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## STRUCTURE AT INFINITY, MODEL MATCHING AND DISTURBANCE REJECTION FOR LINEAR SYSTEMS WITH DELAYS

MICHEL MALABRE AND RABAH RABAH

Structure at infinity for systems with delays is introduced here. A generalization of the Smith–McMillan form at infinity is given with an application to the Model Matching and Disturbance Rejection Problems. With the help of some finite dimensional descriptions, a formulation of a Partial Disturbance Rejection Problem is given for systems with delay and necessary and sufficient structural conditions are provided for the existence of a non anticipative state feedback solution to this problem. These conditions are expressed in terms of structures at infinity and directly extend the corresponding result about Exact Disturbance Rejection for classical systems without delay.

### INTRODUCTION

It is now well known that some particular structures of control systems are fundamental in the solution of some qualitative problems. This is typically the case for the so-called structure at infinity (or structure of zeros at infinity) which is some “measure” of the properness properties. Indeed, for linear classical systems, the structure at infinity gives important characterizations for control problems like the regular row-by-row or block decoupling, the model matching or the disturbance rejection (see for instance [4], [8], [2], . . .). This notion of zero at infinity has been generalized to non linear systems [12]; several definitions are also available for singular linear systems (see for instance [7]) and for linear infinite dimensional systems (see [11] for the particular case of bounded operators).

The aim of this paper is to further extend this structural approach in the direction of linear systems with delays and to use this approach for solving classical control problems like Model Matching or Disturbance Rejection. Results on Disturbance Rejection within an infinite dimensional state space setting may be found in [15], [3] and [21]. Recent results are also available using Ring Theory (see [16]).

The paper is structured as follows. The first part deals with general linear systems with delays described by rational transfer matrices with two unknowns, say  $T(s, e^s)$ . We extend the notion of Smith–McMillan form at infinity and use this extension to solve, in a structural way, the Model Matching Problem for this class of systems. We

also consider the Disturbance Rejection Problem by Dynamic Precompensation (for the case when the disturbance is measured). Structural and geometric necessary and sufficient conditions are given for its solvability. The second part deals with delay systems in the (finite dimensional) state space representation. The Structure at Infinity can easily be derived from a decomposition procedure, which is also used to solve the Partial Disturbance Rejection Problem (rejection on a given finite time horizon).

1. SMITH-MCMILLAN FORM AT INFINITY, MODEL MATCHING AND DISTURBANCE REJECTION

We consider *rational* transfer matrices with two unknowns, say  $T(s, e^s)$ , where  $s$  is the classical Laplace variable, and  $e^s$  stands for the unitary shift operator ( $e^{-s}$  is the unitary delay). These matrices can be written as Laurent powers series expansions in any variable, with coefficients functions of the other variable.

For this class of systems, we can easily extend the notion of *Smith-McMillan Form at infinity* (or Smith Form over the ring of proper functions, see for instance [19] for the classical situation) which is a canonical form under left and right multiplications by biproper transformations. For that purpose, we adopt the following definition:

**Definition.**  $P(s, e^s)$  is proper if and only if  $\lim_{\text{Re}(s) \rightarrow -\infty} P(s, e^s) < \infty$ .

For practical reasons, we shall only describe this canonical form for *proper* systems  $T(s, e^s)$  (in particular we do not consider Laurent powers series with *positive* powers of  $e^s$ , since they obviously induce some anticipation). This form, say  $\Lambda(s, e^s)$ , is structured as follows:

$$\Lambda(s, e^s) = \begin{bmatrix} \Lambda_0(s) & 0 & \dots & \vdots & 0 \\ 0 & e^{-s} \Lambda_1(s) & 0 & \dots & \vdots & 0 \\ 0 & \dots & \dots & \vdots & 0 \\ 0 & \dots & e^{-qs} \Lambda_q(s) & \vdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \vdots & 0 \end{bmatrix}$$

where

$$\Lambda_0(s) = \text{diag} \{ s^{-n_{0j}} \}, \quad j = 1, \dots, p_0; \quad 0 \leq n_{01} \leq \dots \leq n_{0p_0},$$

$$\Lambda_i(s) = \text{diag} \{ s^{-n_{ij}} \}, \quad j = 1, \dots, p_i; \quad n_{i1} \leq n_{i2} \leq \dots \leq n_{ip_i}.$$

That this canonical form exists is simply due to the fact that we can consider such proper functions as polynomials in  $e^{-s}$  with coefficients in the field of rational functions in  $s$  (see [13] as a first reference using this "trick"). Namely,  $T(s, e^s) = T_0(s) + e^{-s}T_1(s) + e^{-2s}T_2(s) + \dots$ . Note that properness of  $T(s, e^s)$

implies properness of  $T_0(s)$ ; however,  $T_i(s)$ , for  $i > 0$  need not to be proper: for instance  $e^{-s} \cdot s^{+2}$  is proper.

More precisely, this canonical form can be obtained as follows: using classical biproper elementary operations, bring  $T_0(s)$  to its classical Smith–McMillan form at infinity, say  $\Lambda_0(s)$ . There then exist biproper transformation  $B_1(s, e^s)$  and  $B_2(s, e^s)$  such that:

$$B_1(s, e^s) T(s, e^s) B_2(s, e^s) = \begin{bmatrix} \Lambda_0(s) & 0 \\ 0 & T^1(s, e^s) \end{bmatrix},$$

where for all  $k \in \mathbb{N} : s^k T^1(s, e^s) \rightarrow 0$  when  $\text{Re}(s) \rightarrow \infty$ .

Note that, as usually,  $B(s, e^s)$  is *biproper* if and only if  $B(s, e^s)$  is proper, invertible and its inverse also is proper. Remark that  $B(s, e^s) = B_0(s) + e^{-s} B_1(s) + e^{-2s} B_2(s) + \dots$  is biproper if and only if  $B_0(s)$  is a *usual* biproper transfer matrix (with respect to the variable  $s$ ), i.e. if and only if  $B_0^0$  is invertible, where  $B_0(s) =: B_0^0 + B_0^1 s^{-1} + B_0^2 s^{-2} + \dots$

Consider next:  $e^s T^1(s, e^s)$  and repeat the previous treatment. Rank considerations and rationality insure that, after a finite number of steps, the desired canonical form is obtained.

Recall that, even though  $T(s, e^s)$  is proper,  $\Lambda_i(s)$  may contain, for  $i \geq 1$ , both negative and positive powers of  $s$  (however,  $\Lambda_0(s)$  only contains non positive powers of  $s$ ).

We say that  $T(s, e^s)$  has:

$p_0$  zeros at infinity of class 0, each of precise order  $n_{0j}$  (this corresponds to the classical zeros at infinity of finite orders for systems without delays), and

$p_i$  zeros at infinity of class  $i$ , for  $i$  varying from 1 to  $q$ , each of precise order  $n_{ij}$ .

The total number of zeros at infinity is, as usually, equal to the rank, say  $r$ , of  $T(s, e^s)$  (otherwise written:  $r = \sum_{i=1}^q p_i$ ).

The overall structure at infinity of  $T(s, e^s)$  is written as

$$\{n_{01}, \dots, n_{0p_0}; n_{11}, \dots, n_{1p_1}; \dots; n_{q1}, \dots, n_{qp_q}\}$$

(note that some integers  $n_{ij}$ ,  $i \geq 1$ , may be negative)

or, more compactly as:

$$\{n_{ij}\}$$

(where  $i$  stands for the class and  $j$  for the order within that class)

or also as:  $\sum_{\infty} \{T(s, e^s)\}$ .

We conclude this first part by showing how this structure plays a fundamental role in the solution of the Model Matching Problem which amounts to finding a *proper* solution  $T_c(s, e^s)$  to the following equation:

$$T(s, e^s) \cdot T_c(s, e^s) = T_m(s, e^s), \tag{1.1}$$

where  $T(s, e^s)$  and  $T_m(s, e^s)$  are two given *proper* systems, respectively called the plant and the model.

Indeed, the following result is an extension of [10] (see also [9] for the original algebraic proof):

**Theorem 1.** Let  $T(s, e^s)$  and  $T_m(s, e^s)$  denote a proper plant and a proper model, there exists a proper precompensator  $T_c(s, e^s)$  which solves the Model Matching Problem (i. e. which solves (1.1)), if and only if:

$$\Sigma_\infty \{T(s, e^s)\} = \Sigma_\infty \{[T(s, e^s) | -T_m(s, e^s)]\}.$$

**Proof only if:** Assume there exists a proper solution, say  $T_c(s, e^s)$ . Equation (1.1) is equivalent to:

$$[T(s, e^s) | -T_m(s, e^s)] \begin{bmatrix} \mathbb{I} & T_c(s, e^s) \\ 0 & \mathbb{I} \end{bmatrix} = [T(s, e^s) | 0].$$

Properness of  $T_c(s, e^s)$  is equivalent to the fact that  $\begin{bmatrix} \mathbb{I} & T_c(s, e^s) \\ 0 & \mathbb{I} \end{bmatrix}$  is a biproper transformation, which immediately provides the desired structural condition.

if: Assume that  $[T(s, e^s) | -T_m(s, e^s)]$  and  $[T(s, e^s) | 0]$  have the same structure at infinity (and thus the same rank, say  $r$ ), and let us denote  $\Lambda(s, e^s)$  their common Smith McMillan Form at infinity. There then exist biproper transformations, say  $B_1(s, e^s)$  and  $B_2(s, e^s)$ , such that:

$$B_1(s, e^s) T(s, e^s) B_2(s, e^s) = \begin{bmatrix} \Lambda(s, e^s) & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence:

$$B_1(s, e^s) T(s, e^s) = \begin{bmatrix} \underline{T}(s, e^s) \\ 0 \end{bmatrix}, \text{ with } \underline{T}(s, e^s) \text{ of full row rank } r.$$

We can thus write:

$$B_1(s, e^s) [T(s, e^s) | -T_m(s, e^s)] = \begin{bmatrix} \underline{T}(s, e^s) & T_{m1}(s, e^s) \\ 0 & T_{m2}(s, e^s) \end{bmatrix}.$$

Now, because of the assumption, we obviously have:

i)  $T_{m2}(s, e^s) = 0$  (otherwise the rank would be increased)

ii) there exists a proper  $P(s, e^s)$  such that:

$$\begin{bmatrix} T_{m1}(s, e^s) \\ 0 \end{bmatrix} = \begin{bmatrix} \Lambda(s, e^s) & 0 \\ 0 & 0 \end{bmatrix} P(s, e^s).$$

Let us then define  $T_c(s, e^s) =: B_2(s, e^s) P(s, e^s)$ . This precompensator is obviously proper and solves the model matching problem. Indeed:

$$\begin{bmatrix} T_{m1}(s, e^s) \\ 0 \end{bmatrix} = \begin{bmatrix} \Lambda(s, e^s) & 0 \\ 0 & 0 \end{bmatrix} B_2^{-1}(s, e^s) B_2(s, e^s) P(s, e^s),$$

otherwise written:

$$B_1(s, e^s) T_m(s, e^s) = \{B_1(s, e^s) T(s, e^s)\} T_c(s, e^s),$$

which ends the proof.  $\square$

The following corollary is related to the existence of *strictly proper* solutions to the Model Matching Problem. Recall that a transfer function matrix is strictly proper if and only if its product by  $s$  is still proper.

**Corollary 1.** Let  $T(s, e^s)$  and  $T_m(s, e^s)$  denote a proper plant and a proper model, there exists a strictly proper precompensator  $T_c(s, e^s)$  which solves the Model Matching Problem (i.e. which solves (1.1)), if and only if:

$$\Sigma_\infty \{s^{-1}T(s, e^s)\} = \Sigma_\infty \{[s^{-1}T(s, e^s) | -T_m(s, e^s)]\}.$$

*Proof.*  $T_c(s, e^s)$  is strictly proper if and only if there exists a proper  $\underline{T}_c(s, e^s)$  such that:  $T_c(s, e^s) = s^{-1}\underline{T}_c(s, e^s)$ . Then  $T_c(s, e^s)$  solves (1.1) if and only if  $\underline{T}_c(s, e^s)$  solves the equation:  $T(s, e^s) \cdot [s^{-1}\underline{T}_c(s, e^s)] = T_m(s, e^s)$ , or equivalently:  $[s^{-1}T(s, e^s)] \cdot \underline{T}_c(s, e^s) = T_m(s, e^s)$ . The result directly follows from Theorem 1.  $\square$

A quite similar result can be derived in the context of Disturbance Rejection. Indeed, it has been known for a long time, at least for classical systems without delays, that these two problems are equivalent (see [5]). Consider now the particular class of system with one delay and with disturbance  $d(t)$ , as described by:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t-1) + B u(t) + E d(t) \\ y(t) = C x(t) \end{cases} \quad (1.2)$$

We shall also assume that  $d(t)$  can be measured and thus incorporated in the compensation scheme. We consider the following:

**Disturbance Rejection Problem with Dynamic Precompensation:** Does there exist a proper (or respectively strictly proper) precompensator,  $u = T_c(s, e^s) d$ , which rejects the effect of  $d$  on the output  $y$ ? Otherwise written, we want to find a proper (resp. strictly proper)  $T_c(s, e^s)$  such that:

$$C (s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} [B T_c(s, e^s) + E] = 0,$$

i.e.

$$T(s, e^s) \cdot T_c(s, e^s) + T_E(s, e^s) = 0, \quad (1.3)$$

with

$$T(s, e^s) =: C (s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} B$$

$$T_E(s, e^s) =: C (s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} E.$$

Thanks to Theorem 1, we immediately conclude that such a solution exists if and only if  $T(s, e^s)$  and  $[T(s, e^s) | T_E(s, e^s)]$  have the same structure at infinity, respectively if and only if  $s^{-1}T(s, e^s)$  and  $[s^{-1}T(s, e^s) | -T_m(s, e^s)]$  have the same structure at infinity.

This problem can also be given a geometric solution, which extends the famous one of [20]. Let  $\mathcal{V}_\Sigma$  denote the following subspace of  $\text{Ker } C$ :

$$\mathcal{V}_\Sigma =: \{x \in \text{Ker } C \mid \exists \xi(s, e^s), \omega(s, e^s) \text{ s. p.}, \\ C \xi(s, e^s) = 0, x = (s\mathbb{I} - A_0 - A_1 e^{-s}) \xi(s, e^s) - B \omega(s, e^s)\},$$

where s. p. means strictly proper.

$\mathcal{V}_\Sigma$  is an extension of Hautus' notion of "supremal frequency invariant subspace" which could also be compared with the infinite version developed in [21].

We then have

**Theorem 2.** The Disturbance Rejection Problem with Proper Dynamic Precompensation is solvable if and only if:

$$\text{Im } E \subset \text{Im } B + \mathcal{V}_\Sigma. \tag{1.4}$$

*Proof.* Suppose that (1.3) is satisfied for some proper  $T_c(s, e^s)$ . This means that:  $C(s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} [B T_c(s, e^s) d + E d] = 0$  for all  $d$ . Since  $T_c(s, e^s)$  is proper, there exists a constant  $G$  such that:  $\lim_{\text{Re}(s) \rightarrow -\infty} T_c(s, e^s) = G$ .

Let:

$$\xi(s, e^s) =: (s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} [B T_c(s, e^s) d + E d] \\ \omega(s, e^s) =: T_c(s, e^s) d - G d,$$

then:

$$\xi(s, e^s) \text{ is strictly proper, with } C \xi(s, e^s) = 0, \\ \omega(s, e^s) \text{ is strictly proper, and} \\ (s\mathbb{I} - A_0 - A_1 e^{-s}) \xi(s, e^s) =: B T_c(s, e^s) d + E d = B \omega(s, e^s) + B G d + E d.$$

Hence, due to the very definition of  $\mathcal{V}_\Sigma$ ,  $(B G + E) d \in \mathcal{V}_\Sigma$ , for all  $d$ ; otherwise written:  $\text{Im } E \subset \text{Im } B + \mathcal{V}_\Sigma$ .

Conversely, from the inclusion:  $\text{Im } E \subset \text{Im } B + \mathcal{V}_\Sigma$ , for any vector  $d_i$  extracted from a basis of the disturbance space  $\mathcal{D}$ , there exist some  $v_i$  in the control space  $\mathcal{V}$ , such that  $(B v_i + E d_i) \in \mathcal{V}_\Sigma$ , i.e. there exist some strictly proper  $\xi_i(s, e^s)$  and  $\omega_i(s, e^s)$  such that  $C \xi_i(s, e^s) = 0$  and:

$$E d_i = (s\mathbb{I} - A_0 - A_1 e^{-s}) \xi_i(s, e^s) - B \omega_i(s, e^s) - B v_i.$$

The following precompensator:

$$T_c(s, e^s) d_i =: \omega_i(s, e^s) + v_i$$

obviously solves the problem. □

**Corollary 2.** The Disturbance Rejection Problem with Strictly Proper Dynamic Precompensation is solvable if and only if:

$$\text{Im } E \subset \mathcal{V}_\Sigma. \quad (1.5)$$

**Proof.** Direct from Theorem 2, just put  $G = 0$ .  $\square$

Note that our definition of Disturbance Rejection is not really standard, in the sense that for "classical" systems without delay, the problem is usually set in terms of static state feedback laws on a given state description of the system. The reason is that in the case of Disturbance Rejection for systems with delays, the equivalence between dynamic solutions and static state feedback solutions has not been established. Moreover, properness of the compensator may even be insufficient for practical purpose. Indeed, despite the convenient setting of our structural approach proper solutions as considered above for the Model Matching Problem or the Disturbance Rejection Problems may still exhibit some unfair "pathologies". For instance, as shown by the following simple example, properness does not mean non anticipation:

$$\begin{aligned} T(s, e^s) &= \frac{s + e^s}{s^2 e^s} && \text{is proper and non anticipative} \\ T_m(s, e^s) &= \frac{1}{s^2 e^s} && \text{is proper and non anticipative, but the solution:} \\ T_c(s, e^s) &= \frac{1}{s + e^s} && \text{is proper but anticipative.} \end{aligned}$$

Since the general study of properness and non anticipation is not sufficiently developed for the time being, we consider in the second part of the paper a modified version of these "exact" control problems for delayed systems. We shall indeed solve some *Partial Disturbance Rejection Problem* by non anticipative static state feedback laws.

## 2. STATE SPACE REPRESENTATION OF SYSTEMS WITH DELAYS AND PARTIAL DISTURBANCE REJECTION

In order to simplify the exposition, we just consider here the previously introduced subclass of systems with delays, those having only one delay in the state:

$$S: \begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t-1) + B u(t) \\ y(t) = C x(t) \end{cases}$$

where  $x(t) \in \mathcal{X} \approx \mathbb{R}^n$ ,  $u(t) \in \mathcal{U} \approx \mathbb{R}^m$ , and  $y(t) \in \mathcal{Y} \approx \mathbb{R}^p$ .

Thanks to the nice *triangular* properties exhibited hereafter, the results of this Section 2 may quite easily be extended to systems with several integer delays in the state, in the input and in the output.



The general results of the first part obviously apply to this particular subclass for which the transfer matrix is analytically known as:

$$\begin{aligned}
 T(s, e^s) &= C (s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} B & (2.1) \\
 &= \sum_{i=0}^{\infty} (e^{-s})^i C (s\mathbb{I} - A_0)^{-1} [A_1 (s\mathbb{I} - A_0)^{-1}]^i B := T_0(s) + \sum_{i=0}^{\infty} e^{-is} T_i(s).
 \end{aligned}$$

Each  $T_i(s)$ ,  $i \geq 0$ , is obviously a proper rational matrix in  $s$ . This property implies that the structure at infinity is only formed with non negative integers.

Moreover, we can associate with this system the following family of classical (without delays) linear systems (see for instance [14],[18]), say  $S_k$ :

$$S_k : \begin{cases} \dot{x}_k(t) = F_k x_k(t) + G_k u_k(t) \\ y_k(t) = H_k x_k(t) \end{cases} \quad (2.2)$$

where  $F_k$  is  $[(k+1)n] \cdot [(k+1)n]$ ,  $G_k$  is  $[(k+1)n] \cdot [(k+1)m]$ , and  $H_k$  is  $[(k+1)p] \cdot [(k+1)n]$ , with:

$$F_k = \begin{bmatrix} A_0 & 0 & \dots & 0 \\ A_1 & A_0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & A_1 & A_0 & 0 \\ 0 & \dots & 0 & A_1 & A_0 \end{bmatrix}, \quad G_k = \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & 0 & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & 0 & B & 0 \\ 0 & \dots & 0 & B \end{bmatrix}$$

and  $H_k = \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & C \end{bmatrix}$ .

For adequate initial conditions for systems  $S_k$ , the outputs of systems  $S_k$  are linked with the output of  $S$  as follows (see [18]):

$$y(t) = [0 \ 0 \ \dots \ 0 \ \mathbb{I}] y_k(t-k), \quad t \in [k, k+1].$$

Indeed, as we are only interested here by the input-output relationship, the initial condition may be taken as zero, and then this relation between the outputs of  $S$  and  $S_k$  holds for every  $k$ .

Let us denote the transfer matrix of  $S_k$  as:

$$\Phi_k(s) = H_k (s\mathbb{I} - F_k)^{-1} G_k.$$

One important connection between  $T(s, e^s)$  and the family  $\Phi_k(s)$  is the following:

$$\Phi_k(s) = \begin{bmatrix} T_0(s) & 0 & \dots & 0 \\ T_1(s) & T_0(s) & 0 & \dots & 0 \\ T_2(s) & T_1(s) & T_0(s) & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ T_{k-1}(s) & \dots & T_1(s) & T_0(s) & 0 \\ T_k(s) & \dots & T_2(s) & T_1(s) & T_0(s) \end{bmatrix}. \quad (2.3)$$

With the help of this relation, it is easily verified that the overall structure at infinity of  $T(s, e^s)$  in (2.1) can be obtained from those of the elements of the family  $S_k$  in (2.2) and after a finite number of steps. Indeed (remember (2.1)):

$$\Sigma_\infty \{\Phi_0(s)\} = \Sigma_\infty \{T_0(s)\} = \{n0_j\}.$$

Let  $k_0$  be any integer such that  $k_0 > \sup\{n0_j\}$ , and let:

$$\Phi_1(s) = \begin{bmatrix} T_0(s) & 0 \\ s^{-k_0}T_1(s) & T_0(s) \end{bmatrix}.$$

Then

$$\Sigma_\infty \{\Phi_1(s)\} = \{\{n0_j\}, \{n0_j\}, \{n1_j + k_0\}\}.$$

The procedure can be continued, choosing any integer  $k_1 > \sup\{\{n0_j\}, \{n1_j + k_0\}\} \dots$ . Note that thanks to the majorant terms  $s^{-k_i}$ , the computation of the structure at infinity of  $T(s, e^s)$  can be done with the classical tools corresponding to transfer matrices which are only rational in the  $s$  variable. The stopping criterion is also given by the rank.  $\square$

Remembering that the  $T_i(s)$ 's are the coefficients of  $e^{-is}$  in the power series expansion of (2.1), the previous result is some generalization of what is classically known for systems without delays: indeed, for such systems, like for instance  $T(s) =: C(sI - A)^{-1}B$ , we can compute the structure at infinity (which only has "zero class" elements) from the Toeplitz matrices (see for instance [17]):

$$\Gamma_i = \begin{bmatrix} CB & 0 & \dots & 0 \\ CAB & CB & 0 & \dots & 0 \\ CA^2B & CAB & CB & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ CA^{i-2}B & \dots & CAB & CB & 0 \\ CA^{i-1}B & \dots & \dots & CAB & CB \end{bmatrix}$$

This geometry of  $S_k$  (namely through the supremal  $(F_k, G_k)$ -invariant subspace contained in the Kernel of  $H_k$ , say  $\mathcal{V}_k^*$ , see for instance [1] or [20]) also has some nice properties. In particular, the following result holds, which will allow us, in the sequel, to use non anticipative feedbacks:

**Lemma.** There always exist maps  $R_k$  such that:

$$R_k = \begin{bmatrix} L_0 & 0 & \dots & 0 \\ L_1 & L_0 & 0 & \dots & 0 \\ L_2 & L_1 & L_0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ L_k & \dots & L_2 & L_1 & L_0 \end{bmatrix} \quad \text{with: } (F_k + G_k R_k) \mathcal{V}_k^* \subset \mathcal{V}_k^*.$$

**Proof (sketch).** We give the proof for  $k = 1$  (trivial for  $k = 0$ ). Let  $\mathcal{V}_1^*$  be the supremal  $(F_1, G_1)$ -invariant subspace contained in  $\text{Ker } H_1$ , then  $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \in \mathcal{V}_1^*$  if

and only if there exist  $\begin{bmatrix} x'_0 \\ x'_1 \end{bmatrix} (\in \mathcal{V}_1^*)$ ,  $u_0$  and  $u_1$  such that  $A_0 x_0 = x'_0 + B u_0$  and  $A_1 x_0 + A_0 x_1 = x'_1 + B u_1$ . Then,  $L_0$  is defined through the relation:  $-L_0 x_0 =: u_0$ . The second equality may be rewritten as:  $A_1 x_0 + A_0 x_1 + B L_0 x_1 = x'_1 + B(u_1 + L_0 x_1)$ .  $L_1$  is thus defined by:  $-L_1 x_0 =: u_1 + L_0 x_1$ . We then obtain:

$$(A_0 B + L_0) x_0 = x'_0 \quad \text{and} \quad (A_0 + B L_0) x_1 + (A_1 + B L_1) x_0 = x'_1.$$

Hence,  $\mathcal{V}_1^*$  is invariant under the particular feedback transformation:

$$\begin{bmatrix} A_0 + B L_0 & 0 \\ A_1 + B L_1 & A_0 + B L_0 \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} L_0 & 0 \\ L_1 & L_0 \end{bmatrix}.$$

For arbitrary  $k$ , the construction of  $L_0, L_1, \dots, L_k$  is quite similar. □

Note that  $\{L_0, L_1, \dots, L_{k-1}, L_k\}$  computed in that way (at step  $k$ ) are such that the subset  $\{L_0, L_1, \dots, L_{k-1}\}$  also satisfies the Lemma for step  $k - 1$ . The reason is that if  $[x_0^T \ x_1^T \ \dots \ x_k^T] \in \mathcal{V}_k^*$  then  $[x_0^T \ x_1^T \ \dots \ x_{k-1}^T]^T \in \mathcal{V}_{k-1}^*$ .

This triangularity property allows us to further study the Disturbance Rejection Problem with disturbance measurement (which is known to be equivalent to the Model Matching Problem, see for instance [5]) for our class of delayed systems, that is (remember (1.2)):

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 x(t - 1) + B u(t) + E d(t) \\ y(t) = C x(t) \end{cases}$$

where  $d(t)$  is some  $q$ -dimensional disturbance (assumed to be zero for  $t \leq 0$ ) which can be measured and thus used in the control law scheme.

Similarly to  $G_k$  and  $H_k$ , let us define the following  $[(k + 1) n] \cdot [(k + 1) q]$  matrix:

$$J_k = \begin{bmatrix} E & 0 & \dots & 0 \\ 0 & E & 0 & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & 0 & E & 0 \\ 0 & \dots & 0 & E \end{bmatrix}.$$

We establish the following structural theorem:

**Theorem 3.** The structural condition:

$$\Sigma_{\infty} \{H_k(s\mathbb{I}_k - F_k)^{-1} G_k\} = \Sigma_{\infty} \{[H_k(s\mathbb{I}_k - F_k)^{-1} G_k \mid H_k(s\mathbb{I}_k - F_k)^{-1} J_k]\}$$

is equivalent to the following (partial) disturbance rejection property:

there exists a non anticipative state feedback law:  
 $u(t) = L_0 x(t) + L_1 x(t - 1) + \dots + L_k x(t - k) + M_0 d(t) + M_1 d(t - 1) + \dots + M_k d(t - k)$  such that the output  $y(t)$  of the compensated system is not affected by  $d(t)$  over the interval  $[0, k + 1]$ .

**Proof.** If the structural condition holds, then  $S_k$  satisfies the classical (exact) disturbance rejection property (recall [8], for instance). This means that (recall Theorem 2):

$$\text{Im } J_k \subset \text{Im } G_k + \mathcal{V}_k^*$$

Then, thanks to the Lemma, there always exist “triangular” feedback matrices which are “friends” of  $\mathcal{V}_k^*$ , and thus which solve the disturbance rejection problem for  $S_k$ . Now, just remember that  $S$  and  $S_k$  have trajectories which are in one to one correspondence all over the time interval  $[0, k + 1]$ .

Conversely, if such a non anticipative feedback law exists which rejects the disturbance on  $S$ , all over the time interval  $[0, k + 1]$ , this is also the same for  $S_k$ . Since  $S_k$  is a “classical” system (without delay), that the forced response between  $d(t)$  and  $y(t)$  be zero on a non zero interval is equivalent to the fact that it is zero for any time. Then, exact disturbance rejection is performed on  $S_k$ , which is exactly expressed by the structural condition.  $\square$

The following corollary is immediate for the case when the disturbance is not measured:

**Corollary 3.** There exists a non anticipative state feedback law  $u(t) = L_0 x(t) + L_1 x(t-1) + \dots + L_k x(t-k)$  such that the output  $y(t)$  of the compensated system is not affected by  $d(t)$  over the interval  $[0, k + 1]$  if and only if the following structural condition holds:

$$\Sigma_\infty \{ s^{-1} H_k (s\mathbb{I}_k - F_k)^{-1} G_k \} = \Sigma_\infty \{ [ s^{-1} H_k (s\mathbb{I}_k - F_k)^{-1} G_k | H_k (s\mathbb{I}_k - F_k)^{-1} J_k ] \}.$$

**Remark 1.** If the structural condition of Theorem 3 is not satisfied for some particular  $j$ , no non anticipative state feedback law may exist for the exact disturbance rejection, even if the following condition holds, which is the structural equivalent of Theorem 2:

$$\begin{aligned} & \Sigma_\infty \{ C(s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} B \} = \\ & = \Sigma_\infty \{ [ C(s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} B | C(s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} E ] \}. \end{aligned}$$

This means that Theorem 3 can be used to build up some procedure for finding on which time horizon ( $k$ ), if any, anticipation becomes necessary for rejecting the disturbance, even in a partial way.

**Remark 2.** Note that the structural condition expressed in Theorem 3 for a given  $k$  is equivalent to the set of conditions:

$$\begin{aligned} & \Sigma_\infty \{ H_j (s\mathbb{I}_j - F_j)^{-1} G_j \} = \\ & = \Sigma_\infty \{ [ H_j (s\mathbb{I}_j - F_j)^{-1} G_j | H_j (s\mathbb{I}_j - F_j)^{-1} J_j ] \}, \quad \text{for any } j, 0 \leq j \leq k. \end{aligned}$$

Indeed, thanks to the Lemma, we have that:

$$\text{Im } J_k \subset \text{Im } G_k + \mathcal{V}_k^* \implies \text{Im } J_{k-1} \subset \text{Im } G_{k-1} + \mathcal{V}_{k-1}^*.$$

Note that this Theorem 3 only solves a *partial* Disturbance Rejection problem: the disturbance can be rejected on a *finite* time interval; “complete” Disturbance Rejection is not guaranteed. However, this partial rejection (as initially introduced in [6]) can be sufficient in many practical situations. This partial rejection property may also be expressed in terms of transfer matrices: that  $y(t)$  is identically zero on  $[0, k + 1]$  is equivalent to the fact that (using Laplace Transforms):  $y(s) = \Theta(s, e^s) d(s)$ , where  $\Theta(s, e^s)$  only has zeros at infinity of class greater than  $k$ , i.e.  $\Theta(s, e^s) = e^{-(k+1)s} \bar{\Theta}(s, e^s)$ , with  $\bar{\Theta}(s, e^s)$  strictly proper (i.e.  $\lim_{\text{Re}(s) \rightarrow \infty} \bar{\Theta}(s, e^s) = 0$ ).

Note also that:

- our compensation, since only based on past information, is *always non anticipative* (which was not easily obtained in the first part of our exposition, for more general situations),
- the structural condition which expresses solvability depends on the horizon  $k$ . When  $k$  is increasing, and if the structural condition still holds for greater horizon, we need to use in the feedback law more and more past information from the state. Some recursive procedure, if any, would of course be welcome to take into account moving horizon.

We shall conclude this paper with some expression for the rank of  $S$ . Remember that this information precises the total number of zeros at infinity of the system. Let  $T(s, e^s)$  be as described in (2.1), and note  $r$  its rank. Consider again the “supremal frequency invariant subspace”,  $\mathcal{V}_\Sigma$ , introduced in Section 1. We have the following result:

**Theorem 4.**

$$r := \text{rank} \{T(s, e^s)\} = \dim \left( \frac{\text{Im } B}{\text{Im } B \cap \mathcal{V}_\Sigma} \right).$$

*Proof.* i) Let  $\{B u_i\}$  denote a basis of  $\text{Im } B \cap \mathcal{V}_\Sigma$ . There then exist strictly proper  $\xi_i(s, e^s)$  and  $\omega_i(s, e^s)$  such that  $\xi_i(s, e^s) \in \text{Ker } C$ , and  $B u_i = (s\mathbb{I} - A_0 - A_1 e^{-s}) \xi_i(s, e^s) - B \omega_i(s, e^s)$ . Let  $\underline{\omega}_i(s, e^s) =: (\omega_i + \omega_i(s, e^s)) s^{-1}$ . We then have  $T(s, e^s) \underline{\omega}_i(s, e^s) = 0$ , for  $i = 1$  to  $\sigma$ , with  $\underline{\omega}_i(s, e^s)$  independent,  $\sigma =: \dim(\text{Im } B \cap \mathcal{V}_\Sigma)$  and  $m =: \dim(\text{Im } B)$ . Hence:  $r \leq m - \sigma$ .

ii) Conversely, let  $r =: \text{rank } T(s, e^s)$  and denote  $\Lambda(s, e^s)$  its Smith-McMillan Form at infinity, i.e. (with ad-hoc biproper matrices):

$$\Lambda(s, e^s) = B_1^{-1}(s, e^s) T(s, e^s) B_2^{-1}(s, e^s).$$

Let  $\{u_i\}$ ,  $i = 1$  to  $m - r$ , denote a basis for the Kernel of  $\Lambda(s, e^s)$ . Choosing:

$$\xi_i(s, e^s) =: (s\mathbb{I} - A_0 - A_1 e^{-s})^{-1} B B_2^{-1}(s, e^s) [u_i + u_i s^{-1}],$$

we have:

$$C \xi_i(s, e^s) = T(s, e^s) B_2^{-1}(s, e^s) [u_i + u_i s^{-1}] = B_1(s, e^s) \Lambda(s, e^s) [u_i + u_i s^{-1}] = 0,$$

i. e.  $\xi_i(s, e^s) \in \text{Ker } C$ . Moreover,  $\xi_i(s, e^s)$  is obviously strictly proper. Then:

$$\begin{aligned} (s\mathbb{I} - A_0 - A_1 e^{-s}) \xi_i(s, e^s) &= B B_2^{-1}(s, e^s) [u_i + u_i s^{-1}] \\ &= B [B_2^{-1}(s, e^s) u_i s^{-1} + D_0 + D(s, e^s) u_i], \end{aligned}$$

with  $D_0$  (constant) invertible, and  $D(s, e^s)$  strictly proper. Finally:

$$(s\mathbb{I} - A_0 - A_1 e^{-s}) \xi_i(s, e^s) - B [B_2^{-1}(s, e^s) u_i s^{-1} + D(s, e^s) u_i] = B D_0 u_i.$$

Hence,  $B D_0 u_i \in \mathcal{V}_\Sigma$ , i. e.  $\sigma =: \dim(\text{Im } B \cap \mathcal{V}_\Sigma) \geq m - r$ , which ends the proof.  $\square$

The following corollary is also immediate:

**Corollary 4.**  $T(s, e^s)$  is left invertible if and only if  $B$  is monic and  $\text{Im } B \cap \mathcal{V}_\Sigma = 0$ .

### 3. CONCLUSION

We have generalized the Smith–McMillan form at infinity to some rather general classes of linear systems with delays. This gives rise to some new structure at infinity, say  $\{n_{ij}\}$ , which is still a list of integers, as for the classical case of systems without delay, but now parametrized with two integers, one precising the class (i. e. the power of  $e^s$ , the other precising the respective order in the class (i. e. the power of  $s^{-1}$ ). This structural information has allowed us to extend to this class of proper systems with delays some familiar results about Dynamic Model Matching or Disturbance Rejection (Theorems 1 and 2). Due to the possible anticipative pathologies of the corresponding solutions (though proper), we have considered the Partial version of the Disturbance Rejection Problem (Theorem 3) which is a generalization to systems with delays of the similar problem previously introduced in [6]. Finally, some geometric characterization of the rank of the system has been given (Theorem 4). The future directions to be explored in that domain concern first some receding horizon technics for the actual computation of the state feedback laws proposed in Section 2, and also, which is necessary for practical applications, the additional requirement of stability.

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