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ON BROADCAST CHANNELS WITH SIDE INFORMATION UNDER FIDELITY CRITERIA*

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In this paper we define the rate distortion functions for the memoryless broadcast channels when side information about the source is provided at both the encoder and the decoder. Basic equations and the Variational equations are obtained under two different situations, the most general situation being the case of fidelity criterion acting along the main channel only whereas in the second situation fidelity criteria act along both the main as well as the side channel. The forms of the Variational equations for Gaussian Channel under squared error fidelity criterion and the convexity of the rate distortion functions in both the cases have also been discussed.

1. INTRODUCTION

The idea of a 'broadcast' channel was first introduced by Cover [3] who defined it as a channel involving simultaneous communication of information from one source to several receivers. The basic problem in such channels is how to send information from a single source to several receivers simultaneously. Cover [3] obtained upper and lower bounds on the capacity region of a broadcast channel. An inner bound to the general broadcast channels for the three communication situations was derived by van der Meulen [5] and Sato [8] obtained an outer bound to the capacity region of broadcast channels. Recently Sharma and Priya [9] extended the concept of fidelity criterion to Multiple Channels viz. Two-User Channels, Broadcast channels and Multiple Access Channels and derived the Basic equations for these channels.

The problem of communication through a channel when side information about the source is provided has been studied by many authors. For work in this direction, one may refer to Wyner and Ziv [12], Sharma and Priya [11] and Priya [6] etc. In this paper we consider the problem of information transmission through a broad-

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cast channel and define the rate distortion function when side information about the source is provided at the encoder and the decoder. In Section 2 Basic equations for the broadcast channels are obtained and the convexity of the rate distortion function is established when fidelity criterion acts on the main channel only. In Section 3 we define a new rate distortion function when the fidelity criteria act along the main channel as well as along the side channel and derive Basic equations for this case also. The study is then extended to the continuous case in Section 4. The Variational equations for Gaussian Channels under squared error fidelity criterion are discussed in Section 5.

2. BASIC EQUATIONS WHEN FIDELITY CRITERION ACTS ON THE MAIN CHANNELS ONLY

A. Preliminaries and Definitions

We consider a 2-receiver discrete memoryless broadcast channel

$$K = [X \times Z,\ Q(y_1, y_2 \mid x, z),\ Y_1 \times Y_2]$$

where X, Z, Y_1 and Y_2 are finite sets and $Q(y_1, y_2 \mid x, z)$ are transition probabilities defined over $Y_1 \times Y_2$ such that

$$Q(B_1, B_2 \mid A, B) = \prod_{t=1}^{n} Q(y_{1t}, y_{2t} \mid x_t, z_t)$$

for all $A = (x_1, \ldots, x_n) \in X^n$, $B = (z_1, \ldots, z_n) \in Z^n$ and $B_r = (y_{r1}, \ldots, y_{rn}) \in Y_r^n$; $r = 1, 2$; $n \geqq 1$. Here X is the input alphabet, Z is the side information alphabet about X and Y_1 and Y_2 are the output alphabets, where for $n \geqq 1$, X^n, Z^n and Y_r^n denote the set of n tuples A, B and B_r $(r = 1, 2)$ respectively. The quantity $Q(y_1, y_2 \mid x, z)$ denotes the transition probability of receiving $y_1 \in Y_1$ and $y_2 \in Y_2$ when $x \in X$ and $z \in Z$ are transmitted through the channel.

The distortion between the source letter $x \in X$ and the reproduced letter $y_i \in Y_i$ shall be denoted by $\varrho_i(x, y_i)$, where as usual $\varrho_i(x, y_i) \geqq 0$ with equality iff $x = y_i$ for $i = 1, 2$.

If $P(x, z)$ denote the joint probability of $x \in X$, $z \in Z$ then clearly the average distortion for the ith output may be defined as

$$d_i(Q) = \sum_{x,z,y_1,y_2} P(x, z)\, Q(y_1, y_2 \mid x, z)\, \varrho_i(x, y_i) \quad (i = 1, 2) \tag{1}$$

Further for $D_1 \geqq 0$, $D_2 \geqq 0$, we define $M(D_1, D_2)$ as the set of transition probability distributions $Q(y_1, y_2 \mid x, z)$, $x \in X$, $z \in Z$, $y_1 \in Y_1$ and $y_2 \in Y_2$ satisfying

$$P(x, z, y_1, y_2) = P(x, z)\, Q(y_1, y_2 \mid x, z) \tag{2}$$

$$P(x, z) = \sum_{y_1, y_2} P(x, z, y_1, y_2) \tag{3}$$

and

$$d_i(Q) \leqq D_i \quad (i = 1, 2) \tag{4}$$

We first confine ourselves to the case of fidelity criterion acting along the main channel only with side information about the source provided at both the encoder and the decoder. We define the rate distortion function $R_{X|Z}(D_1, D_2)$ as

$$R_{X|Z}(D_1, D_2) = \min_{Q(y_1, y_2 | x, z) \in M(D_1, D_2)} I(X; Y_1, Y_2 \mid Z), \tag{5}$$

where $I(X; Y_1, Y_2 \mid Z)$ is given by

$$I(X; Y_1, Y_2 \mid Z) = \sum_{x,z,y_1,y_2} P(x, z)\, Q(y_1, y_2 \mid x, z) \log \frac{Q(y_1, y_2 \mid x, z)}{Q(y_1, y_2 \mid z)} \tag{6}$$

is the mutual information which Y_1 and Y_2 provide about X when side information Z about the source X is provided, $Q(y_1, y_2 \mid z)$ being the transition probability of receiving $y_1 \in Y_1$, $y_2 \in Y_2$ when $z \in Z$ is transmitted.

B. Evaluation of $R_{X|Z}(D_1, D_2)$

Our problem is to minimise $I(X; Y_1, Y_2 \mid Z)$ subject to the constraints:

$$Q(y_1, y_2 \mid x, z) \geqq 0 \tag{7}$$

$$\sum_{y_1, y_2} Q(y_1, y_2 \mid x, z) = 1 \tag{8}$$

and

$$\sum_{x,z,y_1,y_2} P(x, z)\, Q(y_1, y_2 \mid x, z)\, \varrho_i(x, y_i) = D_i \quad (i = 1, 2) \tag{9}$$

We shall employ the technique of Lagrange multipliers for obtaining a solution to this problem.

Ignoring the constraints (7) temporarily, we form the augumented function

$$J(Q) = I(X; Y_1, Y_2 \mid Z) - \sum_{x,z} u(x, z) \sum_{y_1, y_2} Q(y_1, y_2 \mid x, z) -$$
$$- \sum_{i=1}^{2} S_i \sum_{x,z,y_1,y_2} P(x, z)\, Q(y_1, y_2 \mid x, z)\, \varrho_i(x, y_i)$$

where $u(x, z)$ and S_i $(i = 1, 2)$ are Lagrange multipliers. Taking $\log \lambda(x, z) = u(x, z)/P(x, z)$, we can rewrite $J(Q)$ as

(10)

$$J(Q) = \sum_{x,z,y_1,y_2} P(x, z)\, Q(y_1, y_2 \mid x, z) \log \frac{Q(y_1, y_2 \mid x, z)}{\lambda(x, z)\, Q(y_1, y_2 \mid z)} - \sum_{i=1}^{2} S_i\, \varrho_i(x, y_i)$$

For stationary points, we must have

$$\frac{\mathrm{d}\, J(Q)}{\mathrm{d}\, Q(y_1, y_2 \mid x, z)} = 0$$

which gives

$$P(x, z)\left[\log \frac{Q(y_1, y_2 \mid x, z)}{\lambda(x, z) Q(y_1, y_2 \mid z)} - \sum_{i=1}^{2} S_i \varrho_i(x, y_i)\right] = 0$$

and consequently

$$(11) \qquad Q(y_1, y_2 \mid x, z) \quad \lambda(x, z)\, Q(y_1, y_2 \mid z) \exp\left(\sum_{i=1}^{2} S_i \varrho_i(x, y_i)\right)$$

Now summing (11) over y_1, y_2 and using the constraints (8); we get

$$(12) \qquad \lambda(x, z) = \left[\sum_{y_1, y_2} Q(y_1, y_2 \mid z) \exp\left(\sum_{i=1}^{2} S_i \varrho_i(x, y_i)\right)\right]^{1}$$

Further equations (9) and (11) give

$$(13) \qquad D_i = \sum_{x, z, y_1, y_2} \varrho_i(x, y_i)\, P(x, z)\, \lambda(x, z)\, Q(y_1, y_2 \mid z) \exp\left(\sum_{i=1}^{2} S_i \varrho_i(x, y_i)\right)$$

Moreover from (11)· we also have

$$(14) \qquad \log \frac{Q(y_1, y_2 \mid x, z)}{Q(y_1, y_2 \mid z)} = \log \lambda(x, z) + \sum_{i=1}^{2} S_i \varrho_i(x, y_i)$$

so that (6) and (14) yield

$$I(X; Y_1, Y_2 \mid Z) = \sum_{i=1}^{2} S_i D_i + \sum_{x,z} P(x, z) \log \lambda(x, z)$$

Thus the minimum of $I(X; Y_1, Y_2 \mid Z)$ is given by

$$(15) \qquad R_{X|Z}(D_1, D_2) = \sum_{i=1}^{2} S_i D_i + \sum_{x,z} P(x, z) \log \lambda(x, z)$$

Equations (13) and (15) are the Basic equations when side information about the source is provided at the encoder and the decoder with fidelity criterion acting along the main channel only.

When side information about the source is not provided at the encoder and the decoder, then Basic equations (13) and (15) reduce to

$$D_i = \sum_{x, y_1, y_2} \varrho_i(x, y_i)\, P(x)\, \lambda(x)\, Q(y_1, y_2) \exp\left(\sum_{i=1}^{2} S_i \varrho_i(x, y_i)\right) \quad (i = 1, 2)$$

and

$$R_X = \sum_{i=1}^{2} S_i D_i + \sum_{x} P(x) \log \lambda(x)$$

where

$$\lambda(x) = \left[\sum_{y_1, y_2} Q(y_1, y_2) \exp\left(\sum_{i=1}^{2} S_i \varrho_i(x, y_i)\right)\right]^{-1}$$

which are the Basic equations for broadcast channels obtained in [9] when no side information is provided.

C. Convexity of $R_{X|Z}(D_1, D_2)$

We now consider the convexity property of $R_{X|Z}(D_1, D_2)$. For any pair of distortion values (D_1', D_2') and (D_1'', D_2'') and any number $\lambda \in [0, 1]$ we shall show that

$$R_{X|Z}(\lambda D_1' + (1-\lambda) D_1'', \lambda D_2' + (1-\lambda) D_2'') \leqq$$
$$\leqq \lambda R_{X|Z}(D_1', D_2') + (1-\lambda) R_{X|Z}(D_1'', D_2'') .$$

Let $Q'(y_1, y_2 \mid x, z)$ and $Q''(y_1, y_2 \mid x, z)$ achieve the points $(D_1', D_2'; R_{X|Z}(D_1', D_2'))$ and $(D_1'', D_2''; R_{X|Z}(D_1'', D_2''))$ respectively and let

(16) $$Q^*(y_1, y_2 \mid x, z) = \lambda\, Q'(y_1, y_2 \mid x, z) + (1-\lambda)\, Q''(y_1, y_2 \mid x, z)$$

It is easy to see that $\{Q^*(y_1, y_2 \mid x, z)\}$ is a bonafide transition probability distribution. Also we have

(17) $$R_{X|Z}(\lambda D_1' + (1-\lambda) D_1'', \lambda D_2' + (1-\lambda) D_2'') \leqq I(Q^*(y_1, y_2 \mid x, z))$$

where

(18) $$I(Q^*(y_1, y_2 \mid x, z)) =$$
$$= \sum_{x,z,y_1,y_2} P(x, z) [\lambda\, Q'(y_1, y_2 \mid x, z) + (1-\lambda)\, Q''(y_1, y_2 \mid x, z)]$$
$$\log \frac{\lambda\, Q'(y_1, y_2 \mid x, z) + (1-\lambda)\, Q''(y_1, y_2 \mid x, z)}{\lambda\, Q'(y_1, y_2 \mid z) + (1-\lambda) Q''(y_1, y_2 \mid z)}$$

For $a > 0$, $b \geqq 0$; we have the inequality

(19) $$\log (a + b) \leqq \log a + \frac{b}{a}$$

with equality iff $b = 0$.

We shall use the inequality (19) for the set of values a_1, b_2 and a_2, b_1 given by

(20) $$a_1 = \frac{Q'(y_1, y_2 \mid x, z)}{Q'(y_1, y_2 \mid z)}; \quad a_2 = \frac{Q''(y_1, y_2 \mid x, z)}{Q''(y_1, y_2 \mid z)}$$

$$b_1 = \frac{(1-\lambda) [Q'(y_1, y_2 \mid z)\, Q''(y_1, y_2 \mid x, z) - Q''(y_1, y_2 \mid z)\, Q'(y_1, y_2 \mid x, z)]}{Q'(y_1, y_2 \mid z) [\lambda\, Q'(y_1, y_2 \mid z) + (1-\lambda)\, Q''(y_1, y_2 \mid z)]}$$

and

$$b_2 = \frac{\lambda [Q''(y_1, y_2 \mid z)\, Q'(y_1, y_2 \mid x, z) - Q'(y_1, y_2 \mid z)\, Q''(y_1, y_2 \mid x, z)]}{Q''(y_1, y_2 \mid z) [\lambda\, Q'(y_1, y_2 \mid z) + (1-\lambda)\, Q''(y_1, y_2 \mid z)]}$$

Now in view of (19) and (20) equation (18) gives

$$I(Q^*(y_1, y_2 \mid x, z)) \leqq \lambda \sum_{x,z,y_1,y_2} P(x, z)\, Q'(y_1, y_2 \mid x, z) \log \frac{Q'(y_1, y_2 \mid x, z)}{Q'(y_1, y_2 \mid z)} +$$

$$+ \frac{(1-\lambda)\,[Q''(y_1, y_2 \mid x, z)\, Q'(y_1, y_2 \mid z) - Q''(y_1, y_2 \mid z)\, Q'(y_1, y_2 \mid x, z)]}{Q'(y_1, y_2 \mid x, z)\,[\lambda\, Q'(y_1, y_2 \mid z) + (1-\lambda)\, Q''(y_1, y_2 \mid z)]} +$$

$$+ (1-\lambda) \sum_{x,z,y_1,y_2} P(x, z)\, Q''(y_1, y_2 \mid x, z) \log \frac{Q''(y_1, y_2 \mid x, z)}{Q''(y_1, y_2 \mid z)} +$$

$$+ \frac{\lambda[Q''(y_1, y_2 \mid z)\, Q'(y_1, y_2 \mid x, z) - Q'(y_1, y_2 \mid z)\, Q''(y_1, y_2 \mid x, z)]}{Q''(y_1, y_2 \mid x, z)\,[\lambda\, Q'(y_1, y_2 \mid z) + (1-\lambda)\, Q''(y_1, y_2 \mid z)]} =$$

$$= \lambda I(Q'(y_1, y_2 \mid x, z)) + (1-\lambda)\, I(Q''(y_1, y_2 \mid x, z))$$

i.e.

$$(21) \quad I(Q^*(y_1, y_2 \mid x, z)) \leqq \lambda I(Q'(y_1, y_2 \mid x, z)) + (1-\lambda)\, I(Q''(y_1, y_2 \mid x, z)) =$$
$$= \lambda R_{X|Z}(D_1', D_2') + (1-\lambda)\, R_{X|Z}(D_1'', D_2'')$$

Combining equations (17) and (21) we obtain that $R_{X|Z}(D_1, D_2)$ is a convex function of D_1 and D_2.

In the next section we consider the situation when the side information about the source is provided at the encoder and the decoder and in addition to the fidelity criteria D_1 and D_2 acting along the side channel also.

3. BASIC EQUATIONS WHEN FIDELITY CRITERIA ARE OVER MAIN AS WELL AS SIDE CHANNEL

A. Definition of $R_{X|Z}(D_1, D_2; d_1, d_2)$

For $D_1 \geqq 0$, $D_2 \geqq 0$, $d_1 \geqq 0$, $d_2 \geqq 0$; let $M(D_1, D_2; d_1, d_2)$ be the set of transition probability distributions $Q(y_1, y_2 \mid x, z)$ satisfying

$$\sum_{x,z,y_1,y_2} \varrho_i(x, y_i)\, P(x, z)\, Q(y_1, y_2 \mid x, z) \leqq D_i$$

$$\sum_{x,z,y_1,y_2} \varrho_i'(z, y_i)\, P(x, z)\, Q(y_1, y_2 \mid x, z) \leqq d_i \quad (i = 1, 2)$$

where $\varrho_i'(z, y_i)$ is the distortion between $z \in Z$ and $y_i \in Y_i$ $(i = 1, 2)$.

We define the rate distortion function $R_{X|Z}(D_1, D_2; d_1, d_2)$ with fidelity criteria over the main channel and side channel as

$$(22) \qquad R_{X|Z}(D_1, D_2; d_1, d_2) = \min_{Q(y_1,y_2|x,z) \in M(D_1,D_2;d_1,d_2)} I(X; Y_1, Y_2 \mid Z)$$

where $I(X; Y_1, Y_2 \mid Z)$ is the ordinary Shannon's mutual information between the source X and reproduced alphabets Y_1 and Y_2 when side information Z about the source X is prescribed.

B. Evaluation of $R_{X|Z}(D_1, D_2; d_1, d_2)$

Our problem is to minimise $I(X; Y_1, Y_2 \mid Z)$ subject to the constraints (7), (8) and

$$\sum_{x,z,y_1,y_2} \varrho_i(x, y_i)\, P(x, z)\, Q(y_1, y_2 \mid x, z) = D_i \tag{23}$$

$$\sum_{x,z,y_1,y_2} \varrho_i'(z, y_i)\, P(x, z)\, Q(y_1, y_2 \mid x, z) = d_i \quad (i = 1, 2) \tag{24}$$

As earlier ignoring the constraints (7); we form the augmented function $J(Q)$ as

$$\begin{aligned} J(Q) = I(X; Y_1, Y_2 \mid Z) &- \sum_{x,z} \beta(x, z) \sum_{y_1,y_2} Q(y_1, y_2 \mid x, z) - \\ &- \sum_{i=1}^{2} S_i \sum_{x,z,y_1,y_2} \varrho_i(x, y_i)\, P(x, z)\, Q(y_1, y_2 \mid x, z) - \\ &- \sum_{i=1}^{2} S_i' \sum_{x,z,y_1,y_2} \varrho_i'(z, y_i)\, P(x, z)\, Q(y_1, y_2 \mid x, z) \end{aligned}$$

where $\beta(x, z)$, S_i and S_i' are Lagrange multipliers. Taking $\log \eta(x, z) = \beta(x, z)/P(x, z)$; we may rewrite $J(Q)$ as

$$\begin{aligned} J(Q) = \sum_{x,z,y_1,y_2} P(x, z)\, Q(y_1, y_2 \mid x, z) \Bigg[\log \frac{Q(y_1, y_2 \mid x, z)}{\eta(x, z)\, Q(y_1, y_2 \mid z)} &- \\ - \sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) - \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i) \Bigg] \end{aligned}$$

Now for stationary points, we must have

$$\frac{\mathrm{d}J(Q)}{\mathrm{d}Q(y_1, y_2 \mid x, z)} =$$

$$= P(x, z) \left[\log \frac{Q(y_1, y_2 \mid x, z)}{\eta(x, z)\, Q(y_1, y_2 \mid z)} - \sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) - \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i) \right] = 0$$

or

$$Q(y_1, y_2 \mid x, z) = \eta(x, z)\, Q(y_1, y_2 \mid z) \exp \left[\sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) + \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i) \right] \tag{25}$$

In view of equations (23), (24), (6) and (25) we now obtain

$$\begin{aligned} D_i = \sum_{x,z,y_1,y_2} \varrho_i(x, y_i)\, P(x, z)\, \eta(x, z)\, Q(y_1, y_2 \mid z) \exp \Big[\sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) &+ \\ + \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i) \Big] \end{aligned} \tag{26}$$

$$(27) \qquad d_i = \sum_{x,z,y_1,y_2} \varrho_i'(z, y_i)\, P(x, z)\, \eta(x, z)\, Q(y_1, y_2 \mid z) \exp\left[\sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) + \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i)\right]$$

and

$$I(X; Y_1, Y_2 \mid Z) = \sum_{x,z} P(x, z) \log \eta(x, z) + \sum_{i=1}^{2} S_i D_i + \sum_{i=1}^{2} S_i' d_i$$

where

$$(28) \qquad \eta(x, z) = \left[\sum_{y_1,y_2} Q(y_1, y_2 \mid z) \exp\left[\sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) + \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i)\right]\right]^{-1}$$

Clearly, $R_{X|Z}(D_1, D_2; d_1, d_2)$ which is the infimum of $I(X; Y_1, Y_2 \mid Z)$ satisfies

$$(29) \qquad R_{X|Z}(D_1, D_2; d_1, d_2) = \sum_{x,z} P(x, z) \log \eta(x, z) + \sum_{i=1}^{2} S_i D_i + \sum_{i=1}^{2} S_i' d_i$$

These equations (26), (27) and (29) are the required basic equations for the case under consideration.

Remark. It can be easily seen that the results of Section 2 follow as a special case of the above equations.

C. Convexity of the function $R_{X|Z}(D_1, D_2; d_1, d_2)$

We now show that $R_{X|Z}(D_1, D_2; d_1, d_2)$ is a convex U function of $D_1, D_2; d_1$ and d_2. Let the transition probabilities $Q'(y_1, y_2 \mid x, z)$ and $Q''(y_1, y_2 \mid x, z)$ achieve the points $(D_1', D_2', d_1', d_2'; R_{X|Z}(D_1', D_2'; d_1', d_2'))$ and $(D_1'', D_2'', d_1'', d_2''; R_{X|Z}(D_1'', D_2''; d_1'', d_2''))$ respectively and let for any scalar $\lambda \in [0, 1]$,

$$Q^*(y_1, y_2 \mid x, z) = \lambda\, Q'(y_1, y_2 \mid x, z) + (1 - \lambda)\, Q''(y_1, y_2 \mid x, z)$$

It easily follows that $Q^*(y_1, y_2 \mid x, z)$ is a bonafide transition probability distribution and that $D_i(Q^*)$ and $d_i(Q^*)$ are linear functions of D_1, D_2 and d_1, d_2 respectively so that

$$Q^*(y_1, y_2 \mid x, z) \in M(\lambda D_1' + (1 - \lambda)\, D_1'', \lambda D_2' + (1 - \lambda)\, D_2''; \lambda d_1' + (1 - \lambda)\, d_1'', \lambda d_2' + (1 - \lambda)\, d_2'')$$

Next we have

$$R_{X|Z}(\lambda D_1' + (1 - \lambda)\, D_1'', \lambda D_2' + (1 - \lambda)\, D_2''; \lambda d_1' + (1 - \lambda)\, d_1'', \lambda d_2' + (1 - \lambda)\, d_2'') \leqq I(Q^*(y_1, y_2 \mid x, z))$$

where $I(Q^*(y_1, y_2 \mid x, z))$ is given by equation (18).

Again, using the inequality (19) for the values $a_1, b_1; a_2, b_2$ as given by equation (20), we have

$$I(Q^*(y_1, y_2; x, z)) \leqq \lambda I(Q'(y_1, y_2 \mid x, z)) + (1 - \lambda)\, I(Q''(y_1, y_2 \mid x, z))$$

Thus

$$R_{X|Z}(\lambda D_1' + (1-\lambda) D_1'', \lambda D_2' + (1-\lambda) D_2''; \lambda d_1' + (1-\lambda) d_1'', \lambda d_2' + (1-\lambda) d_2'') \leqq$$
$$\leqq \lambda R_{X|Z}(D_1', D_2'; d_1', d_2') + (1-\lambda) R_{X|Z}(D_1'', D_2''; d_1'', d_2'')$$

Hence $R_{X|Z}(D_1, D_2; d_1, d_2)$ is a convex U function of D_1, D_2, d_1 and d_2.

4. CONTINUOUS CASE

We now extend the above investigation for the continuous case.

A. Definitions

Let us denote the source X by the infinite sequence $\{\ldots, X_{t-1}, X_t, X_{t+1}, \ldots\}$ $(-\infty < t < \infty)$; the side information Z about the source X by $\{\ldots, Z_{t-1}, Z_t, Z_{t+1}, \ldots\}$ $(-\infty < t < \infty)$; the message received by the receiver Y_1 by $\{\ldots, y_{1t-1}, y_{1t}, y_{1t+1}, \ldots\}$ $(-\infty < t < \infty)$ and the message received by the receiver Y_2 by $\{\ldots, y_{2t-1}, y_{2t}, y_{2t+1}, \ldots\}$ $(-\infty < t < \infty)$. We shall denote the probability density function (p.d.f.) of the source letter $x \in X$ by $P(x)$, the p.d.f. of the side information letter $z \in Z$ by $P(z)$, the joint p.d.f. of $x \in X$ and $z \in Z$ by $P(x, z)$, the transition p.d.f. of receiving $y_1 \in Y_1$, $y_2 \in Y_2$ when $x \in X$ and $z \in Z$ are transmitted through the channel by $Q(y_1, y_2 \mid x, z)$ and the joint p.d.f. of $x \in X$, $y_1 \in Y_1$, $y_2 \in Y_2$ and $z \in Z$ by $P(x, z, y_1, y_2)$.

Now as usual, the distortion between the source letter $x \in X$ and the reproduced letter $y_i \in Y_i$ $(i = 1, 2)$ will be denoted by $\varrho_i(x, y_i)$ where $\varrho_i(x, y_i) \geqq 0$ with equality iff $x = y_i$ for $i = 1, 2$.

Also the average distortion $d_i(Q)$ between the source alphabet X and the reproduced alphabet Y_i is defined by

$$d_i(Q) = \iiiint \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, P(x, z) \, Q(y_1, y_2 \mid x, z) \, \varrho_i(x, y_i) \quad (i = 1, 2) \tag{31}$$

The transition p.d.f. $Q(y_1, y_2 \mid x, z)$ is said to be D_i admissible if

$$d_i(Q) \leqq D_i \quad (i = 1, 2) \tag{32}$$

We shall denote the set of all D_i admissible transition probabilities $Q(y_1, y_2 \mid x, z)$ by

$$M(D_1, D_2) = \{Q_i(y_1, y_2 \mid x, z) : d_i(Q) \leqq D_i; i = 1, 2\} \tag{33}$$

Further, the mutual information between the source alphabet X and the reproduced alphabets Y_1 and Y_2 when side information Z about the source alphabet is prescribed is given by

$$I(X; Y_1, Y_2 \mid Z) = \tag{34}$$
$$= \iiiint \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, P(x, z) \, Q(y_1, y_2 \mid x, z) \log \frac{Q(y_1, y_2 \mid x, z)}{Q(y_1, y_2 \mid z)}$$

We now define the rate distortion function $R_{X|Z}(D_1, D_2)$ of the source X when side information Z about the source is provided at the encoder and the decoder and there is fidelity criteria acting only over the main channel as

$$R_{X|Z}(D_1, D_2) = \operatorname*{Inf}_{Q(y_1, y_2 | x, z) \in M(D_1, D_2)} I(X; Y_1, Y_2 \mid Z)$$

B. Evaluation of $R_{X|Z}(D_1, D_2)$

We now proceed to evaluate the function $R_{X|Z}(D_1, D_2)$. Our problem is to minimise $I(X; Y_1, Y_2 \mid Z)$ subject to the constraints:

$$Q(y_1, y_2 \mid x, z) \geqq 0\,, \tag{36}$$

$$\iint \mathrm{d}y_1\, \mathrm{d}y_2\, Q(y_1, y_2 \mid x, z) = 1\,, \tag{37}$$

and

$$\iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, P(x, z)\, Q(y_1, y_2 \mid x, z)\, \varrho_i(x, y_i) = D_i \quad (i = 1, 2) \tag{38}$$

In order to solve it, we shall employ the classical methods of multipliers and the calculus of variations. As usual, ignoring the constraints (36) temporarily, we form the augumented function

$$J(Q) = I(X; Y_1, Y_2 \mid Z) - \iint \mathrm{d}x\, \mathrm{d}z\, u(x, z) \iint \mathrm{d}y_1\, \mathrm{d}y_2\, Q(y_1, y_2 \mid x, z) -$$
$$- \sum_{i=1}^{2} S_i \iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, P(x, z)\, Q(y_1, y_2 \mid x, z)\, \varrho_i(x, y_i) =$$
$$= \iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, P(x, z)\, Q(y_1, y_2 \mid x, z) \left[\log \frac{Q(y_1, y_2 \mid x, z)}{\lambda(x, z)\, Q(y_1, y_2 \mid z)} - \right.$$
$$\left. - \sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) \right]$$

where $u(x, z)$ and $S_i's$ $(i = 1, 2)$ are Lagrange multipliers and $\log \lambda(x, z) = u(x, z)/P(x, z)$.
Let us consider a perturbation $Q^*(y_1, y_2 \mid x, z)$ about $Q(y_1, y_2 \mid x, z)$ given by

$$Q^*(y_1, y_2 \mid x, z) = Q(y_1, y_2 \mid x, z) + \varepsilon\, \eta(y_1, y_2 \mid x, z) \tag{39}$$

where $\eta(y_1, y_2 \mid x, z)$ is such that

$$\iint \mathrm{d}y_1\, \mathrm{d}y_2\, \eta(y_1, y_2 \mid x, z) = 0 \tag{40}$$

For stationary points, we must have

$$\left. \frac{\mathrm{d}J(Q^*)}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} = 0$$

and so

$$\iint \mathrm{d}x\,\mathrm{d}z\,P(x,z)\iint \mathrm{d}y_1\,\mathrm{d}y_2\,\eta(y_1,y_2\mid x,z)\left\{\log\frac{Q(y_1,y_2\mid x,z)}{\lambda(x,z)\,Q(y_1,y_2\mid z)} - \sum_{i=1}^{2} S_i\,\varrho_i(x,y_i)\right\} = 0\,.$$

It now follows from (40) and a Fundamental Theorem of the Calculus of Variations (cf. [2]) that

$$\log\frac{Q(y_1,y_2\mid x,z)}{\lambda(x,z)\,Q(y_1,y_2\mid z)} - \sum_{i=1}^{2} S_i\,\varrho_i(x,y_i) = f(x,z) \tag{41}$$

where $f(x, z)$ is a function of x and z only.

Rewriting the equation (41) as

$$Q(y_1,y_2\mid x,z) = Q(y_1,y_2\mid z)\,\lambda'(x,z)\exp\{\sum_{i=1}^{2} S_i\,\varrho_i(x,y_i)\} \tag{42}$$

where

$$\lambda'(x,z) = \lambda(x,z)\exp\{f(x,z)\}\,,$$

and using (42) the equation (38) gives

$$D_i = \iiiint \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}y_2\,\varrho_i(x,y_i)\,P(x,z)\,Q(y_1,y_2\mid z)\,\lambda'(x,z)\,.\,.\exp\{\sum_{i=1}^{2} S_i\,\varrho_i(x,y_i)\}\,. \tag{43}$$

Further (34) and (42) together yield

$$I(X;Y_1,Y_2\mid Z) = \iint \mathrm{d}x\,\mathrm{d}z\,P(x,z)\log\lambda'(x,z) + \sum_{i=1}^{2} S_i D_i$$

and consequently $R_{X|Z}(D_1, D_2)$ is given by

$$R_{X|Z}(D_1,D_2) = \iint \mathrm{d}x\,\mathrm{d}z\,P(x,z)\log\lambda'(x,z) + \sum_{i=1}^{2} S_i D_i \tag{44}$$

where

$$\lambda'(x,z) = \left[\iint \mathrm{d}y_1\,\mathrm{d}y_2\,Q(y_1,y_2\mid z)\exp\left[\sum_{i=1}^{2} S_i\,\varrho_i(x,y_i)\right]\right]^{-1} \tag{45}$$

Equations (43) and (44) are the required forms of the Variational equations when side information about the source is provided at the encoder and the decoder and there is fidelity criteria acting along the main channel only.

C. Definition and Evaluation of $R_{X|Z}(D_1, D_2; d_1, d_2)$

We now consider the situation when in addition to the fidelity criteria D_1 and D_2 acting on the main channel, there ate fidelity criteria d_1 and d_2 acting on the side channel also.

For $D_1 \geqq 0$, $D_2 \geqq 0$, $d_1 \geqq 0$, $d_2 \geqq 0$, we define $M(D_1, D_2; d_1, d_2)$ to be the set of transition probability density function $Q(y_1, y_2 \mid x, z)$ satisfying

$$\iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, \varrho_i(x, y_i)\, P(x, z, y_1, y_2) \leqq D_i$$

$$\iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, \varrho_i'(z, y_i)\, P(x, z, y_1, y_2) \leqq d_i$$

where $\varrho_i'(z, y_i)$ is the distortion between $z \in Z$ and $y_i \in Y_i$ $(i = 1, 2)$.

Also we define the rate distortion function $R_{X|Z}(D_1, D_2; d_1, d_2)$ with fidelity criteria acting on the main as well as the side channel, as

$$(46) \qquad R_{X|Z}(D_1, D_2; d_1, d_2) = \operatorname*{Inf}_{Q(y_1, y_2 | x, z) \in M(D_1, D_2; d_1, d_2)} I(X; Y_1, Y_2 \mid Z)$$

where $I(X; Y_1, Y_2 \mid Z)$ is the ordinary Shannon's mutual information given by (34).

Our problem is to minimise $I(X; Y_1, Y_2 \mid Z)$ subject to the constraints (36), (37), (38) and

$$(47) \qquad \iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, \varrho_i'(z, y_i)\, P(x, z)\, Q(y_1, y_2 \mid x, z) = d_i \quad (i = 1, 2)$$

Ignoring the constraints (36) temporarily, we form the augumented function

$$J(Q) = I(X; Y_1, Y_2 \mid Z) - \iint \mathrm{d}x\, \mathrm{d}z\, u(x, z) \iint \mathrm{d}y_1\, \mathrm{d}y_2\, Q(y_1, y_2 \mid x, z) -$$
$$- \sum_{i=1}^{2} S_i \iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, P(x, z)\, Q(y_1, y_2 \mid x, z)\, \varrho_i(x, y_i) -$$
$$- \sum_{i=1}^{2} S_i' \iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, P(x, z)\, Q(y_1, y_2 \mid x, z)\, \varrho_i'(z, y_i) =$$
$$= \iiiint \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y_1\, \mathrm{d}y_2\, P(x, z)\, Q(y_1, y_2 \mid x, z) \left[\log \frac{Q(y_1, y_2 \mid x, z)}{\lambda(x, z)\, Q(y_1, y_2 \mid z)} - \right.$$
$$\left. - \sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) - \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i)\right]$$

where $u(x, z)$ and S_i, S_i' $(i = 1, 2)$ are Lagrange multipliers and $\log \lambda(x, z) = u(x, z)/P(x, z)$.

Now proceeding on lines considered in substitutions above, we arrive at the following stationary point tiansition probabilities

$$(48) \qquad Q(y_1, y_2 \mid x, z) =$$
$$= Q(y_1, y_2 \mid z)\, \eta(x, z) \exp \left[\sum_{i=1}^{2} S_i\, \varrho_i(x, y_i) + \sum_{i=1}^{2} S_i'\, \varrho_i'(z, y_i)\right]$$

where $\eta(x, z) = \lambda(x, z) \exp \{g(x, z)\}$; $g(x, z)$ being the function of x and z only.

Thus we obtain

$$(49)\qquad D_i = \iiiint \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}y_2\,\varrho_i(x, y_i)\,P(x, z)\,Q(y_1, y_2 \mid z)\,\eta(x, z)\,.$$
$$.\exp\{\sum_{i=1}^{2} S_i\,\varrho_i(x, y_i) + \sum_{i=1}^{2} S_i'\,\varrho_i'(z, y_i)\}\,,$$

$$(50)\qquad d_i = \iiiint \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y_1\,\mathrm{d}y_2\,\varrho_i'(z, y_i)\,P(x, z)\,Q(y_1, y_2 \mid z)\,\eta(x, z)\,.$$
$$.\exp\{\sum_{i=1}^{2} S_i\,\varrho_i(x, y_i) + \sum_{i=1}^{2} S_i'\,\varrho_i'(z, y_i)\}\,,$$

and

$$(51)\qquad R_{X|Z} = \iint \mathrm{d}x\,\mathrm{d}z\,P(x, z)\log\eta(x, z) + \sum_{i=1}^{2} S_i D_i + \sum_{i=1}^{2} S_i' d_i$$

where

$$(52)\qquad \eta(x, z) =$$
$$= \left[\iint \mathrm{d}y_1\,\mathrm{d}y_2\,Q(y_1, y_2 \mid z)\exp\{\sum_{i=1}^{2} S_i\,\varrho_i(x, y_i) + \sum_{i=1}^{2} S_i'\,\varrho_i'(z, y_i)\}\right]^{-1}$$

Equations (49), (50) and (51) are the required forms of the Variational equations when side information about the source is provided at the encoder and the decoder and there are fidelity criteria acting along the main as well as the side channel.

5. GAUSSIAN CHANNEL

We now examine the forms of the Variational equations for Gaussian channel when fidelity criteria are acting along the main channel only.

Let X be the source with mean zero and variance σ_x^2, Z be the side information about the source provided at the encoder and the decoder with mean zero and variance σ_z^2 and Y_i be the message received with mean zero and variance $\sigma_{y_i}^2$ $(i = 1, 2)$. Let the joint p.d.f. of X and Z be given by the following gaussian distribution

$$(53)\quad P(x, z) = \frac{1}{2\pi\sigma_x\sigma_z\sqrt{(1 - \varrho_{xz}^2)}}\exp\left\{-\frac{1}{2(1 - \varrho_{xz}^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2xz\varrho_{xz}}{\sigma_x\sigma_z} + \frac{z^2}{\sigma_z^2}\right)\right\}$$

where ϱ_{xz} is the coefficient of correlation between x and z. Let us suppose that the outputs are statistically independent. The conditional gaussian distribution of output Y_i when side information Z about the source is given, may as well be considered to be given by

$$(54)\quad Q_i(y_i \mid z) = \frac{1}{\sigma_{y_i}\sqrt{(2\pi(1 - \varrho_{zy_i}^2))}}\exp\left\{-\frac{1}{2\pi\sigma_z^2\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)}(y_i\sigma_z - \varrho_{zy_i}z\sigma_{y_i}^2)\right\}$$

where ϱ_{zy_i} is the coefficient of correlation between y_i and z $(i = 1, 2)$.

We shall use the above forms of p.d.f. 's in determining the values of D_i and $R_{X|Z}$ as provided by (43) and (44) for the case when the distortion between x and y_i is given by

$$\varrho_i(x, y_i) = (x - y_i)^2 \quad (i = 1, 2) \tag{55}$$

i.e. when there is the squared error fidelity criterion acting on the main channel.

For distortion measure $\varrho_i(x, y_i)$ and the transition p.d.f. $Q_i(y_i \mid z)$ as given by (55) and (54) respectively, equation (45) gives

$$\lambda'(x, z) = \prod_{i=1}^{2} \sqrt{(1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2))} \exp\left(\frac{S_i'(x\sigma_z - z\sigma_{y_i}\varrho_{zy_i})^2}{\sigma_z^2(1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2))}\right) \tag{56}$$

provided that

$$\frac{1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)}{2\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)} > 0\,, \quad \text{(refer [7])}$$

where $S' = -S$ is a non$-$ve quantity (refer [1]).

Now using (53) and (56), we obtain

$$\iint dx\, dz\, P(x, z) \log \lambda'(x, z) = \tfrac{1}{2}\sum_{i=1}^{2} \log\left(1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)\right) + \\ + \sum_{i=1}^{2} \frac{S_i'[\sigma_x^2 + \varrho_{zy_i}^2\sigma_{y_i}^2 - 2\sigma_x\sigma_{y_i}\varrho_{xz}\varrho_{zy_i}]}{1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)} \tag{57}$$

Thus from (43) and (44) we get

$$D_i = \frac{\sigma_x^2 - 2\sigma_x\sigma_{y_i}\varrho_{xz}\varrho_{zy_i} + \sigma_{y_i}^2 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)^2}{[1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)]^2} \quad (i = 1, 2) \tag{58}$$

and

$$R_{X|Z} = \sum_{i=1}^{2} S_i D_i + \tfrac{1}{2}\sum_{i=1}^{2} \log\left(1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)\right) + \\ + \sum_{i=1}^{2} \frac{S_i'[\sigma_x^2 - \varrho_{zy_i}^2\sigma_{y_i}^2 - 2\sigma_x\sigma_{y_i}\varrho_{xz}\varrho_{zy_i}]}{1 + 2S_i'\sigma_{y_i}^2(1 - \varrho_{zy_i}^2)} \tag{59}$$

Equations (58) and (59) are the required Variational equations for Gaussian Channel under squared error fidelity criterion.

In our opinion the results of this paper can be of significant interest for further work in the direction of evaluating the bounds on rate distortion function for broadcast channel. Source coding theorems for these channels with side information provided at the encoder and the decoder may also be established on lines similar to those considered in [10].

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