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PERTURBATION ANALYSIS OF THE DISCRETE RICCATI EQUATION

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The sensitivity of the discrete-time matrix Riccati equation relative to perturbations in its coefficients is studied. Both local and non-local perturbation bounds are obtained. In particular the conditioning of the equation is determined.

1. INTRODUCTION

Recently there is an increasing interest in the sensitivity analysis of the matrix Riccati equations arising in the solution of quadratic optimization and estimation problems in linear control theory. This interest is motivated by the fact that these equations are usually subject to perturbations in the data reflecting either parameter errors or rounding errors, accompanying the numerical solution [1].

The sensitivity of the continuous Riccati equation is studied in [2]–[7]. The sensitivity of the discrete Riccati equations, however, has not been studied in such depth up to now. Some preliminary results in this area have been published in [1], [3], [4] without proof.

In this paper we study the sensitivity of the non-negative solution of the discrete algebraic matrix Riccati equation (DAMRE) relative to perturbations in its coefficients. Both local and non-local perturbation analysis is done. In the first case we suppose that the perturbations in the data are asymptotically small and the corresponding bound contains first order terms only. In this way the conditioning of the equation is determined as well. In the second case an upper bound for the norm of the perturbation in the solution is obtained without the assumption that the coefficient perturbations are asymptotically small. This bound is a non-linear function of the perturbations in the data, defined in a domain which guarantees the existence of a unique solution of the perturbed equation in the neighbourhood of the unperturbed solution. The latter results are obtained by the method of Lyapunov majorants [8] which is applicable also to many linear control problems [4]. Part of the results have been briefly reported in [3], [4] and are an extension to the discrete-time case of results obtained for a general class of matrix quadratic equations [7]. A sensitivity analysis of the discrete Lyapunov equation, which is a particular case of the discrete Riccati equation, is presented in [9].

2. PROBLEM STATEMENT

Consider the DAMRE

$$\mathbf{X} - \mathbf{A}^{\mathrm{T}}\mathbf{X}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{X}\mathbf{B}\,(\mathbf{I}_m + \mathbf{B}^{\mathrm{T}}\mathbf{X}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{X}\mathbf{A} - \mathbf{C}^{\mathrm{T}}\mathbf{C} = \mathbf{0} \tag{1}$$

arising in the linear-quadratic optimization [10], where $\mathbf{X} \in \mathbb{R}^{n\cdot n}$ is the unknown matrix and $\mathbf{A} \in \mathbb{R}^{n\cdot n}$, $\mathbf{B} \in \mathbb{R}^{n\cdot m}$, $\mathbf{C} \in \mathbb{R}^{r\cdot n}$ are given non-zero matrices. In the sequel we shall write equation (1) in the equivalent form

$$\mathbf{X} - \mathbf{A}^{\mathrm{T}}\mathbf{X}\,(\mathbf{I}_n + \mathbf{S}\mathbf{X})^{-1}\mathbf{A} - \mathbf{Q} = \mathbf{0} \tag{2}$$

where $\mathbf{S} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$ and $\mathbf{Q} = \mathbf{C}^{\mathrm{T}}\mathbf{C}$. Note that equation (2) may be considered also independently of (1) under some requirements for the triple $(\mathbf{Q}, \mathbf{A}, \mathbf{S})$.

We suppose that the triple $(\mathbf{C}, \mathbf{A}, \mathbf{B})$ is regular, i.e. that $(\mathbf{C}, \mathbf{A}]$ is detectable and $[\mathbf{A}, \mathbf{B})$ is stabilizable. This guarantees the existence of a unique non-negative solution $\mathbf{X} = \mathbf{P}$. It is also the unique solution of (1) or (2) such that the closed-loop system matrix $\mathbf{A}_c = \mathbf{A} - \mathbf{B}(\mathbf{I}_m + \mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{A} = (\mathbf{I}_n + \mathbf{S}\mathbf{P})^{-1}\mathbf{A}$ is convergent (i.e. its spectral radius is less than 1). Since $\mathrm{Ker}(\mathbf{P})$ is the unobservable subspace of $(\mathbf{C}, \mathbf{A}]$ then $\mathbf{P}$ is positive definite if $(\mathbf{C}, \mathbf{A}]$ is observable.

We shall refer to (2) as the unperturbed equation and to $\mathbf{P}$ — as the unperturbed solution.

Let $\Delta\mathbf{Q}$, $\Delta\mathbf{A}$, $\Delta\mathbf{S} \in \mathbb{R}^{n\cdot n}$ be perturbations of $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{S}$ in (2) (if matrices $\mathbf{C}$, $\mathbf{A}$, $\mathbf{B}$ in (1) are perturbed, then $\Delta\mathbf{Q} = \Delta\mathbf{C}^{\mathrm{T}}\mathbf{C} + \mathbf{C}^{\mathrm{T}}\Delta\mathbf{C} + \Delta\mathbf{C}^{\mathrm{T}}\Delta\mathbf{C}$, $\Delta\mathbf{S} = \Delta\mathbf{B}\mathbf{B}^{\mathrm{T}} + \mathbf{B}\Delta\mathbf{B}^{\mathrm{T}} + \Delta\mathbf{B}\Delta\mathbf{B}^{\mathrm{T}}$). Consider the perturbed equation

$$\mathbf{Y} - (\mathbf{A} + \Delta\mathbf{A})^{\mathrm{T}}\mathbf{Y}(\mathbf{I}_n + (\mathbf{S} + \Delta\mathbf{S})\mathbf{Y})^{-1}(\mathbf{A} + \Delta\mathbf{A}) - (\mathbf{Q} + \Delta\mathbf{Q}) = \mathbf{0} \tag{3}$$

and denote $\Delta = (\Delta_Q, \Delta_A, \Delta_S)^{\mathrm{T}} \in \mathbb{R}^3_+$, where $\Delta_Q = \|\Delta\mathbf{Q}\|$, $\Delta_A = \|\Delta\mathbf{A}\|$, $\Delta_S = \|\Delta\mathbf{S}\|$ and $\|\cdot\|$ is the Frobenius (F)- or spectral (2)-norm.

Since the Fréchet derivative of the left-hand side of (2) in $\mathbf{X}$ at $\mathbf{X} = \mathbf{P}$ is invertible (see Section 3) then according to the implicit function theorem [11] we get

Theorem 2.1. The perturbed equation (3) has a unique solution $\mathbf{Y} = \mathbf{P} + \Delta\mathbf{P} = \mathcal{P}(\Delta\Sigma)$, $\Delta\Sigma = (\Delta\mathbf{Q}, \Delta\mathbf{A}, \Delta\mathbf{S})$, in the neighbourhood of $\mathbf{P}$, such that $\mathcal{P}(\mathbf{0}) = \mathbf{P}$, whose elements are analytic functions of the elements of the perturbations $\Delta\mathbf{Q}$, $\Delta\mathbf{A}$, $\Delta\mathbf{S}$, at least in certain neighbourhood of the origin (e.g. for $\|\Delta\|$ sufficiently small).

The main problems solved in this paper are formulated as:

(i) Find a local linear estimate for the norm $\Delta_P = \|\Delta\mathbf{P}\|$ of the perturbation $\Delta\mathbf{P}$ as a function of Δ_Q, Δ_A, Δ_S or $\Delta_\Sigma = \|\Delta\Sigma\|$, which is valid for $\|\Delta\|$ asymptotically small.

(ii) Find a convex domain $\mathcal{D} \subset \mathbb{R}^3_+$, $0 \in \mathcal{D}$, such that for each $\Delta\mathbf{Q}$, $\Delta\mathbf{A}$, $\Delta\mathbf{S}$ with $\Delta \in \mathcal{D}$ equation (3) has a unique solution $\mathbf{Y} = \mathbf{P} + \Delta\mathbf{P}$ (in the neighbourhood of $\mathbf{P}$), such that the elements of $\Delta\mathbf{P}$ are analytic functions of the elements of $\Delta\mathbf{Q}$, $\Delta\mathbf{A}$, $\Delta\mathbf{S}$.

(iii) Find an estimate

$$\Delta_P \le f(\Delta) \tag{4}$$

where the function $f: \mathcal{D} \to \mathbb{R}_+$ is analytic, non-decreasing in each component of Δ and $f(0) = 0$.

Note that (4) is a non-local estimate since it holds for all (possibly small but finite)$\Delta \in \mathcal{D}$, i.e. $\|\Delta\|$ needs not to be asymptotically small. If, however, $\|\Delta\|$ is small, then it follows from (iii) that

$$f(\Delta) = C_Q \Delta_Q + C_A \Delta_A + C_S \Delta_S + O(\|\Delta\|^2), \quad \Delta \to 0,$$

where $C_Q = (\partial f / \partial \Delta_Q)(0)$, etc. Hence C_Q, C_A, C_S are estimates of the absolute condition numbers K_Q, K_A, K_S of DAMRE relative to perturbations in $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{S}$ resp. [1] (see also Section 3).

3. MAIN RESULTS

3.1. Local linear estimates

Denote by $\mathbf{F}(\mathbf{X}, \Sigma) = \mathbf{F}(\mathbf{X}, \mathbf{Q}, \mathbf{A}, \mathbf{S})$ the left-hand side of (2), where $\Sigma = (\mathbf{Q}, \mathbf{A}, \mathbf{S}) \in \mathbb{R}^{n \cdot n} \times \mathbb{R}^{n \cdot n} \times \mathbb{R}^{n \cdot n}$. Then

$$\mathbf{F}(\mathbf{P}, \Sigma) = \mathbf{0}. \tag{5}$$

Setting $\mathbf{Y} = \mathbf{P} + \Delta\mathbf{P}$, the perturbed equation (3) may be written as

$$\begin{aligned} &\mathbf{F}(\mathbf{P} + \Delta\mathbf{P}, \Sigma + \Delta\Sigma) = \\ &\mathbf{F}(\mathbf{P}, \Sigma) + \mathbf{F}_X(\Delta\mathbf{P}) + \mathbf{F}_Q(\Delta\mathbf{Q}) + \mathbf{F}_A(\Delta\mathbf{A}) + \mathbf{F}_S(\Delta\mathbf{S}) + \mathbf{G}(\Delta\mathbf{P}, \Delta\Sigma) = \mathbf{0} \end{aligned} \tag{6}$$

where $\mathbf{F}_X(\cdot) \in \mathcal{L}(\mathbb{R}^{n \cdot n}, \mathbb{R}^{n \cdot n})$ is the Fréchet derivative of $\mathbf{F}(\mathbf{X}, \Sigma)$ in $\mathbf{X}$ at $\mathbf{X} = \mathbf{P}$, etc., and $\mathcal{L}(\mathbb{R}^{n \cdot n}, \mathbb{R}^{n \cdot n})$ is the space of linear operators $\mathbb{R}^{n \cdot n} \to \mathbb{R}^{n \cdot n}$ endowed with the induced norm

$$\|\mathcal{P}\|_{\mathcal{L}} = \max\{\|\mathcal{P}(\mathbf{Z})\| : \|\mathbf{Z}\| = 1\}, \quad \mathcal{P} \in \mathcal{L}(\mathbb{R}^{n \cdot n}, \mathbb{R}^{n \cdot n}). \tag{7}$$

We shall also use the notations $\|\cdot\|_{\mathcal{L},F}$ and $\|\cdot\|_{\mathcal{L},2}$ if the F- or 2-norm is used in the right-hand side of (7) respectively.

The function $\mathbf{G}(\cdot,\cdot) : \mathbb{R}^{n \cdot n} \times (\mathbb{R}^{n \cdot n} \times \mathbb{R}^{n \cdot n} \times \mathbb{R}^{n \cdot n}) \to \mathbb{R}^{n \cdot n}$ is non-linear and satisfies $\|\mathbf{G}(\mathbf{Z}, \mathbf{W})\| = O(\omega^2)$, $\omega = \|(\mathbf{Z}, \mathbf{W})\| \to 0$. A straightforward calculation leads to

$$\begin{aligned} \mathbf{F}_X(\mathbf{Z}) &= \mathbf{Z} - \mathbf{A}_c^{\mathrm{T}} \mathbf{Z} \mathbf{A}_c, \quad \mathbf{F}_Q(\mathbf{Z}) = -\mathbf{Z}, \\ \mathbf{F}_A(\mathbf{Z}) &= -(\mathbf{Z}^{\mathrm{T}} \mathbf{P} \mathbf{A}_c + \mathbf{A}_c^{\mathrm{T}} \mathbf{P} \mathbf{Z}), \quad \mathbf{F}_S(\mathbf{Z}) = \mathbf{A}_c^{\mathrm{T}} \mathbf{P} \mathbf{Z} \mathbf{P} \mathbf{A}_c. \end{aligned} \tag{8}$$

Note that the Fréchet derivatives $\mathbf{F}_X$ and $\mathbf{F}_S$ exist (as functions $(\mathbf{P}, \Sigma) \to \mathcal{L}(\mathbb{R}^{n \cdot n}, \mathbb{R}^{n \cdot n})$) at least in a neighbourhood of $\mathbf{P}$ and $\mathbf{S}$ resp., i.e. for Δ_P and Δ_S sufficiently small.

The expression for $\mathbf{G}$ is rather complicated. For $\mathbf{Z} = \mathbf{Z}^{\mathrm{T}}$ we get

$$\begin{aligned} \mathbf{G}(\mathbf{Z}, \Delta\Sigma) &= \mathbf{A}_c^{\mathrm{T}}[\mathbf{Z}(\mathbf{R} + \mathbf{E})^{-1}(\mathbf{S} + \Delta\mathbf{S})\mathbf{Z} + \mathbf{Z}(\mathbf{R} + \mathbf{E})^{-1}\Delta\mathbf{S}\mathbf{P} \\ &- \mathbf{P}\Delta\mathbf{S}\mathbf{P}(\mathbf{R} + \mathbf{E})^{-1}(\mathbf{S} + \Delta\mathbf{S})\mathbf{Z} + \mathbf{P}\Delta\mathbf{Z}(\mathbf{R} + \mathbf{E})^{-1}\mathbf{R} \\ &- \mathbf{P}\Delta\mathbf{S}\mathbf{P}(\mathbf{R} + \mathbf{E})^{-1}\Delta\mathbf{S}\mathbf{P}]\mathbf{A}_c \\ &+ \Delta\mathbf{A}^{\mathrm{T}}\mathbf{R}^{-\mathrm{T}}[\mathbf{P}\Delta\mathbf{S}\mathbf{P} - (\mathbf{I}_n - \mathbf{P}\Delta\mathbf{S})\mathbf{Z}](\mathbf{R} + \mathbf{E})^{-1}\mathbf{A} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \mathbf{A}^{\mathrm{T}}(\mathbf{R}+\mathbf{E})^{-\mathrm{T}}[\mathbf{P}\Delta\mathbf{S}\mathbf{P} - \mathbf{Z}(\mathbf{I}_n - \Delta\mathbf{S}\mathbf{P})]\mathbf{R}^{-1}\Delta\mathbf{A} \\
&- \quad \Delta\mathbf{A}^{\mathrm{T}}(\mathbf{P}+\mathbf{Z})(\mathbf{R}+\mathbf{E})^{-1}\Delta\mathbf{A} \\
&= \quad \mathbf{A}_c^{\mathrm{T}}(\mathbf{Z}\mathbf{R}^{-1}\mathbf{S}\mathbf{Z} + \mathbf{Z}\mathbf{R}^{-1}\Delta\mathbf{S}\mathbf{P} + \mathbf{P}\Delta\mathbf{S}\mathbf{R}^{-\mathrm{T}}\mathbf{Z} \\
&- \quad \mathbf{P}\Delta\mathbf{S}\mathbf{P}\mathbf{R}^{-1}\Delta\mathbf{S}\mathbf{P})\mathbf{A}_c - \mathbf{A}_c^{\mathrm{T}}(\mathbf{P}\Delta\mathbf{S}\mathbf{P} - \mathbf{Z})\mathbf{R}^{-1}\Delta\mathbf{A} \\
&- \quad \Delta\mathbf{A}^{\mathrm{T}}\mathbf{R}^{-\mathrm{T}}(\mathbf{P}\Delta\mathbf{S}\mathbf{P} - \mathbf{Z})\mathbf{A}_c - \Delta\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{R}^{-1}\Delta\mathbf{A} \\
&+ \quad \text{3-rd and higher order terms}
\end{aligned}
$$

where $\mathbf{R} = \mathbf{I}_n + \mathbf{S}\mathbf{P}$ and $\mathbf{E} = (\mathbf{S}+\Delta\mathbf{S})\mathbf{Z} + \Delta\mathbf{S}\mathbf{P}$.

Having in mind (5) and (8), it follows from (6)

$$\mathbf{F}_X(\Delta\mathbf{P}) = \Delta\mathbf{Q} - \mathbf{F}_A(\Delta\mathbf{A}) - \mathbf{F}_S(\Delta\mathbf{S}) - \mathbf{G}(\Delta\mathbf{P}, \Delta\Sigma). \tag{9}$$

The eigenvalues of the operator $\mathbf{F}_X(\cdot)$ are $\mu_{ij} = 1 - \lambda_i\lambda_j$, where the eigenvalues $\lambda_i = \lambda_i(\mathbf{A}_c)$ of $\mathbf{A}_c$ lie inside the unit circle in the complex plane. Hence $0 < |\mu_{ij}| < 2$, the operator $\mathbf{F}_X$ is invertible and (9) yields

$$\Delta\mathbf{P} = \mathbf{F}_X^{-1}(\Delta\mathbf{Q}) - \mathbf{F}_X^{-1}\circ\mathbf{F}_A(\Delta\mathbf{A}) - \mathbf{F}_X^{-1}\circ\mathbf{F}_S(\Delta\mathbf{S}) - \mathbf{F}_X^{-1}(\mathbf{G}(\Delta\mathbf{P}, \Delta\Sigma)). \tag{10}$$

Equation (10) makes possible to obtain exact estimates of the type

$$\Delta_P \leq K_Q\Delta_Q + K_A\Delta_A + K_S\Delta_S + O(\|\Delta\|^2), \quad \Delta \to 0 \tag{11}$$

$$\delta_P \leq k_Q\delta_Q + k_A\delta_A + k_S\delta_S + O(\|\delta\|^2), \quad \delta \to 0 \tag{12}$$

or

$$\Delta_P \leq K_\Sigma\Delta_\Sigma + O(\Delta_\Sigma^2), \quad \Delta_\Sigma \to 0, \tag{13}$$

$$\delta_P \leq k_\Sigma\delta_\Sigma + O(\delta_\Sigma^2), \quad \delta_\Sigma \to 0 \tag{14}$$

where $\delta_P = \Delta_P/\|\mathbf{P}\|$, $\delta_Q = \Delta_Q/\|\mathbf{Q}\|$, $\delta_A = \Delta_A/\|\mathbf{A}\|$, $\delta_S = \Delta_S/\|\mathbf{S}\|$, $\delta_\Sigma = \Delta_\Sigma/\|\Sigma\|$ are the corresponding relative perturbations in the solution and in the coefficient matrices.

The numbers K_Q, K_A, K_S and K_Σ are the absolute condition numbers of DAMRE relative to perturbations in $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{S}$ and Σ respectively, while

$$
\begin{aligned}
k_Q &= K_Q\|\mathbf{Q}\|/\|\mathbf{P}\|, \qquad k_A = K_A\|\mathbf{A}\|/\|\mathbf{P}\|, \\
k_S &= K_S\|\mathbf{S}\|/\|\mathbf{P}\|, \qquad k_\Sigma = K_\Sigma\|\Sigma\|/\|\mathbf{P}\|
\end{aligned}
\tag{15}
$$

are the corresponding relative condition numbers.

If the relative perturbations in $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{S}$ satisfy $\delta_Q \leq \delta^0$, $\delta_A \leq \delta^0$, $\delta_S \leq \delta^0$ for some $\delta^0 > 0$ then

$$\delta_P \leq k\delta^0, \quad k = k_Q + k_A + k_S. \tag{16}$$

Hence the number k may be considered as an overall estimate of the relative conditioning of DAMRE. However, this number would not be a relevant measure of the real perturbation if e. g. some of the numbers k_Q, k_A, k_S is large while the corresponding perturbation is zero.

It follows from (10), (11) that

$$K_Q = \|\mathbf{F}_X^{-1}\|_{\mathcal{L}}, \qquad K_A = \|\mathbf{F}_X^{-1}\circ\mathbf{F}_A\|_{\mathcal{L}}, \qquad K_S = \|\mathbf{F}_X^{-1}\circ\mathbf{F}_S\|_{\mathcal{L}}.$$

However, these expressions are not convenient for calculation of the condition numbers.

As above, let $K_{Q,F} = ||\mathbf{F}_X^{-1}||_{\mathcal{L},F}$, etc., and denote

$$\Lambda = \mathbf{I}_\nu - \mathbf{A}_c^{\mathrm{T}} \otimes \mathbf{A}_c^{\mathrm{T}} \in \mathbb{R}^{\nu \cdot \nu}, \quad \nu = n^2. \tag{17}$$

Using the column-wise representation of $n \times n$ matrices as column ν-vectors, we obtain

$$\begin{aligned} &K_{Q,F} = ||\mathcal{M}_Q||_2, \qquad K_{A,F} = ||\mathcal{M}_A||_2, \qquad K_{S,F} = ||\mathcal{M}_S||_2, \\ &K_{\Sigma,F} = ||\mathcal{M}_\Sigma||_2 \leq (K_{Q,F}^2 + K_{A,F}^2 + K_{S,F}^2)^{1/2} \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\mathcal{M}_Q = \Lambda^{-1}, \qquad \mathcal{M}_A = \Lambda^{-1}[\mathbf{I}_n \otimes \mathbf{A}_c^{\mathrm{T}} + (\mathbf{A}_c^{\mathrm{T}} \otimes \mathbf{I}_n)\Pi], \\ &\mathcal{M}_S = \Lambda^{-1}(\mathbf{A}_c^{\mathrm{T}}\mathbf{P} \otimes \mathbf{A}_c^{\mathrm{T}}\mathbf{P}), \qquad \mathcal{M}_\Sigma = [\mathcal{M}_Q, \mathcal{M}_A, \mathcal{M}_S] \in \mathbb{R}^{\nu \cdot 3\nu} \end{aligned} \tag{19}$$

and $\Pi \in \mathbb{R}^{\nu \cdot \nu}$is the permutation matrix such that $vec(\mathbf{Z}^{\mathrm{T}}) = \Pi\, vec(\mathbf{Z})$ for each $\mathbf{Z} \in \mathbb{R}^{n \cdot n}$ ($vec(\mathbf{Z}) \in \mathbb{R}^{n^2}$ is the column-wise vector representation of $\mathbf{Z} \in \mathbb{R}^{n \cdot n}$).

The case of a 2-norm is more complicated for the following reasons. Let $Mat(\mathcal{P}) \in \mathbb{R}^{n^2 \cdot n^2}$ be the equivalent matrix of the operator $\mathcal{P} \in \mathcal{L}(\mathbb{R}^{n \cdot n}, \mathbb{R}^{n \cdot n})$, i.e. $vec(\mathcal{P}(\mathbf{Z})) = Mat(\mathcal{P})\, vec(\mathbf{Z})$. Then, as used before, $||\mathcal{P}||_{\mathcal{L},F} = ||Mat(\mathcal{P})||_2$ since the F-norm of $\mathbf{Z}$ is the Euclidean norm of $vec(\mathbf{Z})$.

Unfortunately, any p-norm $||\mathbf{Z}||_p$ (the 2-norm in particular) of $\mathbf{Z}$ is a norm of $vec(\mathbf{Z})$, say $|vec(\mathbf{Z})|_p$, but this latter norm is not "natural" and there is not a convenient "explicit" expression for the induced matrix norm $|M|_p = \max\{|Mx|_p : |x|_p = 1\}$. However, it is known [9] that

$$K_{Q,2} = |\Lambda^{-1}|_2 = ||\mathbf{H}||_2 \tag{20}$$

where the positive definite matrix $\mathbf{H} = \mathbf{F}_X^{-1}(\mathbf{I}_n)$ is the unique solution to the discrete Lyapunov equation

$$\mathbf{X} - \mathbf{A}_c^{\mathrm{T}}\, \mathbf{X}\, \mathbf{A}_c - \mathbf{I}_n = \mathbf{0}.$$

Note that in both F- and 2-norms

$$K_A \leq 2K_Q||\mathbf{PA}_c|| \leq 2K_Q||\mathbf{P}||\,||\mathbf{A}_c||, \tag{21}$$

$$K_S \leq K_Q||\mathbf{PA}_c||^2 \leq K_Q||\mathbf{P}||^2\,||\mathbf{A}_c||^2. \tag{22}$$

If $\Delta_{\max} = \max\{\Delta_Q, \Delta_A, \Delta_S\}$ then

$$\Delta_P \leq K_Q(1 + ||\mathbf{PA}_c||)^2 \Delta_{\max}. \tag{23}$$

It must be stressed that the above results are valid without the assumption that $\Delta\mathbf{Q}$, $\Delta\mathbf{S}$ are symmetric and/or $\mathbf{S} + \Delta\mathbf{S}$, $\mathbf{Q} + \Delta\mathbf{Q}$ are non-negative definite.

Hence we have proved the following

Theorem 3.1. For small $||\Delta||$ the estimates (11) – (14) and (23) are valid, where the condition numbers relative to $\mathbf{Q}$, $\mathbf{A}$, $\mathbf{S}$, Σ are determined or estimated from (15) – (22).

3.2. Non-local non-linear estimates (the symmetric case)

Equation (10) may be used to obtain non-local perturbation bounds for the solution. However, a drawback of this form of the equation is the expression for $\mathbf{G}$ which is hard to manipulate. That is why we shall rewrite the perturbed equation (3) as

$$\mathbf{F}(\mathbf{Y}, \Sigma) + \mathbf{F}(\mathbf{Y}, \Sigma + \Delta\Sigma) - \mathbf{F}(\mathbf{Y}, \Sigma) = \mathbf{0} \tag{24}$$

where $\mathbf{Y} = \mathbf{P} + \Delta\mathbf{P}$.

Note that, by inspection,

$$\mathbf{F}(\mathbf{Y}, \Sigma) = \mathbf{F}(\mathbf{P} + \Delta\mathbf{P}, \Sigma) = \mathbf{F}(\mathbf{P}, \Sigma) + \mathbf{F}_X(\Delta\mathbf{P}) + \mathcal{B}(\Delta\mathbf{P}) \tag{25}$$

where

$$\mathcal{B}(\mathbf{Z}) = \mathbf{A}_c^{\mathrm{T}}\mathbf{Z}(\mathbf{I}_n + \mathbf{S}(\mathbf{P} + \mathbf{Z}))^{-1}\mathbf{S}\mathbf{Z}\mathbf{A}_c. \tag{26}$$

The expression $\mathcal{B}(\mathbf{Z})$ is well defined for $\mathbf{P} + \mathbf{Z}$ non-negative definite, for $||\mathbf{Z}||$ sufficiently small, etc.

A straightforward calculation shows that

$$\begin{aligned} &\mathbf{F}(\mathbf{Y}, \Sigma + \Delta\Sigma) - \mathbf{F}(\mathbf{Y}, \Sigma) = \mathbf{A}^{\mathrm{T}}\mathbf{Y}(\mathbf{I}_n + \mathbf{S}\mathbf{Y})^{-1}\Delta\mathbf{S}\mathbf{Y}(\mathbf{I}_n + (\mathbf{S} + \Delta\mathbf{S})\mathbf{Y})^{-1}\mathbf{A} \\ - \ &\mathbf{A}^{\mathrm{T}}\mathbf{Y}(\mathbf{I}_n + (\mathbf{S} + \Delta\mathbf{S})\mathbf{Y})^{-1}\Delta\mathbf{A} - \Delta\mathbf{A}^{\mathrm{T}}(\mathbf{I}_n + (\mathbf{S} + \Delta\mathbf{S})\mathbf{Y})^{-1}\mathbf{A} \\ - \ &\Delta\mathbf{A}^{\mathrm{T}}(\mathbf{I}_n + (\mathbf{S} + \Delta\mathbf{S})\mathbf{Y})^{-1}\Delta\mathbf{A} - \Delta\mathbf{Q}. \end{aligned} \tag{27}$$

Suppose that $\mathbf{S} + \Delta\mathbf{S}$ and $\mathbf{Q} + \Delta\mathbf{Q}$ are symmetric and non-negative definite (this is always the case when (2) is obtained via (1)) and that the triple $\Sigma + \Delta\Sigma$ is regular. Then (24) has a unique symmetric non-negative solution $\mathbf{Y}$.

Note that for each non-negative $\mathbf{S} + \Delta\mathbf{S}$ and $\mathbf{Y}$ one has

$$\begin{aligned} &||\mathbf{Y}(\mathbf{I}_n + \mathbf{S}\mathbf{Y})^{-1}|| = ||(\mathbf{I}_n + \mathbf{Y}\mathbf{S})^{-1}\mathbf{Y}|| \leq ||\mathbf{Y}||, \\ &||\mathbf{Y}(\mathbf{I}_n + (\mathbf{S} + \Delta\mathbf{S})\mathbf{Y})^{-1}|| \leq ||\mathbf{Y}||, \\ &||(\mathbf{I}_n + \mathbf{S}\mathbf{Y})^{-1}\mathbf{S}|| \leq ||\mathbf{S}|| \end{aligned} \tag{28}$$

for both the F- and 2-norms. Indeed, let $\mathbf{Y}$ be non-singular. Then $\mathbf{Y}(\mathbf{I}_n + \mathbf{S}\mathbf{Y})^{-1} = (\mathbf{Y}^{-1} + \mathbf{S})^{-1}$. On the other hand if $\mathbf{T}, \mathbf{H}$ are symmetric and non-singular matrices and $\mathbf{T} - \mathbf{H}$ is non-negative, then $\mathbf{H}^{-1} - \mathbf{T}^{-1}$ is also non-negative and $||\mathbf{T}|| \geq ||\mathbf{H}||$. Now (28) follows by inspection. The case of a singular $\mathbf{Y}$ is considered setting $\mathbf{Y}(\mu) = \mathbf{Y} + \mu\mathbf{I}_n$ ($\mu > 0$) and passing to the limit $\mu \longrightarrow 0$.

It follows from (26) – (28) that

$$||\mathcal{B}(\mathbf{Z})|| \leq ||\mathbf{A}_c||^2\, ||\mathbf{S}||\, ||\mathbf{Z}||^2 \tag{29}$$

and

$$||\mathbf{F}(\mathbf{Y}, \Sigma + \Delta\Sigma) - \mathbf{F}(\mathbf{Y}, \Sigma)|| \leq \Delta_Q + 2||\mathbf{A}||\, ||\mathbf{Y}||\, \Delta_A + ||\mathbf{Y}||\, \Delta_A^2 + ||\mathbf{A}||^2\, ||\mathbf{Y}||^2 \Delta_S. \tag{30}$$

In view of (25) and (5) we may rewrite (24) as an operator equation

$$\begin{aligned} &\Delta\mathbf{P} = \Phi(\Delta\mathbf{P}), \\ &\Phi(\mathbf{Z}) = -\mathbf{F}_X^{-1}(\mathcal{B}(\mathbf{Z}) + \mathbf{F}(\mathbf{P} + \mathbf{Z}, \Sigma + \Delta\Sigma) - \mathbf{F}(\mathbf{P} + \mathbf{Z}, \Sigma)). \end{aligned} \tag{31}$$

We shall show that under some conditions on Δ there exists $\rho = f(\Delta)$ such that Φ is contractive and maps the set

$$\Omega_\rho = \{\mathbf{Z} : \ \|\mathbf{Z}\| \leq \rho, \ \mathbf{Z} = \mathbf{Z}^{\mathrm{T}}, \ \mathbf{P} + \mathbf{Z} \geq 0\}$$

into itself. Let $\mathbf{Z} \in \Omega_\rho$. For $\mathbf{Y} = \mathbf{P} + \mathbf{Z}$ we have $\|\mathbf{Y}\| \leq \|\mathbf{P}\| + \|\mathbf{Z}\| = p + \rho, \ p = \|\mathbf{P}\|$. Now in view of (29) – (31) we get

$$\begin{aligned} \|\Phi(\mathbf{Z})\| &\leq \|\mathbf{F}_X^{-1}\|_{\mathcal{L}}(\|\mathcal{B}(\mathbf{Z})\| + \|\mathbf{F}(\mathbf{P} + \mathbf{Z}, \Sigma + \Delta\Sigma) - \mathbf{F}(\mathbf{P} + \Sigma, \Sigma)\|) \\ &\leq a_0(\Delta) + a_1(\Delta)\rho + a_2(\Delta)\,\rho^2 = h(\rho) \end{aligned} \tag{32}$$

where

$$\begin{aligned} a_0(\Delta) &= K_Q[\Delta_Q + p(2a + \Delta_A)\Delta_A + a^2p^2\Delta_S], \\ a_1(\Delta) &= K_Q[(2a + \Delta_A)\Delta_A + 2a^2p\Delta_S], \\ a_2(\Delta) &= K_Q(a_c^2 s + a^2\Delta_S) \end{aligned} \tag{33}$$

and $a = \|\mathbf{A}\|$, $a_c = \|\mathbf{A}_c\|$, $s = \|\mathbf{S}\|$.

In a similar way we get

$$\|\Phi(\mathbf{Z}_1) - \Phi(\mathbf{Z}_2)\| \leq [a_1(\Delta) + 2a_2(\Delta)]\,\|\mathbf{Z}_1 - \mathbf{Z}_2\| = h'(\rho)\,\|\mathbf{Z}_1 - \mathbf{Z}_2\|, \quad h' = dh/d\rho. \tag{34}$$

Due to (32), (34) the operator Φ is a contraction and maps the compact set Ω_ρ into itself if there exists $\rho > 0$ such that

$$h(\rho) \leq \rho, \qquad h'(\rho) < 1.$$

The last two inequalities hold true iff

$$\Delta \in \mathcal{D} = \left\{\Delta : \ a_1(\Delta) + 2[a_0(\Delta)a_2(\Delta)]^{1/2} < 1\right\}. \tag{35}$$

In this case we may choose $\rho = f(\Delta)$ as the less root of the equation

$$a_2(\Delta)x^2 - (1 - a_1(\Delta))x + a_0(\Delta) = 0,$$

i. e.

$$\begin{aligned} f(\Delta) &= \left[1 - a_1(\Delta) - D^{1/2}(\Delta)\right] / [2a_2(\Delta)], \\ D(\Delta) &= [1 - a_1(\Delta)]^2 - 4a_0(\Delta)a_2(\Delta). \end{aligned} \tag{36}$$

Thus we have proved the following

Theorem 3.2. Let the matrices $\mathbf{Q} + \Delta\mathbf{Q}$ and $\mathbf{S} + \Delta\mathbf{S}$ be symmetric and non-negative definite and let the condition (35) be fulfilled. Then the perturbed equation (3) has a unique solution $\mathbf{Y} = \mathbf{P} + \Delta\mathbf{P}$ in the neighbourhood of $\mathbf{P}$ such that the estimate (4) holds, where the function f is defined via (36), (33).

Since f is analytic, for each $k \geq 1$ we have

$$f(\Delta) = \Sigma_{j=1}^k f_j(\Delta) + O(\|\Delta\|^{k+1}), \quad \Delta \to 0,$$

where $f_j(\Delta)$ is a homogeneous polynomial in Δ_Q, Δ_A, Δ_S. In particular,

$$f(\Delta) = K_Q(\Delta_Q + 2ap\Delta_A + a^2p^2\Delta_S) + O(\|\Delta\|^2), \quad \Delta \to 0. \tag{37}$$

To compare this result with (11), (21), (22) we note that $\|\mathbf{PA}_c\| \le \|\mathbf{P}\|\,\|\mathbf{A}_c\|$ and also $\|\mathbf{PA}_c\| = \|\mathbf{P}(\mathbf{I}+\mathbf{SP})^{-1}\mathbf{A}\| \le \|\mathbf{P}\|\,\|\mathbf{A}\|$. Hence

$$K_A \le 2K_Q p\alpha, \qquad K_S \le K_Q p^2\alpha^2; \qquad \alpha = \min\{a_c, a\}$$

and (37) gives an upper estimate of the conditioning of DAMRE. It must be pointed out that both cases $a < a_c$ and $a > a_c$ are possible.

3.3. Non-local non-linear estimates (the non-symmetric case)

Consider now the perturbed equation (3) without the assumption that $\Delta\mathbf{Q}$, $\Delta\mathbf{S}$ are symmetric and/or that $\mathbf{Q}+\Delta\mathbf{Q}$, $\mathbf{S}+\Delta\mathbf{S}$ are non-negative definite matrices. Then (3) may be written in the form (30), where $\Phi(\mathbf{Z})$ is defined via (25), (26) but the estimates (28) do not hold.

Let $\|\mathbf{Z}\| \le \rho$. Having in mind that for $\mathbf{M}, \mathbf{E} \in \mathbb{R}^{n \cdot n}$ and $\|\mathbf{M}^{-1}\|\,\|\mathbf{E}\| < 1$ it is fulfilled $\|(\mathbf{M}+\mathbf{E})^{-1}\| \le (1/\|\mathbf{M}^{-1}\| - \|\mathbf{E}\|)^{-1}$ we obtain

$$\|(\mathbf{M}+\mathbf{SZ})^{-1}\| \le (1/\mu - s\rho)^{-1} = \mu(1-\mu s\rho)^{-1}, \tag{38}$$

$$\|(\mathbf{I}_n + (\mathbf{S}+\Delta\mathbf{S})(\mathbf{P}+\mathbf{Z}))^{-1}\| \le (1/\mu - p\Delta_S - (s+\Delta_S)\rho)^{-1} \tag{39}$$

for $p\Delta_S + (s+\Delta_S)\rho < 1/\mu$, where $\mu = \|(\mathbf{I}_n + \mathbf{SP})^{-1}\|$.

Suppose that $\theta_1, \theta_2 > 1$ and

$$\delta_S \le (1 - 1/\theta_1)/(\mu p), \tag{40}$$

$$\rho \le (1-1/\theta_2)/(s\mu\theta_1 + (\theta_1 - 1)/p) = \rho_0 < (\mu p)^{-1}. \tag{41}$$

Then

$$\|(\mathbf{I}_n + (\mathbf{S}+\Delta\mathbf{S})(\mathbf{P}+\mathbf{Z}))^{-1}\| \le \theta\mu, \tag{42}$$

where $\theta = \theta_1\theta_2 > 1$. Now it follows from (25) and (38) that

$$\|\mathcal{B}(\mathbf{P},\mathbf{Z})\| \le \mu s a_c^2 \rho^2 (1-\mu s\rho)^{-1}.$$

Similarly, (26) and (39) yield

$$\|\mathbf{F}(\mathbf{Y}, \Sigma+\Delta\Sigma) - \mathbf{F}(\mathbf{Y};\Sigma)\| \le \theta\mu^2 a^2 (p+\rho)^2 \Delta_S (1-\mu s\rho)^{-1} + \theta\mu(p+\rho)(2a+\Delta_A)\Delta_A + \Delta_Q.$$

It follows from (30) and (41), (42) that

$$\begin{aligned} &\|\Phi(\mathbf{Z})\| \le \chi(\rho) = \\ =\ & K_Q[\mu s a_c^2\rho^2 + \theta\mu^2 a^2(p+\rho)^2\Delta_S](1-\mu s\rho)^{-1} + K_Q\theta\mu(p+\rho)(2a+\Delta_A)\Delta_A + K_Q\Delta_Q. \end{aligned}$$

As in Section 3.2, the implication $\|\mathbf{Z}\| \le \rho \Rightarrow \|\Phi(\mathbf{Z})\| \le \rho$ will be valid if the equation $\chi(x) = x$ has a root $x = \rho = g(\Delta) \le \rho_0$. The last condition yields

$$b_2(\Delta)x^2 - (1 - b_1(\Delta))x + b_0(\Delta) = 0$$

where

$$\begin{aligned} b_0(\Delta) &= K_Q\left[\Delta_Q + \theta\mu p(2a\Delta_A + \Delta_A^2) + \theta\mu^2 a^2 p^2 \Delta_S\right], & (43)\\ b_1(\Delta) &= K_Q\mu\left[2\theta p\mu a^2\Delta_S + \theta(2a\Delta_A + \Delta_A^2)(1-\mu ps) - s\Delta_Q\right], & (44)\\ b_2(\Delta) &= \mu\left[K_Q s a_c^2 + K_Q\theta\mu a^2\Delta_S + s - K_Q\theta\mu s(2a\Delta_A + \Delta_A^2)\right]. & (45)\end{aligned}$$

Hence we have proved the following

Theorem 3.3. Let the conditions (40) and

$$\begin{aligned} &b_2(\Delta) > 0, & (46)\\ &b_1(\Delta) + 2[b_0(\Delta)b_2(\Delta)]^{1/2} < 1, &\\ &\varphi(\Delta) \le \rho_0 &\end{aligned}$$

be fulfilled, where

$$\begin{aligned} \varphi(\Delta) &= \left[1 - b_1(\Delta) - E^{1/2}(\Delta)]/[2b_2(\Delta)\right], & (47)\\ E(\Delta) &= [1 - b_1(\Delta)]^2 - 4b_0(\Delta)b_2(\Delta). &\end{aligned}$$

Then the perturbed equation (3) has an unique solution $\mathbf{Y} = \mathbf{P} + \Delta\mathbf{P}$ in the neighbourhood of $\mathbf{P}$ such that the estimate

$$\Delta_P \le \varphi(\Delta)$$

holds, where the function φ is defined via (43) – (47) and (41).

4. NUMERICAL EXAMPLE

Consider a third order DAMRE with matrices $\mathbf{Q} = \mathbf{V}\mathbf{Q}_0\mathbf{V}$, $\mathbf{A} = \mathbf{V}\mathbf{A}_0\mathbf{V}$, $\mathbf{S} = \mathbf{V}\mathbf{S}_0\mathbf{V}$, where $\mathbf{V}$ is an elementary reflection, $\mathbf{V} = \mathbf{I}_3 - 2\mathbf{v}\mathbf{v}^{\mathrm{T}}/3$, $\mathbf{v} = [1,1,1]^{\mathrm{T}}$ and $\mathbf{Q}_0 = \mathrm{diag}(10^{\ell}, 1, 10^{-\ell})$, $\mathbf{A}_0 = \mathrm{diag}(0, 10^{-\ell}, 1)$, $\mathbf{S}_0 = \mathrm{diag}(10^{-\ell}, 10^{-\ell}, 10^{-\ell})$ for some positive integer ℓ. The sensitivity of this equation increases with the increasing of ℓ.

Owing to the diagonal form of the matrices $\mathbf{Q}_0$, $\mathbf{A}_0$ and $\mathbf{S}_0$, the solution is given by $\mathbf{P} = \mathbf{V}\mathbf{P}_0\mathbf{V}$, $\mathbf{P}_0 = \mathrm{diag}(p_1, p_2, p_3)$, where

$$p_i = \left[a_i^2 + q_i s_i - 1 + ((a_i^2 + q_i s_i - 1)^2 + 4q_i s_i)^{1/2}\right]/(2s_i)$$

and q_i, a_i and s_i are the corresponding diagonal elements of $\mathbf{Q}_0$, $\mathbf{A}_0$ and $\mathbf{S}_0$.

The perturbations in the data are taken as $\Delta\mathbf{Q} = \mathbf{V}\Delta\mathbf{Q}_0\mathbf{V}$, $\Delta\mathbf{A} = \mathbf{V}\Delta\mathbf{A}_0\mathbf{V}$, $\Delta\mathbf{S} = \mathbf{V}\Delta\mathbf{S}_0\mathbf{V}$, where

$$\Delta\mathbf{Q}_0 = \begin{bmatrix} 10^{\ell} & -5 & 7\\ -5 & 1 & 3\\ 7 & 3 & 10^{\ell}\end{bmatrix} \times 10^{-j},$$

$$\Delta\,\mathbf{A}_0 = \begin{bmatrix} 3 & -4 & 8 \\ -6 & 2 & -9 \\ 2 & 7 & 5 \end{bmatrix} \times 10^{-j},$$

$$\Delta\,\mathbf{S}_0 = \begin{bmatrix} 10^{-\ell} & -10^{-\ell} & 2\times 10^{-\ell} \\ -10^{-\ell} & 5\times 10^{-\ell} & -10^{-\ell} \\ 2\times 10^{-\ell} & -10^{-\ell} & 3\times 10^{-\ell} \end{bmatrix} \times 10^{-j}$$

for $j = 10, 9, \ldots, 2$.

The perturbed solution $\mathbf{P} + \Delta\mathbf{P}$ of the equation is computed by the generalized Schur method [12] in arithmetic with relative precision $\varepsilon = 2^{-52} \approx 2.22x10^{-16}$.

The relative perturbation δ_P in the solution is estimated by the local bound (12), (18), (19) in accordance with Theorem 3.1 and the non-local bounds predicted by Theorems 3.2 and 3.3.

The results, obtained for different values of ℓ and hence for different conditioning (measured by the quantity k in (16)) of the equation, are shown in Tables 1–4. In all cases the actual relative change in the solution is near to the quantity, predicted by the local sensitivity analysis. The cases when the conditions (35) of Theorem 3.2 or (46) of Theorem 3.3 are violated are denoted by asterisk. The estimate from Theorem 3.3 is given for $\theta_1 = \theta_2 = 1.1$.

Table 1.

$\ell = 0,\ k = 1.9$

j	δ_P	Theorem 3.1	Theorem 3.2	Theorem 3.3
10	7.5×10^{-10}	1.7×10^{-9}	1.02×10^{-8}	8.18×10^{-9}
9	7.5×10^{-9}	1.7×10^{-8}	1.02×10^{-7}	8.18×10^{-8}
8	7.5×10^{-8}	1.7×10^{-7}	1.02×10^{-6}	8.18×10^{-7}
7	7.5×10^{-7}	1.7×10^{-6}	1.02×10^{-5}	8.18×10^{-6}
6	7.5×10^{-6}	1.7×10^{-5}	1.02×10^{-4}	8.18×10^{-5}
5	7.5×10^{-5}	1.7×10^{-4}	1.2×10^{-3}	8.20×10^{-4}
4	7.5×10^{-4}	1.7×10^{-3}	1.6×10^{-2}	8.42×10^{-3}
3	7.5×10^{-3}	1.7×10^{-2}	1.70×10^{-1}	$\star$
2	7.5×10^{-2}	1.7×10^{-1}	$\star$	$\star$

Table 2.

$\ell = 1,\ k = 6.6$

j	δ_P	Theorem 3.1	Theorem 3.2	Theorem 3.3
10	1.1×10^{-9}	2.8×10^{-9}	2.37×10^{-8}	4.09×10^{-8}
9	1.1×10^{-8}	2.8×10^{-8}	2.37×10^{-7}	4.09×10^{-7}
8	1.1×10^{-7}	2.8×10^{-7}	2.37×10^{-6}	4.09×10^{-6}
7	1.1×10^{-6}	2.8×10^{-6}	2.37×10^{-5}	4.09×10^{-5}
6	1.1×10^{-5}	2.8×10^{-5}	2.38×10^{-4}	4.11×10^{-4}
5	1.1×10^{-4}	2.8×10^{-4}	2.43×10^{-3}	4.33×10^{-3}
4	1.1×10^{-3}	2.8×10^{-3}	3.37×10^{-2}	$\star$
3	1.1×10^{-2}	2.8×10^{-2}	$\star$	$\star$
2	1.2×10^{-1}	2.8×10^{-1}	$\star$	$\star$

Table 3.

$\ell = 2,\ k = 51.5$

j	δ_P	Theorem 3.1	Theorem 3.2	Theorem 3.3
10	5.6×10^{-9}	8.9×10^{-9}	2.13×10^{-7}	4.05×10^{-7}
9	5.6×10^{-8}	8.9×10^{-8}	2.13×10^{-6}	4.05×10^{-6}
8	5.6×10^{-7}	8.9×10^{-7}	2.14×10^{-5}	4.07×10^{-5}
7	5.6×10^{-6}	8.9×10^{-6}	2.17×10^{-4}	4.29×10^{-4}
6	5.5×10^{-5}	8.9×10^{-5}	2.82×10^{-3}	$\star$
5	5.4×10^{-4}	8.9×10^{-4}	$\star$	$\star$
4	4.7×10^{-3}	8.9×10^{-3}	$\star$	$\star$
3	3.0×10^{-2}	8.9×10^{-2}	$\star$	$\star$
2	1.7×10^{-1}	8.9×10^{-1}	$\star$	$\star$

Table 4.

$\ell = 2,\ k = 51.5$

j	δ_P	Theorem 3.1	Theorem 3.2	Theorem 3.3
10	5.1×10^{-8}	7.2×10^{-8}	2.12×10^{-6}	4.07×10^{-6}
9	5.1×10^{-7}	7.2×10^{-7}	2.15×10^{-5}	4.29×10^{-5}
8	5.1×10^{-6}	7.2×10^{-6}	2.78×10^{-4}	$\star$
7	4.9×10^{-5}	7.2×10^{-5}	$\star$	$\star$
6	4.2×10^{-4}	7.2×10^{-4}	$\star$	$\star$
5	2.4×10^{-3}	7.2×10^{-3}	$\star$	$\star$
4	9.6×10^{-3}	7.2×10^{-2}	$\star$	$\star$
3	3.7×10^{-2}	7.2×10^{-1}	$\star$	$\star$
2	1.8×10^{-1}	7.2×10^{0}	$\star$	$\star$

For the above example the linear estimate is relatively sharp, while the non-linear ones even do not exist in certain cases. In principle, however, the non-linear estimates are more reliable since they guarantee the corresponding bound as well as the existence of the perturbed solution. At the same time the linear estimate would formally give bounds which may be incorrect or may even correspond to a "solution" which does not exist.

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