\{c_1, \ldots, c_n\}, \quad \{c_{n+1}, \ldots, c_m\}$$

such that

$$\begin{aligned} &c_1 = a, \qquad c_n = b = c_{n+1}, \qquad c_m = c, \\ &c_i \sim c_{i+1} \qquad \text{for } i = 1, \ldots, n-1 \text{ and } i = n+1, \ldots, m-1. \end{aligned} \tag{26}$$

Relations (26) imply that

$$c_i \sim c_{i+1} \qquad \text{for } i = 1, \ldots, m-1$$

and consequently $a \approx c$. □

Theorem 2. Let $a, b, c \in \mathbb{R}$ and let there exist $s_1, \ldots, s_m$, $s'_1, \ldots, s'_n \in \mathbb{S}_0$, $t_1, \ldots, t_m$,
$t'_1, \ldots, t'_n \in \mathbb{T}_1$ such that

$$c = ((\ldots((((a \oplus s_1) \odot t_1) \oplus s_2) \odot t_2) \oplus \ldots) \oplus s_m) \odot t_m \tag{27}$$

and

$$c = ((\ldots((((b \odot t'_1) \oplus s'_1) \odot t'_2) \oplus s'_2) \odot \ldots) \oplus t'_n) \oplus s'_n \tag{28}$$

then $a \approx b \approx c$.

Proof. Let us denote

$$\begin{aligned}
c_{2m} &= (\ldots((((a \oplus s_1) \odot t_1) \oplus s_2) \odot t_2) \oplus \ldots) \oplus s_m, \\
c_{2m-1} &= ((\ldots((((a \oplus s_1) \odot t_1) \oplus s_2) \odot t_2) \oplus \ldots) \oplus s_{m-1}) \odot t_{m-1}, \\
&\vdots \\
c_4 &= (((a \oplus s_1) \odot t_1) \oplus s_2) \odot t_2, \\
c_3 &= ((a \oplus s_1) \odot t_1) \oplus s_2), \\
c_2 &= (a \oplus s_1) \odot t_1, \\
c_1 &= a \oplus s_1,
\end{aligned}$$

and

$$\begin{aligned}
c'_{2n} &= (\ldots((((b \odot t'_1) \oplus s'_1) \odot t'_2) \oplus s'_2) \odot \ldots) \odot t'_n, \\
c'_{2n-1} &= ((\ldots((((b \odot t'_1) \oplus s'_1) \odot t'_2) \oplus s'_2) \odot \ldots) \odot t'_{n-1}) \oplus s'_{n-1}, \\
&\vdots \\
c'_4 &= (((b \odot t'_1) \oplus s'_1) \odot t'_2) \oplus s'_2), \\
c'_3 &= ((b \odot t'_1) \oplus s'_1) \odot t'_2, \\
c'_2 &= (b \odot t'_1) \oplus s'_1, \\
c'_1 &= b \odot t'_1.
\end{aligned}$$

Then obviously

$$\begin{aligned}
& c_1 \sim a,\ c_2 \sim c_1,\ c_3 \sim c_2, \ldots, c_{2m-1} \sim c_{2m},\ c_{2m} \sim c, \\
\text{and} \quad & c'_1 \sim b,\ c'_2 \sim c'_1,\ c'_3 \sim c'_2, \ldots, c'_{2n-1} \sim c'_{2n},\ c'_{2n} \sim c,
\end{aligned}$$

which means that $a \approx c$ and $b \approx c$, hence the statement is proved. □

Remark 5. The Theorem is evidently true if some of relations (27) or

$$c = ((\ldots(((a \oplus s_1) \odot t_1) \oplus s_2) \odot \ldots) \odot t_{m-1}) \oplus s_m, \tag{29}$$

$$\text{or} \quad c = ((\ldots(((a \odot s_1) \oplus s_1) \odot t_2) \oplus \ldots) \oplus s_{m-1}) \odot t_m, \tag{30}$$

$$\text{or} \quad c = ((\ldots(((a \odot s_1) \oplus s_1) \odot t_2) \oplus \ldots) \odot t_m) \oplus s_m \tag{31}$$

and (28) or

$$c = ((\ldots(((b \odot t'_1) \oplus s'_1) \odot t'_2) \oplus \ldots) \oplus s'_{n-1}) \odot t'_n, \tag{32}$$

$$\text{or} \quad c = ((\ldots(((b \oplus s'_1) \odot t'_1) \oplus s'_2) \odot \ldots) \oplus s'_n) \odot t_n,' \tag{33}$$

$$\text{or} \quad c = ((\ldots(((b \oplus s'_1) \odot t'_1) \oplus s'_2) \odot \ldots) \odot t'_{n-1}) \oplus s'_n \tag{34}$$

are fulfilled.

Remark 6. The regular alternation of s_i and t_i, $i = 1, \ldots, m$ (or $1, \ldots, m-1$) and s'_i, t'_i, $i = 1, \ldots, n$ (or $1, \ldots, n-1$) in the statement of Theorem 2 is evident, as in case of sequence $\ldots) \oplus s_i) \oplus s_{i+1} \ldots$ or $\ldots) \odot t_i) \odot t_{i+1} \ldots$ there exist s''_i or t''_i such that $s''_i = s_i \oplus s_{i+1} \in \mathbb{S}_0$ and $t''_i = t_i \odot t_{i+1} \in \mathbb{T}_1$ which can be substituted into the relevant one of formulas (27),...,(34) (cf. [8], Remark 3, and [9], Theorem 2).

Corollary. If a is used instead of c in some of formulas (28), (32), (33), (34) then the validity of that formula implies $a \approx b$. Analogously, if b can be substituted for c in some of formulas (27), (29), (30), (31) then the validity of that formula implies $a \approx b$, too, as follows from Theorem 2.

Lemma 1. If $a, b \in \mathbb{R}$ and $x \in R$, $x \neq 0$ then $a \approx b$ if and only if $x \cdot a \approx x \cdot b$.

Proof. If $a \sim_\oplus b$ then $a \oplus s_1 = b \oplus s_2$ for some $s_1, s_2 \in \mathbb{S}_0$, and for $s'_1 = x \cdot s_1$, $s'_2 = x \cdot s_2$

$$x \cdot (a \oplus s_1) = x \cdot (b \oplus s_2),$$

which implies, together with (13),

$$x \cdot a \oplus s'_1 = x \cdot b + s'_2.$$

As $s'_1 \in \mathbb{S}_0$, $s'_2 \in \mathbb{S}$, equivalence $x \cdot a \sim_\oplus x \cdot b$ holds. Analogously, if $x \cdot a \sim_\oplus x \cdot b$ and consequently $a \sim_\oplus b$.

If $a \sim_\odot b$ then for some $t_1, t_2 \in \mathbb{T}_1$, $a \odot t_1 = b \odot t_2$. This holds iff $x \cdot (a \odot t_1) = x \cdot (b \odot t_2)$ which is equivalent to $(x \cdot a) \odot t_1 = (x \cdot b) \odot t_2$, so that $a \sim_\odot b$ iff $x \cdot a \sim_\odot x \cdot b$. Both equivalences mean that $a \sim b$ iff $x \cdot a \sim x \cdot b$ where $\sim$ is the relativity relation of Definition 2. Generally, if $a, b \in \mathbb{R}$ then there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that $a_1 = c_1 \sim c_2 \sim \cdots \sim c_n = b$ iff $x \cdot a = x \cdot c_1 \sim \cdots \sim x \cdot c_n = x \cdot b$. □

Theorem 3. Let $a, b \in \mathbb{R}$, $s \in \mathbb{S}_0$, $x \in R_0$, $c \in \mathbb{T}_x$. Then $a \approx b$ if and only if

$$c \odot (a \oplus s) \approx c \odot (b \oplus s) \tag{35}$$

Proof. If $a \approx b$ and $c \in \mathbb{T}_x$ then $c = x \cdot t$ for some $t \in \mathbb{T}_1$ (cf. [9]), and for $s' = x \cdot s \in \mathbb{S}_0$

$$\begin{aligned} c \odot (a \oplus s) &= (t \cdot x) \odot (a \oplus s) = t \odot (x \cdot a + x \cdot s) \sim_\odot x \cdot a \oplus s' \sim_\oplus \\ &\sim_\oplus x \cdot a \approx x \cdot b \sim_\oplus x \cdot b \oplus s' \sim_\odot t \odot (x \cdot b + x \cdot s) = \\ &= (t \cdot x) \odot (b \oplus s) = c \odot (b \oplus s), \end{aligned}$$

where Lemma 1 was used. On the other hand, if (35) then

$$\begin{aligned} & x \cdot a \sim_\oplus x \cdot a \oplus s' = x \cdot (a \oplus s) \sim_\odot (t \cdot x) \odot (a \oplus s) = \\ = \;& c \odot (a \oplus s) \approx c \odot (b \oplus s) = (t \cdot x) \odot (b \oplus s) \sim_\odot x \cdot (b \oplus x) = \\ = \;& x \cdot b \oplus x \cdot s \sim_\oplus x \cdot b \end{aligned}$$

so that $x \cdot a \approx x \cdot b$ and by Lemma 1 $a \approx b$. □

Theorem 4. Let $a, b \in \mathbb{R}$, $s \in \mathbb{S}_0$, $x \in R_0$, $c \in \mathbb{T}_x$. Then $a \approx b$ if and only if

$$s \oplus (a \odot c) \approx s \oplus (b \odot c). \tag{36}$$

Proof. By Theorem 3 $a \approx b$ iff $a \odot c \approx b \odot c$ iff $s \oplus (a \odot c) \approx s \oplus (b \odot c)$. □

6. DISTRIBUTIVITY

Some results concerning distributivity of the multiplication $\odot$ and addition $\oplus$ are presented in [2], some others concerning the multiplication by crisp real number (12) were derived in [8]. Namely, by [2,5,6,8], equality (13) is true and for symmetric normal fuzzy quantity $a \in \mathbb{S}$ also the equivalence relation (17) holds. If the weak equivalence $\approx$ is considered then also the following statements can be derived.

Lemma 2. If $x \in R$, $t \in \mathbb{T}_1$, $a \in \mathbb{S}_x$ then

$$t \odot a \approx (x \cdot t) = \langle x \rangle \odot t.$$

Proof. The statement follows from Lemma 1 as by [8] (Lemma 3) there exists $s \in \mathbb{S}_0$ such that $a = \langle x \rangle \oplus s$, and

$$t \odot a = t \odot (\langle x \rangle \oplus s) \approx (t \odot \langle x \rangle) \oplus (t \odot s) \sim_\oplus \langle x \rangle \odot t = x \cdot t.$$ □

Remark 7. If $y \in R_0$, $s \in \mathbb{S}_0$, $b \in \mathbb{T}_y$ then there exists $t \in \mathbb{T}_1$ such that $b \odot s = t \odot (s \odot \langle y \rangle) = t \odot (y \cdot s) \sim_\oplus s$, and $b \odot s \in \mathbb{S}_0$ as follows from [9] (Lemma 9 and Lemma 4).

The distributivity of $\oplus$ and $\odot$ is not generally guaranteed for arbitrary fuzzy quantities even under the weak equivalence $\approx$. However, some useful special cases can be found.

Theorem 5. If $a, b \in \mathbb{R}$, $y \in R_0$, $c \in \mathbb{T}_y$ then

$$c \odot (a \oplus b) \sim_\odot (y \cdot a) \oplus (y \cdot b) = (\langle y \rangle \odot a) \oplus (\langle y \rangle \odot b).$$

Proof. Preserving the notation of the statement, there exists $t \in \mathbb{T}_1$ such that $c = \langle y \rangle \odot t$ and then

$$\begin{aligned} c \odot (a \oplus b) &= t \odot \langle y \rangle (a \oplus b) \sim_\odot \langle y \rangle \odot (a \oplus b) = \\ &= y \cdot (a \oplus b) = y \cdot a \oplus y \cdot b = \\ &= (\langle y \rangle \odot a) \oplus (\langle y \rangle \odot b). \end{aligned}$$ □

Lemma 1 can be extended in the following way.

Theorem 6. Let $a \in \mathbb{R}$, $s \in \mathbb{S}_0$, $c \in \mathbb{T}$. Then

$$c \odot (a \oplus s) \approx (c \odot a) \oplus (c \odot s) \sim_{\oplus} c \odot a.$$

Proof. Let $y \in R_0$ be the real number for which $c \in \mathbb{T}_y$, i.e. $c = y \cdot t = \langle y \rangle \odot t$ for some $t \in \mathbb{T}_1$. Then $s' = y \cdot s \in \mathbb{S}_0$ and also $c \odot s \in \mathbb{S}_0$ (cf. [9]). Hence $c \odot (a \oplus s) = t \odot \langle y \rangle \odot (a \oplus s) = t \odot (y \cdot (a \oplus s)) = t \odot (y \cdot a \oplus y \cdot s) \sim_{\odot} y \cdot a \oplus s' \sim_{\oplus} y \cdot a \sim_{\odot} t \odot (y \cdot y) = (t \odot \langle y \rangle) \odot a = c \odot a \sim_{\oplus} c \odot a + c \odot s$. It means that the statement is true. □

The results presented in this section contribute to the effective arithmetic manipulation with normal fuzzy quantities. Nevertheless, recollecting the previous results a few conclusions appear obvious.

7. CONCLUSIONS

The asymmetry of addition $\oplus$ and multiplication $\odot$ cannot be eliminated even by the weak equivalence relation $\approx$ in which both, $\sim_{\oplus}$ and $\sim_{\odot}$, act in a very symmetric way.

It means that the validity of the distributivity condition cannot be generally achieved even for so specific fuzzy quantities like the transversible and symmetric ones.

The objectively existing possibilities of even weakened, distributivity seem to be nearly exhausted by the results presented in [2,5,8] and above.

Respecting the limitations commented in the previous sections, the results derived in this paper and in some of the referred works [5,6,7,8,9] offer certain tools adequate to the character of vagueness and uncertainty present in fuzzy quantities. The possibilities of arithmetic operations cannot be evidently as complex and rich as those ones usual in the deterministic case. In the previous sections we could recognize some of their limits as well as some of possible weakening modifications of the fuzzy quantities theory under which those limits can be rather shifted.

Anyhow, the remaining space for arithmetic manipulation with fuzzy quantities is not too narrow. Especially the n.f.q. representing some crisp data contaminated by additive or multiplicative fuzzy noise can be handled in a frequently sufficient degree. It concerns namely crisp data $x \in R$ contaminated by a symmetrical additive noise to the form $\langle x \rangle \oplus s$, $s \in \mathbb{S}_0$ or transversible multiplicative noise expressed by $\langle x \rangle \odot t$, $t \in \mathbb{T}_1$, whose simple structure allows (cf. [8,9] and the previous sections) to derive acceptable practical results valid up to certain type of additive, multiplicative or weak equivalence.

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