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HANKEL-MATRIX APPROACH TO INVERTIBILITY OF LINEAR MULTIVARIABLE SYSTEMS

K. B. DATTA

The transfer-function matrix $R(s)$ of a linear multivariable system can be represented by the matrix fraction description $R(s) = P(s)Q^{-1}(s) = \tilde{Q}^{-1}(s)\tilde{P}(s)$ or by $R(s) = \bar{P}(s)/q(s)$ where $q(s)$ is the common denominator of all entries in $R(s)$. Based on either of these descriptions, new criteria of k-integral invertibility of linear multivariable systems are derived which are expressed as a rank condition of matrices formed by the parameters in the numerator matrix $P(s)$ (or $\tilde{P}(s)$ or $\bar{P}(s)$) and denominator matrix $Q(s)$ (or $\tilde{Q}(s)$ or the denominator polynomial $q(s)$). A new method based on the Hankel-matrix is used to construct the inverse, to parametrize the set of all minimal order inverses and to identify the stable minimal inverse if one such exists.

1. INTRODUCTION

In the last three decades the invertibility of linear systems has been extensively studied with a view to applications in diversified fields of control systems such as coding and decoding problems, filtering and prediction theory, decoupling and synthesis of linear systems, and so on and so forth. Current interest in investigating the fundamental questions of the existence, properties and construction of inverse systems was initiated by Brockett and Mesarović [4] by introducing the concept of functional reproducibility which is a characteristic of right invertible systems. Subsequently Brockett [3] took the help of what is known as left inverse system to offer a state-space interpretation of system zeros. In the following years the invertibility of linear multivariable systems drew the attention of a large number of research workers who were apparently divided principally into three classes. One class of workers (Sain and Massey [15], [18] Dorato [7]) were led by Sain and Massey who brought in the concept of k-integral or k-delay inverse and stated the invertibility criteria as a rank condition of a matrix of Toeplitz type formed by the system's Markov parameters although state-variable description plays an important role in their construction of inverse system. Silverman pioneered the second class by presenting his "structure algorithm" ([19], [20]) which could be acclaimed as an important tool for studying the diversified aspects of multivariable systems. An interesting feature of this "structure algorithm" is that it can be applied to the construction of an inverse of time-invariant as well as time-varying multivariable systems starting from their state-variable descriptions. In the third approach (Forney [10]. Hautus [12], Borukhov [2], Wyman and Sain [24]) the system is described by a transfer-function matrix in which the elements are looked upon as elements over a PID $\mathbb{R}[[s^{-1}]]$, $\mathbb{R}$ is the field of real numbers or over a PID Q where Q is a ratio of polynomials in $\mathbb{R}[s]$ and invertibility criteria are stated in terms of what is called invariant factors or $\mathscr{F}$-MacMillan form of the transfer-function matrix. Or a theory of inverse system can be laid down with the concept of zero module and pole module.

There are a number of works devoted primarily to the study of stable inversion (Antsaklis [1] and Moylan [16]) or to the study of inverse system of minimal order (Kučera [14], Wang and Davison [22], Forney [11] and Yuan [25]). Sain-Massey criterion of invertibility was improved by tightening the bounds on inherent integrations or delays in (Willsky [23]) and additional criteria of invertibility were given in (Wang and Davison [22]).

In deriving the invertibility criteria in all above cases one has to start with the state-space description (A, B, C, D) of the system or with the manipulation of the elements in transfer-function matrix $R(s)$ which is considered as having elements over the field $\mathbb{R}(s)$ of fractions of the ring $\mathbb{R}[s]$ of polynomials with coefficients from the real-number field $\mathbb{R}$. By expressing $R(s)$ as $P(s)/q(s)$ where $q(s)$ is common denominator of all the elements in $R(s)$, Orner [17], however, related in a straight-forward fashion the Sain-Massefy invertibility criterion given in terms of Markov parameters to a similar criterion expressed in terms of coefficient matrices P_i's in the polynomial matrix $P(s)$. With the same system description $P(s)/q(s)$ Emre and Hüseyin [9] gave invertibility criteria in terms of P_i's but different from the above.

From a careful scrutiny of the foregoing references it seems justified to set up a procedure for the study of k-integral invertibility of linear multivariable systems when they are expressed as a (right or left) matrix fraction description in the form $R(s) = P(s)\,Q^{-1}(s) = \tilde{Q}^{-1}(s)\,\tilde{P}(s)$. The purpose of this paper is, therefore, to establish an invertibility theory based on the matrix fraction description by deriving *a new invertibility criterion* and a constructive procedure for an inverse system, and making an effort to answer such questions as how to obtain an inverse of minimal order and how to achieve a stable inverse. It is a consequence of the proposed theoretical approach to generate a sufficient number of Markov parameters associated with the inverse system from which its state-space realization is determined in a bid to construct the inverse by applying the existing realization theories. The above invertibility criterion and constructive procedure can be specialized to the case when the transfer-function matrix is expressed as $R(s) = P(s)/q(s)$ from which follows Orner's criterion. One can apply the invertibility results established in this paper to discrete-time system as well although the underlying system with the help of which the foregoing results will be established is a continuous time system.

After a preliminary description in Section 2 of the multivariable system and its inverse with which we shall mainly concern ourselves in this paper, we present the main results in Section 3. The construction of an inverse possibly of a minimal order is described in Section 4.

2. SYSTEM DESCRIPTION

Let a linear time-invariant multivariable system $\mathscr{S}$ be described by:

(2.1a) $$\mathscr{S}: \; \dot{x}(\tau) = A\, x(\tau) + B\, u(\tau)\,,$$

(2.1b) $$y(\tau) = C\, x(\tau) + D\, u(\tau)\,,$$

where $u(\tau) \in \mathbb{R}^m$ is the input, $x(\tau) \in \mathbb{R}^{\bar{n}}$ is the state, $y(\tau) \in \mathbb{R}^r$ is the output, A, B, C and D are real constant matrices of compatible dimensions. The transfer function matrix $R(s)$ of the system $\mathscr{S}$ is a rational function matrix of order $r \times m$ given by

(2.2) $$R(s) = C(sI - A)^{-1}\, B + D\,,$$

which as well can be written as

(2.3) $$\bar{\mathscr{S}}: \; R(s) = P(s)\, Q^{-1}(s)\,,$$

where $P(s) = [p_{ij}(s)]$ is a polynomial matrix of order $r \times m$ expressed as:

(2.4) $$P(s) = P_0 s^l + P_1 s^{l-1} + \ldots + P_l\,,$$

P_i's $(i = 0, 1, \ldots, l)$ being $r \times m$ real constant matrices, $P_0 \neq 0$, $l = \max_{i,j} [\deg p_{ij}(s)]$ and $Q(s)$ is a matrix of order m expressed as

(2.5) $$Q(s) = Q_n s^n + Q_{n-1} s^{n-1} + \ldots + Q_1 s + Q_0\,,$$

Q_i's $(i = 0, 1, \ldots, n)$ being $m \times m$ matrices and $l \leqq n$. At the beginning we shall study the invertibility of $\bar{\mathscr{S}}$ by assuming that Q_n is nonsingular.

Now $\hat{R}(s)$, a causal transfer function matrix, is defined to be a k-integral (left) inverse of the system $\bar{\mathscr{S}}$ if

(2.6) $$\hat{R}(s)\, R(s) = s^{-k} I_m\,,$$

which means that $R(s)$ has rank m over the field of rational functions in s. The minimum value of k for which (2.6) is satisfied is called the inherent integrations of the invertible system and is denoted by k_0. The inverse $\hat{R}(s)$ is expressed as a series in s^{-1} as

(2.7) $$\hat{R}(s) = R_0 + R_1 s^{-1} + R_2 s^{-2} + \ldots,$$

where R_i's $(i = 0, 1, 2, \ldots)$ are $m \times r$ real constant matrices. We now insert (2.3) to

(2.5) and (2.7) into (2.6) to get:

$$(2.8)\qquad R_0P_0 + (R_0P_1 + R_1P_0)\,s^{-1} + (R_0P_2 + R_1P_1 + R_2P_0)\,s^{-2} + \ldots = = (Q_n + Q_{n-1}s^{-1} + \ldots + Q_{n-l}s^{-l} + \ldots + Q_0s^{-n})\,s^{n-(k+l)},$$

and observing that $k \geqq n - l$, equating coefficients of various powers of s^{-1} for $i = 0, 1, 2, \ldots$ where $i = k + l - n$ on both sides of the above equation (2.8) we have

(2.9)

$$\begin{bmatrix} R_0 & R_1 & \cdots & R_{n+i+1-l} & \cdots & R_{n+i+1} \\ R_1 & R_2 & \cdots & R_{n+i+2-l} & \cdots & R_{n+i+2} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ R_{n+i+1} & R_{n+i+2} & \cdots & R_{2(n+i+1)-l} & \cdots & R_{2(n+i+1)} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ R_{t-(n+i+1)} & R_{t-(n+i)} & \cdots & R_{t-l} & \cdots & R_t \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P_l \\ \vdots \\ P_0 \end{bmatrix} \begin{matrix} n+i+1-l \\ \text{rows} \\ l+1 \\ \text{rows} \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$(2.10)\qquad X_jA_j = \bar{Q}_j,$$

where

$$(2.11)\qquad X_j = (R_0\ R_1\ \ldots\ R_j),$$

$$(2.12)\qquad T_j = \begin{bmatrix} P_0 & P_1 & \cdots & P_j \\ & P_0 & \cdots & P_{j-1} \\ & & & \vdots \\ & & & P_0 \end{bmatrix}, \quad j = 0, 1, \ldots, l;$$

$$(2.13)\qquad A_j = T_j, \quad j \leqq l;$$

$$(2.14)\qquad A_j = \begin{bmatrix} P_0 & P_1 & P_2 & \cdots & P_l & & & & & \\ & P_0 & P_1 & \cdots & P_{l-1} & P_l & & & & \\ & & \ddots & & \vdots & \vdots & & & & \\ & & & & P_0 & P_1 & \cdots & P_l & 0 & \\ & & & & & P_0 & \cdots & P_{l-1} & P_l & \\ & & & & & & & & \vdots & \\ & & & & & & & & P_0 & \end{bmatrix}, \quad j > l;$$

$\bar{Q}_j$ is a matrix of order $1 \times (j + 1)$ having $(m \times m)$ submatrices as elements and is given by

$$\bar{Q}_j = \begin{cases} (0, \ldots, 0), & 0 \leqq j < i; \quad (2.15) \\ (0, \ldots, 0, Q_n), & j = i, \quad (2.16) \\ \quad \text{(0th col. } \ldots \text{ } i\text{th column)} & \\ (0, \ldots, 0, Q_n, Q_{n-1}, \ldots, Q_{n-j+i}), & j > i \quad (2.17) \end{cases}$$

(2.18) $$Q_j = 0\,, \quad j > n \quad \text{and} \quad j > 0\,.$$

Note that columns are counted as $0, 1, \ldots, j$.

2.19. Remark. In view of (2.8) it is interesting to observe that the number of inherent integrations k_0 is lower bounded by $n - l$, a new insight offered by transfer-function approach.

2.20. Example. Let us consider the transfer function

$$R(s) = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+1 & s+2 \\ s^2+s+3 & 2s \\ 3s & 0 \end{bmatrix}$$

which when put in the form (2.3) gives us

$$P_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = I_2\,, \quad Q_1 = 3I_2 \quad \text{and} \quad Q_0 = 2I_2\,.$$

It is obvious that here $n = 2$, $l = 2$ and $i = 0$ if k is chosen to be zero. Then

$$\bar{Q}_j = [Q_2\ Q_1]\,, \quad j = 1 > i = 0$$

and

$$\bar{Q}_j = [Q_2\ Q_1\ Q_0\ 0\ 0]\,, \quad j = 4 > i = 0\,;$$

and the set (2.10) is inconsistent because

(2.21) $$A_j \quad \text{and} \quad \begin{bmatrix} A_j \\ \bar{Q}_j \end{bmatrix}$$

have not the same rank. Choosing $k = 1$ when also $i = 1$ we have

$$\bar{Q}_j = [0\ Q_2\ Q_1\ Q_0\ 0]\,, \quad j = 4 > i = 1$$

and the matrices in (2.21) have the same rank. Consequently (2.10) is now a consistent set of equations and when solved gives R_i's which describe the inverse $\hat{R}(s)$ of $R(s)$.

2.22. Remark. We claim that for a consistent system of linear equations described by (2.10) for a suitable choice of i the sequence of $m \times r$ matrices $\{R_0\ R_1\ \ldots\ R_t\}$ in (2.9) is realizable for some finite value of t (cf. [13], Problem 2) if $r \times m$ matrix $P(s)$ is full column rank. To justify our claim we write (2.10) in an expanded form as

$$(2.23)\qquad [R_0\, R_1 \dots R_j] \overset{\mathscr{P}}{\begin{bmatrix} P_0 & P_1 & \dots & P_l & & & & \\ & P_0 & \dots & P_{l-1} & P_l & & & \\ & & & \vdots & \vdots & & & \\ & & & P_0 & P_1 & \dots & P_l & 0 \\ & & & & P_0 & \dots & P_{l-1} & P_l \\ & & & & & \ddots & & \vdots \\ & & & & & & & P_0 \end{bmatrix}}$$

$$\overset{\mathscr{Q}}{= [0 \dots 0\; Q_n\; Q_{n-1} \dots Q_0 \,|\, 0 \dots 0]}$$
$$|\longleftarrow n + i + 1 \text{ cols.} \longrightarrow|$$

If rank $P_0 = q_0 < m$, there exist a nonsingular matrix S_0 such that

$$P_{00} = P_0 S_0 = [\bar{P}_{00} \,|\, 0]$$

where $\bar{P}_{00}$ has q_0 columns and rank q_0 and

$$P_{j0} = P_j S_0 = [\bar{P}_{j0}\; \tilde{P}_{j0}]\,, \quad j = 1, 2, \dots, l\,;$$

and

$$Q_{k0} = Q_k S_0 = [\bar{Q}_{k0}\; \tilde{Q}_{k0}]\,, \quad k = 0, 1, \dots, n\,.$$

We now add the last $m - q_0$ columns in any block column to the last $m - q_0$ columns in the block immediately preceding it in an order starting from the second column block both in $\mathscr{P}$ and $\mathscr{Q}$ matrices. Then starting from the first column block we substract last $m - q_0$ columns in any column block from the last $m - q_0$ columns in the next column block again in an order both in $\mathscr{P}$ and $\mathscr{Q}$ matrix after which we get the elements in these matrices as

$$[\bar{P}_{j0}, \tilde{P}_{j+1,0}]\,, \quad j = 0, 1, \dots, l - 1\,, \quad \text{and} \quad [\bar{P}_{l0}\; 0]$$

and the last $m - q_0$ columns in last column block in $\mathscr{P}$ are zero. If $i = 0$ the $(n + i + 2)$nd column block in $\mathscr{Q}$ matrix is nonzero, otherwise it is zero, after one step of these two sequences of operations. Now if

$$\text{rank}\, [\bar{P}_{00}\, \tilde{P}_{10}] = q_1 < m$$

we can find a constant nonsingular matrix S_1 such that

$$P_{01} = [\bar{P}_{00}\, \tilde{P}_{10}]\, S_1 = [\bar{P}_{01}\; 0]\,,$$

where $\bar{P}_{01}$ has q_1 columns and rank q_1. The above step is then repeated till at the end of α steps we have

$$\text{rank}\, [\bar{P}_{0,\alpha-1}\; \tilde{P}_{1,\alpha-1}] = q_\alpha = m$$

where

$$\alpha \leqq \delta = \text{rank} \begin{bmatrix} P_0 & P_1 & \dots & P_{l-1} \\ & P_0 & \dots & P_{l-2} \\ & & \ddots & \vdots \\ & & & P_0 \end{bmatrix}$$

This result follows with a trivial modification of the method given in [8].

After these α steps, the set of equations (2.23) becomes

$$
(2.24) \qquad [R_0\, R_1 \dots R_{\beta-l} \mid R_{\beta-l+1} \dots R_j] =
$$

$$
= \left[\begin{array}{ccc|c|c}
\multicolumn{3}{c|}{\longleftarrow \beta+1 \longrightarrow} & \leftarrow j-\alpha-\beta \rightarrow & \longleftarrow \alpha \longrightarrow \\
N_0 N_1 \dots N_l & & & & \\
\ddots & \ddots & & & \\
N_0 \dots N_{l-1} & N_l & & & \\
\hline
\ddots\ \vdots & \ddots & & & \\
 & N_{l-1} & & & \\
 & \vdots & & & \\
N_0 & N_1 & \dots N_l & & 0 \\
 & \ddots & & & \\
 & N_0 & \dots N_{l-1} & & N_{l1} \\
 & & \ddots \ \vdots & & \vdots \ \ddots \\
 & & N_0 & & \vdots \qquad N_{l\alpha} \\
 & & & & N_{01} \quad \vdots \\
 & & & & \ddots \quad \vdots \\
 & & & & N_{0\alpha}
\end{array}\right] \begin{array}{c} \uparrow \\ \beta - l + 1 \\ \downarrow \end{array} =
$$

$$
= [M_0\, M_1 \dots M_l \dots M_\beta \mid \underbrace{0 \dots 0}_{j-\alpha-\beta} \mid \underbrace{\dots 0}_{\alpha \text{ columns}}]
$$

where

$$\beta = n + i \quad \text{if} \quad \alpha \leqq i ,$$

and

$$\beta = n + \alpha \quad \text{if} \quad \alpha > i ,$$

$N_0 = [\bar{P}_{0,\alpha-1}, \tilde{P}_{1,\alpha-1}]$ has full rank m, N_{ab}'s $(a = 0, \dots, l, b = 1, \dots \alpha)$ have some zero columns and $N_{01} \dots N_{0\alpha}$ have rank less than m.

Choosing $j > (\alpha + \beta)$ the set of $(j - \alpha - \beta)$ number of equations in the middle of (2.24) as shown can be written as

$$
(2.25) \qquad \underbrace{\begin{bmatrix}
R_1 \dots R_{\beta-l} & R_{\beta-l+1} \dots R_{\beta+1} \\
R_2 \dots R_{\beta-l+1} & R_{\beta-l+2} \quad R_{\beta+2} \\
\vdots \ \dots \ \vdots & \vdots \ \dots \ \vdots \\
R_{j-\alpha-\beta}\ R_{j-\alpha-l-1} & R_{j-\alpha-l} \quad R_{j-\alpha}
\end{bmatrix}}_{\mathscr{R}_{j-\alpha-\beta,\beta+1}}
\begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline N_l \\ \vdots \\ N_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

which shows as N_0 has full column rank that the last block column of the Hankel matrix $\mathscr{R}_{j-\alpha-\beta,\beta+1}$ in (2.25) can be expressed as a linear combination of the remaining columns for all values of $j > (\alpha + \beta)$. This implies that rank $\mathscr{R}_{j-\alpha-\beta,\beta} =$

$= \operatorname{rank} \mathscr{R}_{j-\alpha-\beta,\beta+1}$. If we choose j to be the least positive integer such that

$$m(j-\alpha-\beta) \geqq \beta r \tag{2.26}$$

then obviously

$$\operatorname{rank} \mathscr{R}_{j-\alpha-\beta+1,\beta} = \operatorname{rank} \mathscr{R}_{j-\alpha-\beta,\beta}$$

These two rank conditions would imply (cf. [13]) that $\{R_1, R_2, \ldots, R_t\}$ where $t \leqq (j-\alpha)$ is a realizable sequence for the choice of j given by (2.26).

3. CRITERIA OF INVERTIBILITY

With the above introduction we now present our first Theorem which is concerned with a criterion of k-integral invertibility of the system $\bar{\mathscr{S}}$.

3.1. Theorem. The linear time-invariant multivariable system (LTIMS) $\bar{\mathscr{S}}$ when Q_n in (2.5) is nonsingular has a k-integral (left) inverse if and only if for some non-negative integer $\bar{k} \leqq l$

$$\operatorname{rank}(T_{\bar{k}}) - \operatorname{rank}(T_{\bar{k}-1}) = m\,, \quad (\bar{k} = 0, 1, \ldots, l) \tag{3.2}$$

$$\operatorname{rank}(T_{-1}) := 0\,.$$

Proof. To prove the claim we note that the inverse of the polynomial matrix $Q(s)$ represented by (2.5) when Q_n is nonsingular can be expressed in the form

$$L(s) = L_0 s^{-n} + L_1 s^{-n-1} + L_2 s^{-n-2} + \ldots \tag{3.3}$$

where L_0 is nonsingular and we redefine $P(s)$ in (2.4) to write

$$P(s) = \bar{P}_0 s^n + \bar{P}_1 s^{n-1} + \ldots + \bar{P}_{n-1} s + \bar{P}_n\,, \tag{3.4}$$

where some higher degree coefficient matrices may be zero. It now follows from (2.2) and (2.3) after substituting (3.3) and (3.4) into (2.3) that

$$\begin{bmatrix} \bar{P}_0 & \bar{P}_1 & \ldots & \bar{P}_j \\ & \bar{P}_0 & \ldots & \bar{P}_{j-1} \\ & & \ddots & \vdots \\ & & & \bar{P}_0 \end{bmatrix} \begin{bmatrix} L_0 & L_1 & \ldots & L_j \\ & L_0 & \ldots & L_{j-1} \\ & & \ddots & \vdots \\ & & & L_0 \end{bmatrix} = \begin{bmatrix} D & CB & \ldots & CA^{j-1}B \\ & D & \ldots & CA^{j-2}B \\ & & \ddots & \vdots \\ & & & D \end{bmatrix} =: M_{D_j}, j \leqq n$$

whence, since L_0 is nonsingular, it is evident that

$$\operatorname{rank}(\bar{T}_j) = \operatorname{rank}(M_{D_j})\,, \quad j \leqq n\,,$$

where $\bar{T}_j$ is a matrix of the form (2.12) where P_i's are now replaced by $\bar{P}_i$'s given in (3.4). Moreover $\operatorname{rank} T_j = \operatorname{rank} \bar{T}_{j+n-l}$, $j = 0, 1, \ldots, l$. By interchanging rows and columns one can derive from M_{D_j} the matrix M_i as defined in [18], from which follows the condition given in (3.2). □

3.5. Remark. The least integer $k_{\min}$ of $\bar{k}$ satisfying (3.2) is the minimum value of i for which (2.10) has a solution. Therefore from (2.8) the inherent integrations k_0 of the invertible system are $k_0 = k_{\min} + n - l \leqq n$. Moreover $\operatorname{rank}(A_{\bar{k}}) - \operatorname{rank}(A_{\bar{k}-1}) = m$ for all $\bar{k} \geqq k_{\min}$ which shows that if a k_0-integral inverse exists so does an inverse with integrations $k > k_0$. The finite sequence $\{R_0, R_1, \ldots, R_t\}$, $t \leqq (j - \alpha)$ obtained as a solution of (2.10) determines the state-space description $\{\bar{A}, \bar{B}, \bar{C}, \bar{D} = R_0\}$ while for $v > t$, we set $R_v := \bar{C}\bar{A}^{v-1}\bar{B}$ which specifies (2.7) completely with the above state-space description.

3.6. Remark. By making $Q(s)$ column proper in (2.3) and then augmenting the column degrees of $P(s)$ and $Q(s)$ appropriately, Q_n in (2.5) can always be made nonsingular. Alternatively to achieve a nonsingular Q_n we can write (2.3) in the form $R(s) = P(s)\, q^{-1}(s)$ where $q(s) = s^n + q_{n-1}s^{n-1} + \ldots + q_0$. Setting $Q(s) = q(s)\, I_m$ it is obvious that $Q_n = I_m$ is nonsingular.

From the proof of Theorem (3.1) it follows easily

3.7. Theorem. The LTIMS $\bar{\mathscr{S}}$ with Q_n nonsingular has a k-integral (left) inverse iff for some nonnegative integer $\bar{k} \leqq n$

$$\operatorname{rank}(\bar{T}_{\bar{k}}) - \operatorname{rank}(\bar{T}_{\bar{k}-1}) = m\,, \quad \bar{k} \leqq n\,; \qquad \operatorname{rank}(\bar{T}_{-1}) = 0\,, \tag{3.8}$$

and the minimum value of $\bar{k}$ satisfying (3.8) gives inherent integrations of the inverse system.

The invertibility criterion contained in Theorem 3.7 when specialized to $R(s) = P(s)\, q^{-1}(s)$ gives Orner [17] criterion. Now in (2.5) if Q_n happens to be singular then we have the following theorem which can be easily verified.

3.9. Theorem. The LTIMS when Q_n in (2.5) is not necessarily nonsingular and has a k-integral (left) inverse iff for some nonnegative integer $\bar{k} \leqq l$

$$\operatorname{rank}\begin{bmatrix} A_{n+\bar{k}} \\ Q_{n+\bar{k}} \end{bmatrix} = \operatorname{rank}(A_{n+\bar{k}}) \tag{3.10}$$

and the number of inherent integrations is $\bar{k}_{\min} + n - l$ where $\bar{k}_{\min}$ is the minimum value of $\bar{k}$ satisfying (3.10).

Let the transfer-function matrix $R(s)$ of the given system $\mathscr{S}$ be expressed by the (left) matrix fraction representation

$$\tilde{\mathscr{S}}:\quad R(s) = \tilde{Q}^{-1}(s)\, \tilde{P}(s) \tag{3.11}$$

where

$$\tilde{P}(s) = \tilde{P}_0 s^l + \tilde{P}_1 s^{l-1} + \ldots + \tilde{P}_l\,,$$
$$\tilde{Q}(s) = \tilde{Q}_{\tilde{n}} s^{\tilde{n}} + \tilde{Q}_{\tilde{n}-1} s^{\tilde{n}-1} + \ldots + \tilde{Q}_0\,,$$

and $\tilde{Q}_{\tilde{n}}$ is nonsingular and let $\tilde{R}(s)$ be the k-integral (right) inverse of $\tilde{\mathscr{S}}$ defined by

$$R(s)\,\tilde{R}(s) = s^{-k} I_r\,.$$

This implies that $R^{\mathrm{T}}(s)$ must have a k-integral (left) inverse where $R(s)$ is given by (3.11). This introduction leads, in a manner similar to the derivation of Theorem 3.1, to the following:

3.12. Theorem. The LTIMS $\tilde{\mathscr{S}}$ when $\tilde{Q}_{\tilde{n}}$ is nonsingular has a k-integral (right) inverse iff for some nonnegative integer $\bar{k} \leqq \bar{l}$

$$\text{rank}\,(\tilde{T}_{\bar{k}}) - \text{rank}\,(\tilde{T}_{\bar{k}-1}) = r\,, \qquad \text{rank}\,(\tilde{T}_{-1}) = 0\,, \tag{3.13}$$

where

$$\tilde{T}_j = \begin{bmatrix} \tilde{P}_0 & & & \\ \tilde{P}_1 & \tilde{P}_0 & & \\ \vdots & \vdots & \ddots & \\ \tilde{P}_j & \tilde{P}_{j-1} & \dots & \tilde{P}_0 \end{bmatrix}, \quad j \leqq \bar{l}$$

and the number of inherent integrations is $\bar{k}_{\min} + \tilde{n} - \bar{l}$ where $\bar{k}_{\min}$ is the minimum value of $\bar{k}$ satisfying (3.13).

The assumption that $\tilde{Q}_{\tilde{n}}$ is nonsingular in the foregoing discussion is not too stringent. In fact $\tilde{Q}(s)$ can always be made row proper and then appropriately augmenting the row degrees of $\tilde{Q}(s)$ and $\tilde{P}(s)$ we can choose $\tilde{Q}_{\tilde{n}}$ invertible. In case $\tilde{Q}_{\tilde{n}}$ is singular we can put forward a theorem for (right) invertibility similar to Theorem 3.9.

4. CONSTRUCTION OF INVERSES

The inverse of the given system is described by (2.7) where the Markov parameters $R_0, R_1, \dots$ are obtained as a solution of (2.10). As noted in Remark 3.10 at most $j - \alpha$ number of R_i's are needed to realize the state-space description of the inverse system. The realization algorithms given in [5], [6] are well suited for this purpose. Moreover the state-space description of the inverse system by employing these algorithms are given in multivariable companion forms. If the solution matrices R_i's of (2.10) contain arbitrary parameters then these parameters can possibly be adjusted to give a Hankel matrix formed by these R_i's which is of minimum rank. The state-space realization corresponding to this Hankel matrix of minimum rank is the state-variable description of the inverse system having minimal order. These concepts will be illustrated by the following Example taken from [17] before finally formulating the theory of a minimal order inverse system.

4.1. Example. Let

$$R(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s + 1 & s + 2 \\ s + 3 & s^2 + 2s \\ s^2 + 3s & 0 \end{bmatrix},$$

which when considered to be in the form (2.3) gives us

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.2}$$

$$Q_2 = I_2, \quad Q_1 = 3I_2, \quad Q_0 = 2I_2. \tag{4.3}$$

Applying Theorem 3.1 we note that the system is invertible. Also we have $n = 2$, $l = 2$, $\bar{k}_{\min} = 0$, $k_0 = \bar{k}_{\min} + n - l = 0$ and that a maximum number of $j - \alpha = 5$. R_i's are to be computed for realizing the state space description of the inverse system. We denote R_i by

$$R_i = \begin{bmatrix} r^i_{11} & r^i_{12} & r^i_{13} \\ r^i_{21} & r^i_{22} & r^i_{23} \end{bmatrix},$$

and then solve (2.10) successively with P_i's and Q_i's given by (4.2) and (4.3) to get

$$R_0 = \begin{bmatrix} r^0_{11} & 0 & 1 \\ r^0_{21} & 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} R_1 & R_2 \\ R_2 & R_3 \end{bmatrix} = \tag{4.4}$$

$$= \left[\begin{array}{ccc|ccc} r^1_{11}, & -r^0_{11}, & -r^0_{11} & r^2_{11}, & -r^1_{11}, & 2 - r^1_{11} + 3r^0_{11} \\ r^1_{21}, & -r^0_{21} + 1, & -r^0_{21} - 1 & r^2_{21}, & -r^1_{21}, & -1 - r^1_{21} + 3r^0_{21} \\ \hline r^2_{11}, & -r^1_{11}, & 2 - r^1_{11} + 3r^0_{11} & r^3_{11}, & -r^2_{11}, & -r^2_{11} + 3r^1_{11} - 6r^0_{11} - 6 \\ r^2_{21}, & -r^1_{21}, & -1 - r^1_{21} + 3r^0_{21} & r^3_{21}, & -r^2_{21}, & -r^2_{21} + 3r^1_{21} - 6r^0_{21} \end{array}\right]$$

where the parameters r^0_{11}, r^0_{21}, r^1_{11}, etc. are arbitrary and can be assigned any value whatsoever to make the rank of (4.4) a minimum. By elementary column transformations (4.4) is found to be equivalent to

$$\left[\begin{array}{ccc|ccc} r^1_{11}, & -r^0_{11}, & 0 & r^2_{11} & 0 & 2 \\ r^1_{21}, & -r^0_{21} + 1, & -2 & r^2_{21} & 0 & -2 \\ \hline r^2_{11}, & -r^1_{11}, & 2 + 3r^0_{11} & r^3_{11} & 0 & -2 \\ r^2_{21}, & -r^1_{21}, & -1 + 3r^0_{21} & r^3_{21} & 0 & -2 \end{array}\right] \tag{4.4a}$$

in which we select $r^0_{21} = 1$ and all other unassigned parameters equal to zero for a minimum rank of (4.4). This is an obvious choice, although there are other choices for the rank of (4.4) to be a minimum, one of which we shall see later. With the above choice of arbitrary parameters in R_0, R_1, R_2, R_3, we now solve (2.10) for R_4 an R_5 assigning zero values to arbitrary parameters to get (see [6])

$$\begin{bmatrix} & & R_1 \\ & R_1 & R_2 \\ \hline R_1 & R_2 & R_3 \\ R_2 & R_3 & R_4 \\ R_3 & R_4 & R_5 \end{bmatrix} T = \left[\begin{array}{ccccccccc} & & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 & -2 \\ & & & 0 & 0 & 0 & 0 & 0 & 2 \\ & & & 0 & 0 & -2 & 0 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -6 \\ 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & -6 \\ 0 & 0 & 2 & 0 & 0 & -6 & 0 & 0 & 18 \\ 0 & 0 & 2 & 0 & 0 & -6 & 0 & 0 & 18 \\ 0 & 0 & -6 & 0 & 0 & 18 & 0 & 0 & -54 \\ 0 & 0 & -6 & 0 & 0 & 18 & 0 & 0 & -54 \end{array}\right]$$

$$\underset{T}{\left[\begin{array}{cccccc|ccc} & & & & & & \downarrow & \downarrow & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & 1 & 0 & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 & 3 \\ \hline & & & & & & 1 & 0 & 0 \\ & & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 0 & 1 \end{array}\right]} \begin{array}{c} \\ \leftarrow \\ \leftarrow \\ A_1 \\ \leftarrow \\ \leftarrow \\ A_2 \\ \leftarrow \\ \leftarrow \\ A_3 \end{array} = \left[\begin{array}{cccccc|ccc} & & & & & & \downarrow & \downarrow & \\ & & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 & -2 \\ \hline & & & 0 & 0 & 0 & 0 & 0 & 2 \\ & & & 0 & 0 & -2 & 0 & 0 & -4 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 18 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 18 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ C_2 \\ \\ C_1 \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

Fig. 1. Computation of state-space realization of the inverse system. Arrows indicate the rows and columns which are of no consequence in constructing state-space realization because $d_1 = 0$, $d_2 = 0$.

Following the algorithm and the notation given in [6] we find that controllability indices are $d_1 = 0$, $d_2 = 0$ and $d_3 = 2$ and the inverse system is of minimal dimension 2 represented by

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 2 & 0 \\ -4 & -2 \end{bmatrix}, \quad \bar{D} = R_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

If, however, we assign values different from above to the arbitrary parameters in the solution matrices R_i's, e.g.,

$$R_0 = \begin{bmatrix} -\frac{2}{3} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{4}{3} & 0 & -2 \end{bmatrix}, \quad R_2 = \begin{bmatrix} -\frac{2}{3}, & -\frac{2}{3}, & -\frac{2}{3} \\ -\frac{8}{3}, & -\frac{4}{3}, & \frac{2}{3} \end{bmatrix}$$

$$R_3 = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 4 & \frac{8}{3} & \frac{2}{3} \end{bmatrix}$$

we have Fig. 2

$$(4.4b)\qquad \begin{bmatrix} & R_1 \\ \hline R_1 & R_2 \\ R_2 & R_3 \end{bmatrix} T = \begin{bmatrix} & & & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ & & & \frac{4}{3} & 0 & -2 \\ \hline \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{4}{3} & 0 & -2 & -\frac{8}{3} & -\frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{8}{3} & -\frac{4}{3} & \frac{2}{3} & 4 & \frac{8}{3} & \frac{2}{3} \end{bmatrix}$$

$$\downarrow$$

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & -1 & 1\cdot5 & 2 & 1 & 0 \\ & 1 & -2\cdot5 & -1 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ \hline & & & 1 & 0 & 1\cdot5 \\ & & & & 1 & -2\cdot5 \\ & & & & & 1 \end{array}\right]}_{T} \begin{matrix} A_1 \\ \leftarrow \\ A_2 \end{matrix}$$

$$\downarrow$$

$$= \left[\begin{array}{ccc|ccc} & & & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ & & & \frac{4}{3} & 0 & 0 \\ \hline \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} & -\frac{4}{3} & 0 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ -\frac{8}{3} & \frac{4}{3} & 0 & 0 & 0 & 0 \end{array}\right] C_1$$

Fig. 2. Computation of an alternative realization for the inverse system.

where $d_1 = 1$, $d_2 = 1$, $d_3 = 0$ (see [6] for notation). Observing that

$$A_2^{-1} = \begin{bmatrix} 1 & 0 & -1\cdot5 \\ 0 & 1 & 2\cdot5 \\ 0 & 0 & 1 \end{bmatrix}$$

it now follows from Fig. 2 that state-space description of the inverse of minimal order is

$$\bar{A} = \left[\begin{array}{c|c} -2 & -1 \\ \hline 1 & 0 \end{array}\right], \quad \bar{B} = \begin{bmatrix} 1 & 0 & -1\cdot5 \\ 0 & 0 & 2\cdot5 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{4}{3} & 0 \end{bmatrix}$$

$$\bar{D} = R_0 = \begin{bmatrix} -\frac{2}{3} & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

which has all the poles on the left half of the s-plane.

We shall now try to theorise the ideas contained in the above example as how to construct an inverse of a minimal order. The first step to construct an inverse $\hat{R}(s)$ by the proposed method is to determine the Markov parameters $R_0, R_1, \ldots,$ associated with $\hat{R}(s)$ by solving a consistent system of equations of the form

$$(4.5)\qquad R_0 N_0 = M_0,\ R_1 N_0 + R_0 N_1 = M_1, \ldots$$

as given by (2.24) which is derived from (2.10) where N_0 is a matrix of full column rank. We now find nonsingular matrices U and V of order r and m respectively such that

(4.6) $$UN_0V = \begin{bmatrix} I_m \\ \hdashline 0 \end{bmatrix}$$

We next define

(4.7) $$\bar{R}_i := R_iU^{-1}\,, \quad \bar{M}_j := M_jV \quad \text{and} \quad \bar{N}_k := UN_kV$$

for all $i = 0, 1, \ldots, j = 1, 2, \ldots, \beta$ and $k = 0, 1, 2, \ldots l$ so that solving (4.5) for $R_0, R_1, \ldots$ is equivalent to solving the set of equations

(4.8) $$\bar{R}_0\bar{N}_0 = \bar{M}_0, \bar{R}_1\bar{N}_0 + \bar{R}_0\bar{N}_1 = \bar{M}_1, \ldots$$

for $\bar{R}_0, \bar{R}_1, \ldots$ i.e. (2.24) with all elements replaced by those in (4.7) respectively. If the sequence $\{R_0, R_1, \ldots\}$ represents the realizable system

$$\Sigma : \{\bar{A}, \bar{B}, \bar{C}, \bar{D} = R_0\}$$

so does $\{\bar{R}_0, \bar{R}_1, \ldots\}$ represent the system

$$\Sigma_u : \{\bar{A}, \bar{B}U^{-1}, \bar{C}, \bar{D}U^{-1}\}\,.$$

Let $\bar{\mathscr{R}}_{ab}$ denote the Hankel matrix formed by the elements $\{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_{a+b-1}\}$ i.e., a Hankel matrix similar to that given in (2.25) with R_i's replaced by $\bar{R}_i$'s.

With the above introduction we can put forward the following two assertions which can be easily verified:

4.9. Theorem. Each element in the sequence $\{\bar{R}_0, \bar{R}_1, \bar{R}_2, \ldots\}$ obtained as a solution of (4.8) has $(r - m)\,m$ number of arbitrary parameters occupying columns $m + 1, \ldots, r$. Consequently the columns $m + 1 + kr, \ldots, r + kr$, for $k = 0, 1, 2, \ldots \ldots, b - 1$ in the Hankel matrix $\bar{\mathscr{R}}_{ab}$ have all elements arbitrarily assignable.

4.10. Corollary. The Hankel matrix $\mathscr{R}_{ab}$ formed by the matrices in $\{R_1, R_2, \ldots \ldots, R_{a+b-1}\}$ which are solutions of (2.10) has $m(r - m)(a + b - 1)$ number of arbitrary parameters occupying the columns $i_1 + kr, i_2 + kr, \ldots, i_{r-m} + kr$ $(k = 0, 1, \ldots, b - 1)$. The arbitrary parameters in R_j are designated by

(4.11) $$R_\zeta^j = \begin{bmatrix} r^j_{1i_1} & r^j_{1i_2} & \cdots & r^j_{1i_{r-m}} \\ \vdots & \vdots & & \vdots \\ r^j_{mi_1} & r^j_{mi_2} & \cdots & r^j_{mi_{r-m}} \end{bmatrix}$$

where $i_1, i_2, \ldots, i_{r-m}$ identifies the columns of R_j.

The above Theorem 4.9 and Corollary 4.10 give information concerning the structure of the Hankel matrix associated with the inverse system $\hat{R}(s)$. This structural information will prove crucial very soon in establishing the construction of a minimal order inverse system. Intuitively it is obvious if all the arbitrary parameters are set equal to zero we would get a fixed value for the rank of $\mathscr{R}_{j-\alpha-\beta,\beta+1}$. This value of rank can possibly be reduced by suitably selecting the so-called arbitrary parameters. This observation gives us:

4.12. Theorem. The minimal rank of the Hankel matrix $\mathscr{R}_{j-\alpha-\beta,\beta+1}$ is bounded below by

$$\varrho_m = \operatorname{rank} \mathscr{R}_{j-\alpha-\beta,\beta+1} \quad \text{with} \quad R_\zeta^p = 0 \quad (p = 0, 1, \ldots, j - \alpha)$$

where j is given by (2.26) and the dimension of a minimal order inverse system is lower-bounded by ϱ_m.

The bound given in Theorem 4.12 is easy to obtain but is not strong enough. A stronger bound is given below. This will be given first in terms of the matrix $\overline{\mathscr{R}}_{ab}$ with which the mathematical formulation is easily done and the criterion derived very simply.

Taking into account (4.6), (4.8) and (2.24) it is not difficult to write the solution of the sequence $\{\overline{R}_0, \overline{R}_1, \ldots, \overline{R}_l, \ldots, \overline{R}_k\}$:

$$\overline{R}_0 = [\overline{M}_0, \overline{R}_\zeta^0], \quad \overline{R}_1 = [\overline{M}_1 - \overline{R}_0\overline{N}_1, \overline{R}_\zeta^1],$$

$$\overline{R}_2 = [\overline{M}_2 - \overline{R}_0\overline{N}_2 - \overline{R}_1\overline{N}_1, \overline{R}_\zeta^2], \ldots, \overline{R}_l = [\overline{M}_l - \overline{R}_0\overline{N}_l - \ldots - \overline{R}_{l-1}\overline{N}_1, \overline{R}_\zeta^l],$$

and

$$\overline{R}_k = [\overline{M}_k - \overline{R}_{k-l}\overline{N}_l \ldots - \overline{R}_{k-1}\overline{N}_1, \overline{R}_\zeta^k], \quad k \geqq l; \quad \overline{M}_k = 0, \quad k > \beta;$$

since $\beta \geqq l$, where the arbitrary parameters are represented by $\overline{R}_\zeta^j$ as in (4.11) but with an overhead bar with each element and $i_1 = m + 1, \ldots, i_{r-m} = r$. Inserting these solutions in $\overline{\mathscr{R}}_{j-\alpha-\beta,\beta+1}$ we have

$$\overline{\mathscr{R}}_{j-\alpha-\beta,\beta+1} = \left[\begin{array}{ccc|cc} \overline{R}_1 & \ldots & \overline{R}_\beta, & -\overline{R}_{\beta+1-l}\overline{N}_l - \ldots - \overline{R}_\beta\overline{N}_1, & \overline{R}_\zeta^{\beta+1} \\ \overline{R}_2 & \ldots & \overline{R}_{\beta+1}, & -\overline{R}_{\beta+2-l}\overline{N}_l - \ldots - \overline{R}_{\beta+1}\overline{N}_1, & \overline{R}_\zeta^{\beta+2} \\ \vdots & & \vdots & \vdots & \\ \overline{R}_{j-\alpha-\beta} & \ldots & \overline{R}_{j-\alpha-1}, & -\overline{R}_{j-\alpha-l}\overline{N}_l - \ldots - \overline{R}_{j-\alpha-1}\overline{N}, & \overline{R}_\zeta^{j-\alpha} \end{array}\right]$$

Adding βth, $(\beta - 1)$st, $\ldots$, $(\beta + 1 - l)$th block columns post-multiplied respectively by $\overline{N}_1, \ldots \overline{N}_l$ to the $(\beta + 1)$st, $\ldots$, 1st block column post-multiplied by $\overline{N}_1$ to the second we have

(4.13)
$$\overline{\mathscr{R}}_{j-\alpha-\beta,\beta+1} W =$$

$$= \left[\begin{array}{c|cc|} \overline{R}_1 & \overline{M}_2 - \overline{R}_0\overline{N}_2 & , \overline{R}_\zeta^2 \\ \overline{R}_2 & \overline{M}_3 - \overline{R}_0\overline{N}_3 - \overline{R}_1\overline{N}_2 & , \overline{R}_\zeta^3 \\ \vdots & \vdots & \vdots \\ \overline{R}_{j-\alpha-\beta} & \overline{M}_{j-\alpha-\beta+1} - \overline{R}_0\overline{N}_{j-\alpha-\beta+1} - \ldots \overline{R}_{j-\alpha-\beta-1}\overline{N}_2 & , \overline{R}_\zeta^{j-\alpha-\beta+1} \end{array}\right. \ldots$$

$$\left|\begin{array}{cc|cc|} \overline{M}_{l-1} - \overline{R}_0\overline{N}_{l-1} & , \overline{R}_\zeta^{l-1} & \overline{M}_l - \overline{R}_0\overline{N}_l & , \overline{R}_\zeta^l \\ \overline{M}_l - \overline{R}_0\overline{N}_l - \overline{R}_1\overline{N}_{l-1} & , \overline{R}_\zeta^l & \overline{M}_{l+1} - \overline{R}_1\overline{N}_l & , \overline{R}_\zeta^{l+1} \\ \vdots & & \vdots & \vdots \\ \overline{M}_{l+j-\alpha-\beta-2} - \overline{R}_{j-\alpha-\beta-2}\overline{N}_l & & \overline{M}_{l+j-\alpha-\beta-1} - \overline{R}_{j-\alpha-\beta-1}\overline{N}_l & , \overline{R}_\zeta^{l+j-\alpha-\beta-1} \\ -\overline{R}_{j-\alpha-\beta-1}\overline{N}_{l-1} & , \overline{R}_\zeta^{l+j-\alpha-\beta-2} & & \end{array}\right. \ldots$$

$$\left|\begin{array}{cc|cc} \overline{M}_\beta, & \overline{R}_\zeta^\beta & 0, & R_\zeta^{\beta+1} \\ 0 & , \overline{R}_\zeta^{\beta+1} & 0, & R_\zeta^{\beta+2} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & , \overline{R}_\zeta^{j-\alpha-1} & 0, & R_\zeta^{j-\alpha} \end{array}\right]$$

where W is an elementary matrix.

The following simple Lemma is instrumental in establishing the structure of an inverse system.

4.14. Lemma. Let G be the 2×2 matrix

$$G = \begin{bmatrix} p & ap + bq + c \\ q & dp + fq + g \end{bmatrix}$$

where p and q are two arbitrary parameters, and a, b, c, d, g, and f are variables independent of p and q. Define

$$\varrho := \min_{p,q} \operatorname{rank} G\,.$$

Then (a) $\varrho < 2$, (b) $\varrho = 0$, iff $c = g = 0$ simultaneously and (c) $\varrho = 1$ iff both c and g are not equal to zero.

The minimum value $\varrho = 1$ in the above Lemma is attained when p and q satisfy

$$dp^2 + (f - a)\,pq - bq^2 + gp - qc = 0$$

which is true for $p = q = 0$ or an infinite number of nonzero values of p and q.

By repeated application of this Lemma 4.14 to (4.13) reveals that if $\bar{R}_\zeta^p$'s $(p = 1, 2, \ldots, j - \alpha)$ are set equal to zero then $\varrho(\bar{R}_\zeta^0) = \operatorname{rank} \bar{\mathscr{R}}_{j-\alpha-\beta,\beta+1}$ is a minimum with respect to these arbitrary parameters $\bar{R}_\zeta^1, \ldots, \bar{R}_\zeta^{j-\alpha}$ and is a function of $\bar{R}_\zeta^0$ only. With respect to the parameters in $\bar{R}_\zeta^0$, $\varrho(\bar{R}_\zeta^0)$ is now minimized to give us a stronger lower bound for the rank $\bar{\mathscr{R}}_{j-\alpha-\beta,\beta+1}$. Denoting by

$$\bar{R}_\zeta = [\bar{R}_\zeta^1, \bar{R}_\zeta^2, \ldots, \bar{R}_\zeta^{j-\alpha}]$$

and R_ζ by a similar matrix without bar, from the preceding discussion it is easy to state

4.15. Theorem. The rank of the Hankel matrix $\bar{\mathscr{R}}_{j-\alpha-\beta,\beta+1}$ is bounded below by

$$\varrho_{\min} = \min_{\bar{R}_\zeta^0} \operatorname{rank} \bar{\mathscr{R}}_{j-\alpha-\beta,\beta+1} \quad \text{at} \quad \bar{R}_\zeta = 0$$

where minimization is effected with respect to the parameters in $\bar{R}_\zeta^0$.

Since $\bar{\mathscr{R}}$ is obtained from $\mathscr{R}$ by a nonsingular transformation given in (4.7) we have

4.16. Corollary. The dimension of the minimal order inverse system is bounded below by

$$\varrho_{\min} = \min_{R_\zeta^0} \operatorname{rank} \mathscr{R}_{j-\alpha-\beta,\beta+1} \quad \text{at} \quad R_\zeta = 0$$

where minimization is effected with respect to the parameters in R_ζ^0.

We would now like to observe that such minimization would cause no computational problem. It is not hard to see that $\mathscr{R}_{j-\alpha-\beta,\beta+1}$ at $R_\zeta = 0$ has elements linear in the parameters of R_ζ^0 having the form

$$\beta_p = \alpha_{pk0} + \alpha_{pk1} r_{pi_1}^0 + \alpha_{pk2} r_{pi_2}^0 + \ldots + \alpha_{pkr-m} r_{pi_{r-m}}^0 \tag{4.17}$$

where α_{pk0}'s are real numbers. The values of the parameters in R_ζ^0 are sought in such a way that $\beta_p = 0$ if β_p contains unknown parameters reducing as many elements as possible in a column of $\mathscr{R}_{j-\alpha-\beta,\beta+1}$ to zero within limits of consistency. There are many solutions for these parameters which are then used in $\mathscr{R}_{j-\alpha-\beta,\beta+1}$ (at $R_\zeta = 0$) for a minimum rank. But however the values of parameters in R_ζ^0 for a minimal rank of $\mathscr{R}_{j-\alpha-\beta,\beta+1}$ is not unique. Even nonzero values of R_ζ can give rise to innumerable number of minimal order inverses (see Example 4.1).

4.18. Example. In (4.4) of Example 4.1 to get a lower-bound for the rank we set

$$r_{11}^1 = r_{21}^1 = r_{11}^2 = r_{21}^2 = r_{11}^3 = r_{21}^3 = 0$$

and for parameters r_{11}^0 and r_{21}^0 we solve equations of the form $\beta_p = 0$ as given in (4.17) if β_p contains an unknown parameter. This operation gives

$$r_{11}^0 = 0, -\tfrac{2}{3}, -1 \quad \text{and} \quad r_{21}^0 = 1, -1, \tfrac{1}{3}, 0\,,$$

but only the solution (0, 1) gives a minimum value of rank for (4.4) as already indicated.

After deciding upon a minimal order inverse by the foregoing method we shall now parametrize all the solutions of such an inverse. This parametrization will be facilitated by the following theorem given in [22].

4.19. Theorem. All minimal order k-integral inverses of an invertible system have the same set of observability indices.

The set of observability indices $d_1, d_2, \ldots, d_m$ of the minimal order inverse obtained in view of Theorem 4.15 will give a corresponding number of independent rows in $\bar{\mathscr{R}}_{j-\alpha-\beta,\beta}$ which can be identified. With the help of these earmarked number of rows, Gaussian elimination procedure is applied either to $\bar{\mathscr{R}}_{j-\alpha-\beta,\beta}$ or to $\bar{\mathscr{R}}_{j-\alpha-\beta,\beta}\bar{W}$ (see (4.13), $\bar{W}$ is W without last r columns) to reduce to zero as many elements as possible on the dependent rows in it. Elements, in the earmarked independent rows, which are real numbers or are in terms of arbitrary parameters in the solution $\bar{R}_i$'s of (4.8) will take part in this reduction process but preference would be given to real number elements. The nonzero elements on the dependent rows are now set equal to zero, the existence of solutions for the arbitrary parameters from these equations being quaranteed by Theorem 4.19. If there are more arbitrary parameters than the number of equations, these *excess* arbitrary *parameters* will parametrize all the minimal order inverses. The row operations in the Gaussian elimination process will generate an elementary matrix Z such that $Z\bar{\mathscr{R}}_{j-\alpha-\beta,\beta}\bar{W}$ will contain only $d_1 + d_2 + \ldots + d_m$ number of independent rows. Following a dual version of the realization algorithm given in [6], $\bar{A}$ in the inverse system Σ can be directly obtained from the elements in last m rows of Z. The matrix $\bar{A}$ is therefore given in terms of some arbitrary parameters from which the permissible number of excess parameters will then determine how many eigenvalues of $\bar{A}$ which are consequently poles of Σ can be arbitrarily assigned and will therefore generate a minimal order stable inverse if one such exists.

4.20. Example. Choosing $r^i_{j1} = x_{2i+j}$, $i = 0, 1, 2, 3$ and $j = 1, 2$ the matrix in (4.4a) can be transformed by Gaussian process to

(4.21)

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 + 3x_1/2 & 1 + 3x_1/2 & 1 & 0 \\ (1 + 3x_2)/2 & (-1 + 3x_2)/2 & 0 & 1 \end{bmatrix}}_{A_1 \mid A_2} \begin{bmatrix} x_3 & -x_1 & 0 & x_5 & 0 & 2 \\ x_4 & -x_2 + 1 & -2 & x_6 & 0 & -2 \\ x_5 & -x_3 & 2 + 3x_1 & x_7 & 0 & -2 \\ x_6 & -x_4 & -1 + 3x_2 & x_8 & 0 & -2 \end{bmatrix} =$$

$$= \begin{bmatrix} x_3 & -x_1 & 0 & x_5 & 0 & 2 \\ x_3 + x_4 & -x_1 - x_2 + 1 & -2 & x_5 + x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

when solutions for

$$\begin{aligned} &x_1 + x_3 + \alpha\delta = 0\,, \quad x_1 + x_4 + \alpha\varepsilon = 0\,, \\ &x_3 + x_5 + \beta\delta = 0\,, \quad x_3 + x_6 + \beta\varepsilon = 0\,, \\ &x_5 + x_7 + \gamma\delta = 0\,, \quad x_5 + x_8 + \gamma\varepsilon = 0\,, \end{aligned} \tag{4.22}$$

must exist where

$$\alpha = x_1 + x_2 - 1\,, \quad \beta = x_3 + x_4\,, \quad \gamma = x_5 + x_6\,, \quad \delta = 1 + \tfrac{3}{2}x_1\,,$$
$$\varepsilon = (-1 + 3x_2)/2\,.$$

The set of six equations (4.22) with eight unknowns gives rise to two excess parameters which can be identified in the following manner. Employing A_1 and A_2 in (4.21) the matrix $\bar{A}$ of the inverse Σ, with the help of a dual version of the algorithm in [6], is

$$\bar{A} = \begin{bmatrix} -(4 + 3x_1)/2 & -(2 + 3x_1)/2 \\ -(1 + 3x_2)/2 & (1 - 3x_2)/2 \end{bmatrix},$$

whose characteristic polynomial is

$$s^2 + \tfrac{3}{2}(x_1 + x_2 + 1)\, s + \tfrac{3}{2}(x_2 - x_1 - 1) = 0\,.$$

Seeking the poles of the inverse Σ to be the solutions of $s^2 + 2s + 1 = 0$ we determine $x_1 = -\frac{2}{3}$ and $x_2 = 1$. These values when substituted in (4.22) gives rise to (4.4b) from which a stable inverse follows as already outlined in Example 4.1.

5. CONCLUSION

We have shown how an invertibility theory of linear time invariant multivariable systems can be set up based on the matrix fraction description of the system. The inverse system is initially represented by a sequence of Markov parameters from which its state-space description or transfer-function representation can be obtained by using the theory of realization. If one insists on an inverse system of minimal order, it can be achieved by adjusting the arbitrary parameters in the Hankel matrix associated with the inverse system. The same technique can be applied to obtain a stable inverse if one such exists. An interesting question is how minimality and stability of the inverse system can be predicted with some test involving P_i's and Q_i's in (2.4) and (2.5) without solving (2.10) which needs further investigation. The results of this paper can obviously be extended to study the right invertibility of multivariable system invoking the principle of duality.

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Dr. K. B. Datta, Department of Electrical Engineering, IIT, Kharagpur — 721302. India.