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MARTINGALE METHODS IN DISCRETE STATE RANDOM PROCESSES

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ACADEMIA
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INTRODUCTION

The article is based on lectures given in the Seminar on the Theory of Random Processes at the Faculty of Mathematics and Physics, Charles University. In the martingale approach to continuous time random processes with a discrete state space the basic notion is that of a rate. In general probability theory the rates are encountered in the renewal theory as failure rates and in Markov processes where we have rates of transition between the states. In Chapter II the mutual densities of probability distributions are expressed by means of the rates. Using the densities and the Bayes formula, the problem of estimating the rate from the observation is dealt with in Chapter IV. The control of discrete state random processes consists in changing the rates. Limit theorems for martingales are employed in Chapter III to examine the asymptotic properties of controlled processes.

Beyond the basic probability theory only general knowledge of the martingales and of the stopping times is expected from the reader. For information about the literature on the subject, references [4], [5], [6] should be consulted.

I. OCCURRENCE RATE OF RANDOM EVENTS

1. Renewal processes

The classical model of the renewal theory is the following. Imagine a machine and a particular component of it which is subject to failure. There is an infinite stock of spare components whose life times (operation times) σ are mutually independent, identically distributed

$$P(\sigma \leqq t) = F(t)\,, \quad t \geqq 0\,.$$

After failure the component in the machine is instantaneously replaced by a new one. The failure times, coinciding with the replacement times, constitute a random sequence of points $\tau = \{\tau_n, n = 1, 2, \ldots\}$ in the time interval $(0, \infty)$. The distribution

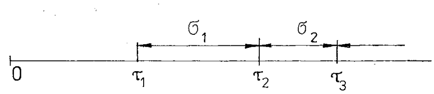

Fig. 1.

of the first failure time τ_1 depends on the age of the machine component in time 0. It can therefore be different from F. τ is equivalently defined by its counting process

$$N_t = \sum_{n=1}^{\infty} \chi\{\tau_n \leqq t\}\,, \quad t \geqq 0\,.$$

χ is the indicator of the random event in the curly bracket. In words, N_t is the number of replacements made until time t.

Assume that F has probability density f, and define the failure rate of a component of age $t \geqq 0$ to be

$$q(t) = f(t)/\bar{F}(t), \quad \text{where} \quad \bar{F}(t) = 1 - F(t) = \int_t^\infty f(s)\,\mathrm{d}s .$$

It holds then

$$P(\sigma \in (t, t + \Delta) \mid \sigma > t) = q(t)\Delta + o(\Delta) \quad \text{as} \quad \Delta \to 0+ ,$$

provided that q is right continuous.

Returning to the renewal process $\boldsymbol{\tau}$, let us denote by $Q = \{Q_t, t \geqq 0\}$ the evolution

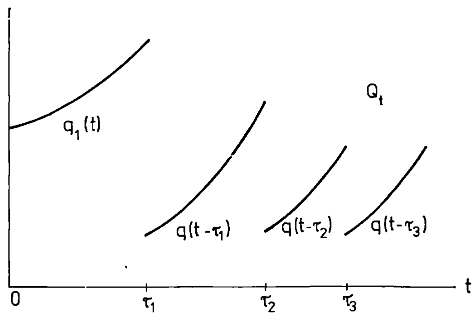

Fig. 2.

of the failure rate in time (Fig. 2). Define X_t to be the age of the component in operation at time t,

$$X_t = t\,, \quad 0 \leqq t \leqq \tau_1\,, \quad X_t = t - \tau_n\,, \quad \tau_n < t \leqq \tau_{n+1}\,, \quad n = 1, 2, \ldots$$

Let $X^+ = \{X_t^+, t \geqq 0\}$ be the right continuous version of $X = \{X_t, t \geqq 0\}$. Since the failure rate depends solely on the age of the component, we have

$$Q_t = q(X_t^+)\,, \quad t \geqq 0 .$$

Random function Q has the property that the probability of failure in time interval $(t, t + \Delta)$ is $Q_t\Delta + o(\Delta)$ as $\Delta \to 0+$. We shall call Q a failure rate as well. The mentioned probability is conditioned on the entire past of the process. At time t this is the collection of the events defined on N_s, $s \leqq t$, in symbols,

$$\mathscr{F}_t = \sigma a(N_s, 0 \leqq s \leqq t) .$$

Example 1. Let $f(t) = q\,\mathrm{e}^{-qt}$, $t \geqq 0$. Then N is a Poisson process, $q(t) = q$, $t \geqq 0$. Define

$$M_t = N_t - qt\,, \quad t \geqq 0\,.$$

$M = \{M_t, t \geqq 0\}$ is a martingale with respect to $\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$. Namely, for $0 \leqq s \leqq t$,

$$\mathsf{E}\{M_t \mid \mathscr{F}_s\} = \mathsf{E}\{(N_t - N_s) - q(t-s) \mid \mathscr{F}_s\} + M_s = M_s\,. \qquad \square$$

Let us generalize Example 1. Consider a renewal process with life time density $f(t)$. We shall show that

$$M_t = N_t - \int_0^t Q_u\,\mathrm{d}u\,, \quad t \geqq 0\,,$$

is a martingale. This amounts to verifying

$$\mathsf{E}\{N_t - N_s \mid \mathscr{F}_s\} = \mathsf{E}\left\{\int_s^t Q_u\,\mathrm{d}u \mid \mathscr{F}_s\right\}, \quad 0 \leqq s < t\,. \tag{1}$$

Let l be the age of the component in operation at time s. The replacement times after s form a renewal process with initial density

$$f_1(t) = f(t + l)/(1 - F(l))\,.$$

Thus to establish (1) it suffices to prove

$$\mathsf{E}N_t = \mathsf{E}\int_0^t Q_u\,\mathrm{d}u\,, \quad t \geqq 0\,, \tag{2}$$

for arbitrary initial density $f_1(t)$. Applying the Laplace-Stieltjes transform to (2) we get

$$\int_0^\infty \mathrm{e}^{-pt}\,\mathrm{d}\mathsf{E}N_t = \int_0^\infty \mathrm{e}^{-pt}\,\mathrm{d}\mathsf{E}\int_0^t Q_u\,\mathrm{d}u\,, \quad p > 0\,,$$

or

$$\mathsf{E}\int_0^\infty \mathrm{e}^{-pt}\,\mathrm{d}N_t = \mathsf{E}\int_0^\infty \mathrm{e}^{-pt} Q_t\,\mathrm{d}t\,. \tag{3}$$

We have on one hand

$$\mathsf{E}\int_0^\infty \mathrm{e}^{-pt}\,\mathrm{d}N_t = \mathsf{E}\sum_{n=1}^\infty \mathrm{e}^{-p\tau_n} = \sum_{n=1}^\infty \mathscr{L}(f_1)\,\mathscr{L}(f)^{n-1} = \frac{\mathscr{L}(f_1)}{1 - \mathscr{L}(f)}\,,$$

where $\mathscr{L}$ means the Laplace transform. On the other hand,

$$\mathsf{E}\int_0^\infty \mathrm{e}^{-pt}\,Q_t\,\mathrm{d}t = \mathsf{E}\left[\int_0^{\tau_1} \mathrm{e}^{-ps}\,q_1(s)\,\mathrm{d}s + \sum_{n=1}^\infty \mathrm{e}^{-p\tau_n}\int_{\tau_n}^{\tau_{n+1}} \mathrm{e}^{-p(s-\tau_n)}\,q(s-\tau_n)\,\mathrm{d}s\right] =$$

$$= \int_0^\infty \int_0^t e^{-ps} q_1(s)\,ds\, f_1(t)\,dt + \sum_{n=1}^{\infty} \mathscr{L}(f_1)\,\mathscr{L}(f)^{n-1} \int_0^\infty \int_0^t e^{-ps} q(s)\,ds\, f(t)\,dt =$$

$$= \mathscr{L}(f_1) + \sum_{n=1}^{\infty} \mathscr{L}(f_1)\,\mathscr{L}(f)^n = \frac{\mathscr{L}(f_1)}{1 - \mathscr{L}(f)}.$$

Hence, (3) holds.

Renewal process with preventive replacements. Assume that in the renewal process besides the replacements after the failure of the component preventive replacements are made according to a certain rule. Most simple is the age replacement. An $x > 0$ is given. Whenever the operation time of the component reaches x, it is replaced by a new one (Fig. 3).

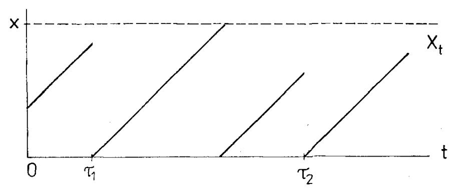

Fig. 3.

The replacement rule can be rather arbitrary. It has to employ only the information about the past of the process (nonanticipativity), and must not imply an infinite number of replacements in finite time. We denote by $N = \{N_t,\ t \geqq 0\}$ the counting process of the failures. Note that the times of preventive replacements can be reconstructed from N and from the replacement rule.

For any admissible replacement rule the failure rate satisfies

$$Q_t = q(X_t^+),\ t \geqq 0.$$

Further,

$$M_t = N_t - \int_0^t Q_u\,du, \quad t \geqq 0,$$

is a martingale with respect to $\mathscr{F}$. This can be seen heuristically as follows. Take $u \geqq 0$. There are no preventive replacements in $(u, u + \Delta)$, if Δ is sufficiently small. Hence,

$$P(N_{u+\Delta} - N_u \geqq 1 \mid \mathscr{F}_u) = q(X_u^+)\,\Delta + o(\Delta),$$

$$P(N_{u+\Delta} - N_u \geqq 2 \mid \mathscr{F}_u) = \int_0^\Delta F(\Delta - y)\, f(X_u^+ + y)\, \bar{F}(X_u^+)^{-1}\,dy = o(\Delta)$$

as $\Delta \to 0+$. Thus,

$$\mathsf{E}\{N_{u+\Delta} - N_u \mid \mathscr{F}_u\} = q(X_u^+)\,\Delta + o(\Delta).$$

Further,

$$\mathsf{E}\left\{\int_u^{u+\Delta} q(X_s^+)\,\mathrm{d}s \;\middle|\; \mathscr{F}_u\right\} = q(X_u^+)\,\Delta + o(\Delta)\,.$$

We conclude that

$$\mathsf{E}\{M_{u+\Delta} - M_u \mid \mathscr{F}_u\} = o(\Delta)\,.$$

Let $0 \leqq s < t$. Set $\Delta = (t - s)/n$. It holds

$$\mathsf{E}\{M_t - M_s \mid \mathscr{F}_s\} = \sum_{k=0}^{n-1} \mathsf{E}\{M_{s+(k+1)\Delta} - M_{s+k\Delta} \mid \mathscr{F}_s\} =$$

$$= \mathsf{E}\{\sum_k \mathsf{E}\{M_{s+(k+1)\Delta} - M_{s+k\Delta} \mid \mathscr{F}_{s+k\Delta}\} \mid \mathscr{F}_s\} =$$

$$= \mathsf{E}\{\sum_k o(\Delta) \mid \mathscr{F}_s\} \to 0\,,$$

as $n \to \infty$. This yields the martingale property of M.

Let us observe that N as a nondecreasing process is a submartingale, and that

$$N_t = M_t + \int_0^t Q_u\,\mathrm{d}u\,, \quad t \geqq 0\,,$$

is its *Doob-Meyer decomposition*. The process

$$A_t = \int_0^t Q_u\,\mathrm{d}u\,, \quad t \geqq 0\,,$$

is called the compensator of N.

2. Markov processes

The probability distribution of a Markov process $X = \{X_t, t \geqq 0\}$ with finite state space I is specified by the initial distribution $P(X_0 = i)$, $i \in I$, and by the transition rates $q(i, j, t)$ which, for $i \neq j$, are related to the transition probabilities by the formula

$$P(X_{t+\Delta} = j \mid X_t = i) = q(i, j, t)\,\Delta + o(\Delta), \quad \Delta \to 0+\,.$$

We set

$$-\sum_{j \neq i} q(i, j, t) = q(i, i, t)\,, \quad i \in I\,. \tag{4}$$

We imagine process X defined on the set Ω of all paths (trajectories) $\omega(t)$, $t \geqq 0$, with values in I, piecewise constant and right continuous. We let $X_t(\omega) = \omega(t)$, (canonical representation of X). X generates a nondecreasing family of σ-algebras

$$\mathscr{F}_t = \sigma a(X_s, s \leqq t)\,, \quad t \geqq 0\,.$$

The basic σ-algebra on Ω is $\mathscr{F}_\infty = \bigvee_t \mathscr{F}_t$.

Take any state $i \in I$, and consider the process $\chi\{X_t = i\}$, $t \geqq 0$. A martingale is obtained by subtracting from $\chi\{X_t = i\}$ the integral of the transition rate,

$$^iM_t = \chi\{X_t = i\} - \int_0^t q(X_s, i, s)\,\mathrm{d}s\,, \quad t \geqq 0\,.$$

An intuitive verification that ^iM is a martingale is the following. The relations

$$\mathsf{E}\{\chi\{X_{u+\Delta} = i\} - \chi\{X_u = i\} \mid \mathscr{F}_u\} = \begin{cases} q(X_u, i, u)\,\Delta + o(\Delta)\,, & X_u \neq i\,, \\ -\sum_{j \neq i} q(i, j, u)\,\Delta + o(\Delta), & X_u = i\,, \end{cases}$$

imply with regard to (4)

$$\mathsf{E}\{^iM_{u+\Delta} - {}^iM_u \mid \mathscr{F}_u\} = o(\Delta)\,, \quad \Delta \to 0+\,.$$

In § 1 we saw that this implies the martingale property.

Proposition 1. X on $(\Omega, \mathscr{F}_\infty, P)$ is a Markov process with piecewise continuous transition rates $q(i, j, t)$, $i, j \in I$, $t \geqq 0$, if and only if ^iM, $i \in I$, are martingales with respect to $\mathscr{F}$.

Proof. Let X be a Markov process with transition rates $q(i, j, t)$. Denote $p(i, j; s, t)$ its transition probabilities. Then for $0 \leqq s \leqq t$

$$\mathsf{E}\{^iM_t - {}^iM_s \mid \mathscr{F}_s\} = \mathsf{E}\left\{\chi\{X_t = i\} - \chi\{X_s = i\} - \int_s^t q(X_u, i, u)\,\mathrm{d}u \mid \mathscr{F}_s\right\} =$$

$$= p(X_s, i; s, t) - \chi\{X_s = i\} - \int_s^t \sum_k p(X_s, k; s, u)\, q(k, i, u)\,\mathrm{d}u =$$

$$= p(X_s, i; s, t) - p(X_s, i; s, s) - \int_s^t \frac{\partial}{\partial u}\, p(X_s, i; s, u)\,\mathrm{d}u = 0$$

by the forward system of Kolmogorov differential equations.

Let ^iM, $i \in I$, be martingales. Then for $0 \leqq s \leqq t$

$$0 = \mathsf{E}\{^iM_t - {}^iM_s \mid \mathscr{F}_s\} = P(X_t = i \mid \mathscr{F}_s) - \chi\{X_s = i\} -$$

$$- \int_s^t \sum_k P(X_u = k \mid \mathscr{F}_s)\, q(k, i, u)\,\mathrm{d}u\,.$$

This means for fixed s and variable t

(5) $$\frac{\partial}{\partial t} P(X_t = i \mid \mathscr{F}_s) = \sum_k P(X_t = k \mid \mathscr{F}_s)\, q(k, i, t)\,,$$

$$P(X_s = i \mid \mathscr{F}_s) = \chi\{X_s = i\}\,, \quad i \in I\,.$$

On $\{X_s = j\}$ (5) is the forward system of differential equations for $p(j, i; s, t)$. We conclude that

$$P(X_t = i \mid \mathscr{F}_s) = p(X_s, i; s, t), \quad 0 \leqq s \leqq t, \quad i \in I.$$

Thus, X is Markovian with transition rates $q(i, j, t)$. □

The martingale characterization of the probability distribution is suitable for controlled Markov processes. Before considering them let us give an example.

Example 2. Customers of two types 1, 2 arrive at a single server system. The arrival times of the customers of type $i = 1, 2$, form a Poisson process with rate q_i, their service times are exponentially distributed with mean $1/r_i$, and the server gains the amount $a_i > 0$ per unit time of the service. The customer is lost, if the server is not free at his arrival. It is therefore undesirable to have the server blocked by the customers yielding little gain. It is to be decided, whether to accept both types of customers, or to reject the customers of one type always or in a part of the time period in consideration.

The state of the system at time t is

$$X_t = \begin{cases} 0 \ldots \text{the server is idle,} \\ 1 \ldots \text{the server serves a customer of type 1,} \\ 2 \ldots \text{the server serves a customer of type 2.} \end{cases}$$

There are three possible decisions: $z = 0$ not to reject any customer, $z = 1$ reject the customers of type 1, $z = 2$ reject the customers of type 2. Since $a_i > 0$, $i = 1, 2$, the rejection of both types of customers is not advantageous. The decisions are efficient only when $X_t = 0$. The matrix of transition rates is

$$\begin{matrix} \\ 0 \\ \\ 1 \\ 2 \end{matrix} \begin{pmatrix} -(q_1 + q_2), & q_1, & q_2 \\ -q_2, & 0, & q_2 \\ -q_1 & q_1, & 0 \\ r_1, & -r_1, & 0 \\ r_2, & 0, & -r_2 \end{pmatrix} \begin{matrix} z = 0 \\ z = 1 \\ z = 2 \\ \\ \end{matrix} .$$

□

The dynamics of a *controlled Markov process* with state space I is defined by the transition rates

$$q(i, j; z), \quad i, j \in I,$$

depending on a control parameter $z \in J$. Let J be a compact set, and let q be continuous in z. We limit ourselves to the time-homogeneous. case. The initial distribution $P(X_0 = i) = p_i$, $i \in I$, is assumed to be fixed.

The control parameter at time t is chosen on the basis of the observation until time t, i.e. on the basis of the events from $\mathscr{F}_t$. A *control* is a random function $Z = \{Z_t,$

$t \geqq 0\}$ on $(\Omega, \mathscr{F}_\infty)$ with values in J, left-continuous and such that Z_t is $\mathscr{F}_t$-measurable for $t \geqq 0$. Closed loop controls

$$(6) \qquad Z_t = \bar{z}(X_t^-), \quad t \geqq 0,$$

are called stationary or homogeneous Markovian controls. In (6) $\bar{z}(i)$ is a mapping from I into J, $\{X_t^-, t \geqq 0\}$ is the left-continuous version of X.

With any control Z we want to associate a probability measure P^Z on $(\Omega, \mathscr{F}_\infty)$, being the probability distribution of X under the control Z. Recall that we have the canonical representation of X. Thus, P^Z is a measure on the set of trajectories. If (6) holds, we define P^Z so that X is a Markov process with transition rates $q(i, j; \bar{z}(i))$, $i, j \in I$. From Proposition 1 follows that then

$$(7) \qquad {}^iM_t = \chi\{X_t = i\} - \int_0^t q(X_s, i; Z_s)\,\mathrm{d}s, \quad t \geqq 0, \quad i \in I,$$

are martingales with respect to $\mathscr{F}$. The extension to general controls is evident.

The probability distribution of the controlled process X under the control Z is the probability measure P^Z on $(\Omega, \mathscr{F}_\infty)$ such that ${}^iM$, $i \in I$, defined by (7) are martingales with respect to $\mathscr{F}$, and $P^Z(X_0 = i) = p_i$, $i \in I$.

The existence and the unicity of P^Z will be established in § 7.

3. The rate of a point process

A *point process* is a random sequence of points $\boldsymbol{\tau} = \{\tau_n, n = 1, 2, \ldots\}$ in the time interval $(0, \infty]$ satisfying the following conditions:

1. $\tau_1 > 0, \quad \lim_{n \to \infty} \tau_n = \infty$.
2. $\tau_n < \tau_{n+1}$ whenever $\tau_n < \infty$.

In addition, we shall assume

3. The counting process

$$N_t = \sum_{n=1}^{\infty} \chi\{\tau_n \leqq t\}, \quad t \geqq 0,$$

has finite expectation, i.e. $\mathsf{E}N_t < \infty$, $t \geqq 0$. In applications the point processes are constituted by the occurrences of a repetitive random event (earthquakes, vehicles passing a point etc.).

First we shall consider the point process alone, *isolated* from other events. Only the the random sequence $\boldsymbol{\tau}$ or, which is the same, the counting process N is observed. The increase of the field of events in time represents the nondecreasing family of σ-algebras

$$\mathscr{F}_t^N = \sigma a(N_s, s \leqq t), \quad t \geqq 0.$$

The trajectory of τ is a nondecreasing sequence of numbers $\omega = \{s_1, s_2, \ldots\}$, positive or ∞, with the properties:

$$1.\ \lim_{n\to\infty} s_n = \infty\,. \quad 2.\ s_n < s_{n+1} \quad \text{whenever} \quad s_n < \infty\,.$$

Let Ω denote the set of all such sequences. We take $(\Omega, \mathscr{F}^N_\infty)$ for the basic space with $\tau_n(\omega) = s_n$, $n = 1, 2, \ldots$.

The probability distribution of τ is defined by means of conditional distribution functions

$$\begin{aligned} F_1(t) &= P(\tau_1 \leqq t)\,, \\ F_n(t; t_1, \ldots, t_{n-1}) &= P(\tau_n \leqq t \mid \tau_1 = t_1, \ldots, \tau_{n-1} = t_{n-1})\,, \\ 0 < t_1 &< t_2 < \ldots < t_{n-1} < \infty\,. \end{aligned} \tag{8}$$

Obviously, $F_n(t_{n-1}; t_1, \ldots, t_{n-1}) = 0$ for $t_{n-1} < \infty$. We shall assume that conditional distributions (8) have densities $f_n(t; t_1, \ldots, t_{n-1})$, $n = 1, 2, \ldots$. Then the rate of the point process can be constructed analogously to the rate of a renewal process. To the failure rate $q(t) = f(t)/\int_t^\infty f(s)\,ds$ correspond the conditional rates

$$q_n(t; t_1, \ldots, t_{n-1}) = f_n(t; t_1, \ldots, t_{n-1}) / \int_t^\infty f_n\,ds\,.$$

In fact,

$$P(\tau_n \in (t, t + \Delta) \mid \tau_1 = t_1, \ldots, \tau_{n-1} = t_{n-1}, \tau_n < t) =$$
$$= \int_t^{t+\Delta} f_n\,ds / \int_t^\infty f_n\,ds = q_n(t; t_1, \ldots, t_{n-1})\,\Delta + o(\Delta), \quad \Delta \to 0+\,.$$

As in § 1, the evolution of the rate in time is of interest (Fig. 4). The occurrence rate

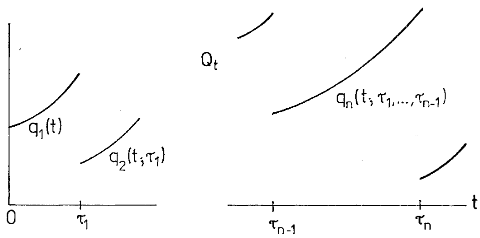

Fig. 4.

of the n-th event is $\chi\{\tau_n > t\}\, q_n(t; \tau_1, \ldots, \tau_{n-1})$. Hence,

$$Q_t = \sum_{n=1}^\infty \chi\{\tau_n > t\}\, q_n(t; \tau_1, \ldots, \tau_{n-1})\,, \quad t \geqq 0\,.$$

The compensator is the integral of the rate,

$$\bar{A}_t = \int_0^t Q_u \, \mathrm{d}u = \sum_{n=1}^{\infty} \int_0^{t \wedge \tau_n} q_n(u; \tau_1, \ldots, \tau_{n-1}) \, \mathrm{d}u \,, \quad t \geqq 0 \,.$$

$t \wedge \tau_n$ means $\min(t, \tau_n)$.

Proposition 2. $M_t = N_t - \bar{A}_t$, $t \geqq 0$, is a martingale with respect to $\mathscr{F}^N$.

Proof. Let $0 \leqq s < t$. We demonstrate

$$\mathsf{E}\{N_t - N_s - (\bar{A}_t - \bar{A}_s) \mid \mathscr{F}_s^N\} = 0 \,,$$

or

$$\mathsf{E}\left\{\sum_{n=1}^{\infty}\left(\chi\{s < \tau_n \leqq t\} - \int_{s \wedge \tau_n}^{t \wedge \tau_n} q_n \, \mathrm{d}u\right) \middle| \mathscr{F}_s^N\right\} = 0 \,.$$

Thus, it suffices to prove for n arbitrary

(9) $$P(s < \tau_n \leqq t \mid \mathscr{F}_s^N) = \mathsf{E}\left\{\int_{s \wedge \tau_n}^{t \wedge \tau_n} q_n \, \mathrm{d}u \mid \mathscr{F}_s^N\right\}.$$

Specifying the observation up to time s to be

$$\tau_1 = t_1, \ldots, \tau_k = t_k \leqq s \,,$$

we rewrite (9) as

(10) $$P(s < \tau_n \leqq t \mid \tau_1 = t_1, \ldots, \tau_k = t_k, \tau_{k+1} > s) = \\ = \mathsf{E}\left\{\int_{s \wedge \tau_n}^{t \wedge \tau_n} q_n \, \mathrm{d}u \mid \tau_1 = t_1, \ldots, \tau_k = t_k, \tau_{k+1} > s\right\}.$$

For $k \geqq n$ both sides of (10) equal 0. For $k < n$ (10) follows, if we prove

(11) $$P(s < \tau_n \leqq t \mid \tau_1 = t_1, \ldots, \tau_k = t_k, \ldots, \tau_{n-1} = t_{n-1}) = \\ = \mathsf{E}\left\{\int_{s \wedge \tau_n}^{t \wedge \tau_n} q_n \, \mathrm{d}u \mid \tau_1 = t_1, \ldots, \tau_k = t_k, \ldots, \tau_{n-1} = t_{n-1}\right\}$$

for arbitrary $t_{k+1} < \ldots < t_{n-1}$. The right-hand side of (11) equals

(12) $$\int_s^t \int_s^y q_n \, \mathrm{d}u \, f_n \, \mathrm{d}y + \int_s^t q_n \, \mathrm{d}u \int_t^{\infty} f_n \, \mathrm{d}y \,.$$

Integrating the first term by parts we transform (12) into $\int_s^t f_n \, \mathrm{d}y$ equal to the left-hand side of (11). □

The formula for $\bar{A}$ can be generalized to include the case when the conditional distributions are not absolutely continuous. Namely,

$$\bar{A}_t = \sum_{n=1}^{\infty} \int_0^{t \wedge \tau_n} \frac{\mathrm{d}F_n(u; \tau_1, \ldots, \tau_{n-1})}{1 - F_n(u-; \tau_1, \ldots, \tau_{n-1})} \,, \quad t \geqq 0 \,.$$

The proof that $N_t - \bar{A}_t$, $t \geqq 0$, is a martingale proceeds as above. It involves partial integration of Stieltjes integrals.

In the preceding the point process was investigated separately, the rate was related to the family of σ-algebras $\mathscr{F}^N$. Compensator $\bar{A}$ is therefore called *the minimal compensator* of N. Let us turn to the general situation.

Let $(\Omega, \mathscr{A}, P)$ be a probability space equipped with a nondecreasing family of σ-algebras $\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$. A point process on $(\Omega, \mathscr{A}, P)$ is a sequence $\tau = \{\tau_n, n = 1, 2, \ldots\}$ of stopping times with respect to $\mathscr{F}$ having Properties 1, 2, 3 stated at the beginning of this section. Recall that by definition τ is a stopping time with tespect to $\mathscr{F}$ if $\{\tau \leqq t\} \in \mathscr{F}_t$, $t \geqq 0$. Hence,

$$N_t = \sum_{n=1}^{\infty} \chi\{\tau_n \leqq t\}$$

is $\mathscr{F}_t$-measurable for $t \geqq 0$. A right or left-continuous process with this property is called *nonanticipative* (with respect to $\mathscr{F}$). Thus, the counting process N is nonanticipative, and $\mathscr{F}_t^N \subset \mathscr{F}_t$, $t \geqq 0$. τ defines N and vice versa. We may call N a point process as well.

Example 3. Let $W = \{W_t, t \geqq 0\}$, $W_0 = 0$, be a Wiener process, $\mathsf{E}(\mathrm{d}W_t)^2 = \mathrm{d}t$. W generates a nondecreasing family of σ-algebras $\mathscr{F}^W = \mathscr{F}$. Let $\tau = \{\tau_n, n = 1, 2, \ldots\}$

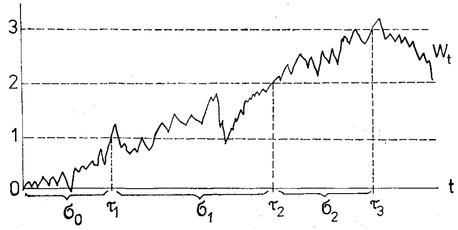

Fig. 5.

be the sequence of the first times at which W has reached the levels 1, 2, ... (see Fig. 5). In symbols,

$$\tau_n = \inf\{t : W_t = n\}, \quad n = 1, 2, \ldots$$

τ isolated is a renewal process with

$$P(\tau_{n+1} - \tau_n \leqq t) = 2\left(1 - \Phi\left(\frac{1}{\sqrt{t}}\right)\right) = F(t),$$

where

$$\Phi(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{x} e^{-y^2/2} \, dy .$$

F has density

$$f(t) = \frac{1}{\sqrt{(2\pi)}} t^{-3/2} e^{-1/(2t)} .$$

The rate of τ with respect to $\mathscr{F}^N$ is obtained from the formulae of § 1.

Consider now the rate Q of τ with respect of $\mathscr{F}^W$. Take $t \geqq 0$, and assume known the trajectory W_s, $s \leqq t$. Let n levels be passed before time t, i.e. $\tau_1 < \tau_2 < \ldots < \tau_n < t$, and let $W_t = x < n + 1$. Then

$$P(\tau_{n+1} \in (t, t + \Delta) \mid W_s, s \leqq t) = 2\left(1 - \Phi\left(\frac{n + 1 - x}{\sqrt{\Delta}}\right)\right) \leqq$$

$$\leqq \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\Delta}}{n + 1 - x} \exp\left\{- \frac{(n + 1 - x)^2}{2\Delta}\right\} = o(\Delta) , \quad \Delta \to 0+.$$

We see that for each t not coinciding with any τ_n holds $Q_t = 0$. In points τ_n Q should be the Dirac function. Consequently,

$$Q_t = \sum_{n=1}^{\infty} \delta(t - \tau_n) , \quad t \geqq 0 . \tag{13}$$

The integral of the rate is

$$A_t = \int_0^t Q_s \, ds = N_t , \quad t \geqq 0 .$$

With respect to $\mathscr{F}^W$ process N is its own compensator. □

A point process having the rate (13) will be called *predictable* (with respect to the family $\mathscr{F}$ in consideration).

Example 4. Let us now present the construction of a point process, which with respect to $\mathscr{F}^N$ is the Poisson process with rate q, and which is predictable with respect to $\mathscr{F}^W$. Take $\sigma_0 = \tau_1$ from Example 3 having distribution function $F(t)$. Hence,

$$\sigma^* = - \frac{1}{q} \log \bar{F}(\sigma_0) ,$$

with $\bar{F}(t) = 1 - F(t)$ has exponential distribution with density $q\, e^{-qt}$. It holds

$$\sigma_0 = \bar{F}^{-1}(e^{-q\sigma^*})$$

or, from the definition of σ_0,

$$\sigma^* = \inf \{t : W_{\bar{F}^{-1}(e^{-qt})} = 1\} .$$

σ^* is not a stopping time with respect to $\mathscr{F}^W$ because $\bar{F}^{-1}(e^{-qt}) \leqq t$, $t \geqq 0$, is not true. To improve this, we shall employ a property of stochastic integrals. Namely,

$$\int_0^t h(s)\,\mathrm{d}W_s = \mathscr{W}_{\int_0^t h(s)^2\mathrm{d}s},\ t \geqq 0\,,$$

where $\mathscr{W}$ is also a Wiener process. Choose $h(t)$ so that

$$\int_0^t h(s)^2\,\mathrm{d}s = \bar{F}^{-1}(\mathrm{e}^{-qt}),\ \text{i.e. } h(t) = \left(\frac{q\,\mathrm{e}^{-qt}}{f(\bar{F}^{-1}(\mathrm{e}^{-qt}))}\right)^{1/2}.$$

Then

$$\sigma_0^* = \inf\left\{t : \int_0^t h(s)\,\mathrm{d}W_s = 1\right\} = \inf\{t : \mathscr{W}_{\bar{F}^{-1}(\mathrm{e}^{-qt})} = 1\}$$

is a stopping time with respect to $\mathscr{F}^W$ having exponential distribution with mean $1/q$.

Repeating the construction from σ_0^* onwards we get σ_1^*, further $\sigma_2^*, \sigma_3^*, \ldots$, mutually independent. Consequently,

$$\tau_n^* = \sigma_0^* + \ldots + \sigma_{n-1}^*, \quad n = 1, 2, \ldots,$$

is a Poisson process with rate q. The argument presented in Example 3 shows that τ^* is predictable with respect to $\mathscr{F}^W$. □

Predictable processes are an extreme case. Let us therefore illustrate the dependence of the rate on the family $\mathscr{F}$ on two further examples.

Example 5. As in § 2, let $X = \{X_t, t \geqq 0\}$ be a Markov process with finite state space I and transition rates $q(i, j, t)$, $t \geqq 0$. Choose a state i, and observe the times at which the trajectory jumps into i. The times form a point process $\tau = \{\tau_n, n = 1, 2, \ldots\}$. Denote by $N = \{N_t, t \geqq 0\}$ the counting process of τ. The rate of τ with respect to the family $\mathscr{F}^X$ associated with the observation of the entire trajectory of X is

(14) $$Q_t = (1 - \chi\{X_t = i\})\, q(X_t, i, t)\,, \quad t \geqq 0\,.$$

(14) is intuitively evident. When $X_t = i$, the rate vanishes, when $X_t \neq i$, Q_t equals to the rate of the jump into state i. We refer to § 7 for a more elaborate argument.

Let us find $\bar{Q} = \{\bar{Q}_t, t \geqq 0\}$, the rate of τ with respect to $\mathscr{F}^N$ whose integral is the minimal compensator. Take $t \geqq 0$. It holds

$$\begin{aligned} P(N_{t+\Delta} - N_t > 0 \mid \mathscr{F}_t^N) &= \mathsf{E}\{P(N_{t+\Delta} - N_t > 0 \mid \mathscr{F}_t^X) \mid \mathscr{F}_t^N\} = \\ &= \sum_j P(X_t = j \mid \mathscr{F}_t^N)\,.\,P(N_{t+\Delta} - N_t > 0 \mid X_t = j) = \\ &= \sum_{j \neq i} P(X_t = j \mid \mathscr{F}_t^N)\, q(j, i, t)\,\Delta + o(\Delta) = \mathsf{E}\{Q_t \mid \mathscr{F}_t^N\}\,\Delta + o(\Delta) \end{aligned}$$

as $\Delta \to 0+$. We arrived at the relation

(15) $$\bar{Q}_t = \mathsf{E}\{Q_t \mid \mathscr{F}_t^N\}\,, \quad t \geqq 0\,.$$

The general validity of (15) will be verified in the sequel.

Let us show, how the aposteriori probabilities $P(X_t = j \mid \mathscr{F}_t^N)$ are calculated. For the sake of denotational simplicity set $\tau_0 = 0$, $P(X_0 = i) = 1$. Let $\tau_n \leqq t < \tau_{n+1}$. The observation of τ until time t thus says that $X_{\tau_n} = i$ and that there are no jumps into i during $(\tau_n, t]$. The Markovian character of X implies that

$$(16) \qquad P(X_t = j \mid \mathscr{F}_t^N) = {}_ip(i, j; \tau_n, t) / \sum_k {}_ip(i, k; \tau_n, t) ,$$

where ${}_ip(i, j; s, t)$ are transition probabilities with exclusion of jumps into state i. They fulfil the system of differential equations

$$(17) \qquad \frac{\partial}{\partial t}\, {}_ip(i, i; s, t) = {}_ip(i, i; s, t)\, q(i, i, t) ,$$

$$\frac{\partial}{\partial t}\, {}_ip(i, j; s, t) = \sum_k {}_ip(i, k; s, t)\, q(k, j, t) , \quad j \neq i , \quad t \geqq s ,$$

$${}_ip(i, i; s, s) = 1 , \quad {}_ip(i, j; s, s) = 0 , \quad j \neq i .$$

□

Example 6. The queuing model $M/M/1/1$ provides a special case of the preceding situation. Let q be the rate of the incoming Poisson stream of customers, and let r be the service completion rate, $r \neq q$. Let $X_t = 0, 1$ be the number of customers in service at time t. Assume $X_0 = 0$. Consider the outgoing stream of customers $\tau = \{\tau_n, n = 1, 2, \ldots\}$ identical with the moments of jumps into state 0.

The rate of τ with respect to $\mathscr{F}^X$ is

$$Q_t = rX_t , \quad t \geqq 0 .$$

Equations (17) have the form

$$\frac{\partial}{\partial t}\, {}_0p(0, 0; s, t) = -\, {}_0p(0, 0; s, t)\, q ,$$

$$\frac{\partial}{\partial t}\, {}_0p(0, 1; s, t) = {}_0p(0, 0; s, t)\, q - {}_0p(0, 1; s, t)\, r .$$

Solving them and using (16) we get the rate of τ with respect to $\mathscr{F}^N$,

$$\bar{Q}_t = \mathsf{E}\{Q_t \mid \mathscr{F}_t^N\} = rP(X_t = 1 \mid \mathscr{F}_t^N) =$$
$$= rq(\mathrm{e}^{-r(t-\tau_n)} - \mathrm{e}^{-q(t-\tau_n)}) / (q\, \mathrm{e}^{-r(t-\tau_n)} - r\, \mathrm{e}^{-q(t-\tau_n)}) .$$

τ_n is the last departure time of a customer not exceeding t. □

We pass now to the proof of (15) in the general situation. First we have to deal with the unicity of the rate or of the compensator of a point process. Let $(\Omega, \mathscr{A}, P)$ be the underlying probability space with a nondecreasing family of σ-algebras $\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$.

Lemma 1. Let $Y = \{Y_t, t \geqq 0\}$, $Y_0 = 0$, be a martingale with respect to $\mathscr{F}$ whose trajectories are continuous and have finite variation on finite intervals. Then, with probability 1, $Y_t = 0$, $t \geqq 0$.

Let us present only the main ideas of the proof. Assume $\mathsf{E}Y_t^2 < \infty$, $t \geqq 0$. Y has orthogonal increments. Namely, for $s < t < u < v$,

$$(18) \qquad \mathsf{E}(Y_v - Y_u)(Y_t - Y_s) = \mathsf{E}\mathsf{E}\{Y_v - Y_u \mid \mathscr{F}_u\}(Y_t - Y_s) = 0\,.$$

Take $t > 0$, n positive integer. In virtue of (18),

$$\mathsf{E}Y_t^2 = \mathsf{E}\Big(\sum_{k=0}^{n-1}(Y_{t(k+1)/n} - Y_{tk/n})\Big)^2 = \mathsf{E}\sum_k (Y_{t(k+1)/n} - Y_{tk/n})^2 \leqq$$
$$\leqq \mathsf{E}\big(\max_j \left| Y_{t(j+1)/n} - Y_{tj/n} \right| \sum_k \left| Y_{t(k+1)/n} - Y_{tk/n} \right|\big)\,.$$

In the last bracket the first term tends to 0 as $n \to \infty$ by the continuity of the trajectories while the second term stays bounded. This yields $\mathsf{E}Y_t^2 = 0$, $t \geqq 0$, from which, taking the continuity of Y into account, we get the assertion. □

Proposition 3. Let $N = \{N_t, t \geqq 0\}$ be a point process possessing compensators $A = \{A_t, t \geqq 0\}$, $A' = \{A'_t, t \geqq 0\}$ with continuous trajectories. Then A and A' are indistinguishable, i.e. their trajectories coincide with probability 1.

Proof. The difference

$$A_t - A'_t = (N_t - A'_t) - (N_t - A_t) = M'_t - M_t\,, \quad t \geqq 0\,,$$

is a martingale fulfilling the hypotheses of Lemma 1. Consequently, with probability 1, $A_t = A'_t$, $t \geqq 0$. □

Recall that the rate of point process N with respect to $\mathscr{F}$ is a nonnegative random function $Q = \{Q_t, t \geqq 0\}$ such that Q_t is $\mathscr{F}_t$-measurable for $t \geqq 0$, and

$$M_t = N_t - \int_0^t Q_s \,\mathrm{d}s\,, \quad t \geqq 0\,,$$

is a martingale with respect to $\mathscr{F}$. The first property is evident, since Q_t comes from conditional probabilities with respect to $\mathscr{F}_t$. Proposition 3 implies that $Q_t(\omega)$ is defined uniquely up to a (t, ω)-set of $\mathrm{d}t \times \mathrm{d}P$-measure 0.

Next proposition includes (15).

Proposition 4. Let a point process N have the rate Q with respect to $\mathscr{F}$. Let $\mathscr{F}^*$ be another nondecreasing family of σ-algebras satisfying

$$\mathscr{F}_t \supset \mathscr{F}_t^* \supset \mathscr{F}_t^N\,, \quad t \geqq 0\,.$$

Assume

$$Q_t^* = \mathsf{E}\{Q_t \mid \mathscr{F}_t^*\}\,, \quad t \geqq 0\,,$$

where Q^* has right (left)-continuous trajectories. Then Q^* is the rate of N with respect to $\mathscr{F}^*$.

Proof. We have to prove that

$$M_t^* = N_t - \int_0^t Q_s^* \, ds , \quad t \geqq 0 ,$$

is a martingale with respect to $\mathscr{F}^*$. In fact for $0 \leqq s < t$

$$\mathsf{E}\{M_t^* - M_s^* \mid \mathscr{F}_s^*\} = \mathsf{E}\{M_t - M_s \mid \mathscr{F}_s^*\} + \mathsf{E}\left\{\int_s^t (Q_u - Q_u^*) \, du \mid \mathscr{F}_s^*\right\} =$$

$$= \int_s^t \mathsf{E}\{\mathsf{E}\{Q_u - Q_u^* \mid \mathscr{F}_u^*\} \mid \mathscr{F}_s^*\} \, du = 0 . \qquad \square$$

Point processes recording the occurrence of several random events are called *labelled point processes*. They are double sequences $\tau_\lambda = \{(\tau_n, \lambda_n), \ n = 1, 2 \ldots\}$. τ_n is the n-th time at which an event occurred, and λ_n is the label stating the kind of the event. The set of admissible labels will be denoted by I. As example take vehicles passing a point classified as cars, trucks, and motorcycles.

If I is finite, $I = \{1, 2, \ldots, m\}$, then the labelled point process is adequately represented by an m-dimensional counting process

$$N_t = ({}^1N_t, {}^2N_t, \ldots, {}^mN_t) , \quad t \geqq 0 ,$$

where

$${}^iN_t = \sum_{n=1}^{\infty} \chi\{\tau_n \leqq t , \quad \lambda_n = i\} , \quad t \geqq 0 ,$$

is the counting process of events with label i.

4. Point processes with deterministic rate

Let $\tau = \{\tau_n, n = 1, 2, \ldots\}$ or $N = \{N_t, t \geqq 0\}$ be a point process with respect to family $\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$ defined on a probability space $(\Omega, \mathscr{A}, P)$. A nonnegative locally integrable function $q(t)$, $t \geqq 0$, is a *deterministic rate* of N if

$$M_t = N_t - \int_0^t q(s) \, ds , \quad t \geqq 0 ,$$

is a martingale with respect to $\mathscr{F}$. The compensator of N is then $a(t) = \int_0^t q(s) \, ds$, $t \geqq 0$.

Let us give an example.

Example 7. N is a Poisson process with respect to $\mathscr{F}$ with variable rate $q(t)$, $t \geqq 0$, if for $0 \leqq s < t$ the increments $N_t - N_s$ are independent of (the events from) $\mathscr{F}_s$, and have Poisson distribution with mean $a(t) - a(s)$. In symbols,

$$P(N_t - N_s = k \mid \mathscr{F}_s) = \frac{(a(t) - a(s))^k}{k!} e^{-(a(t)-a(s))}, \; k = 0, 1, \ldots \tag{19}$$

The characteristic function of the probability distribution (19) is

$$\mathsf{E}\{e^{iu(N_t - N_s)} \mid \mathscr{F}_s\} = \exp\{(e^{iu} - 1)(a(t) - a(s))\}, \quad u \in (-\infty, \infty). \tag{20}$$ □

Next we show that Example 7 embraces all point processes with deterministic rate.

Proposition 5. Let N have rate $q(t)$, $t \geqq 0$. Then N is a Poisson process with respect to $\mathscr{F}$.

Proof. We shall establish (20) for $0 \leqq s < t$. Simple transformations give

$$e^{iu(N_t - N_s)} - 1 = \sum_{s<z\leqq t} (e^{iu(N_z - N_s)} - e^{iu(N_{z-} - N_s)}) =$$

$$= \sum_{s<z\leqq t} e^{iu(N_{z-} - N_s)} (e^{iu} - 1)(N_z - N_{z-}) = (e^{iu} - 1) \int_s^t e^{iu(N_{z-} - N_s)} \, dN_z =$$

$$= (e^{iu} - 1) \int_s^t e^{iu(N_{z-} - N_s)} \, d(N_z - a(z)) + (e^{iu} - 1) \int_s^t e^{iu(N_z - Ns)} q(z) \, dz .$$

Assume s fixed, t variable. The before last term being the integral of a bounded left-continuous nonanticipative random function with respect to a martingale is itself a martingale. Applying conditional expectation we obtain

$$\mathsf{E}\{e^{iu(N_t - N_s)} \mid \mathscr{F}_s\} = 1 + (e^{iu} - 1) \int_s^t \mathsf{E}\{e^{iu(N_z - N_s)} \mid \mathscr{F}_s\} q(z) \, dz, \quad t \geqq s. \tag{21}$$

(21) has unique solution

$$\mathsf{E}\{e^{iu(N_t - N_s)} \mid \mathscr{F}_s\} = \exp\{(e^{iu} - 1)(a(t) - a(s))\}.$$

Hence, (20) is valid. □

Assume $a(t)$ strictly increasing, $a(\infty) = \infty$. Let $\sigma(t)$ be its inverse function,

$$a(\sigma(t)) = t, \quad t \geqq 0 .$$

Then, as it is readily seen, ${}^\sigma N = \{N_{\sigma(t)}, t \geqq 0\}$ is a Poisson process with respect to ${}^\sigma\mathscr{F} = \{\mathscr{F}_{\sigma(t)}, t \geqq 0\}$ having constant rate 1. This is true also for processes with a nondeterministic rate. Before presenting this result due to S. Watanabe, we recall some facts about martingales and stopping times.

Let σ be a stopping time with respect to $\mathscr{F}$. In the usual interpretation $\mathscr{F}_t$ is the σ-algebra of random events up to time t. Similarly, $\mathscr{F}_\sigma$ includes the events up to time σ. Its definition is

$$\mathscr{F}_\sigma = \{A \in \mathscr{A} : A \cap \{\sigma \leqq t\} \in \mathscr{F}_t, \; t \geqq 0\} .$$

In words, whenever $\sigma \leqq t$, then at time t it is known whether event A took place or not. Inequality $\sigma \leqq \sigma'$ between stopping times implies $\mathscr{F}_\sigma \subset \mathscr{F}_{\sigma'}$.

Lemma 2. Let $Y = \{Y_t, t \geqq 0\}$ be a nonanticipative right (left)-continuous random function, σ a stopping time, $\sigma < \infty$. Then Y_σ is an $\mathscr{F}_\sigma$-measurable random variable.

Lemma 3. (Extension of the martingale property.) Let Y be a right-continuous martingale with respect to $\mathscr{F}$. Let σ, σ' be stopping times satisfying $\sigma \leqq \sigma' \leqq T < \infty$ where T is a constant. Then

$$\mathsf{E}\{Y_{\sigma'} \mid \mathscr{F}_\sigma\} = Y_\sigma . \tag{22}$$

Lemma 4. Let Y be as in Lemma 3. Let there exist a random variable η such that $|Y_t| \leqq \eta$, $t \geqq 0$, $\mathsf{E}\eta < \infty$. Then (22) holds for arbitrary stopping times $\sigma \leqq \sigma'$.

Corollary. Let Y be as in Lemma 3. Then $\{Y_{\sigma \wedge t}, t \geqq 0\}$ is a martingale with respect to $\mathscr{F}$.

Proof. For $0 \leqq s < t$, using Lemmas 2, 3,

$$\begin{aligned} \mathsf{E}\{Y_{\sigma \wedge t} \mid \mathscr{F}_s\} &= \mathsf{E}\{\chi\{\sigma \leqq s\}\, Y_{\sigma \wedge s} \mid \mathscr{F}_s\} + \\ &+ \mathsf{E}\{\chi\{\sigma > s\}\, Y_{\max(t \wedge \sigma, s)} \mid \mathscr{F}_s\} = \chi\{\sigma \leqq s\}\, Y_{\sigma \wedge s} + \\ &+ \chi\{\sigma > s\}\, Y_s = Y_{\sigma \wedge s} . \end{aligned}$$

□

Proposition 6. Let the point process $N = \{N_t, t \geqq 0\}$ have a continuous strictly increasing compensator $A = \{A_t, t \geqq 0\}$ satisfying $\lim_{t \to \infty} A_t = \infty$. Define $\{\sigma(t), t \geqq 0\}$ by the relation

$$A_{\sigma(t)} = t , \quad t \geqq 0 .$$

Then ${}^\sigma N = \{N_{\sigma(t)}, t \geqq 0\}$ is a Poisson process with respect to ${}^\sigma\mathscr{F} = \{\mathscr{F}_{\sigma(t)}, t \geqq 0\}$ having rate 1.

Proof. By the definition of a compensator

$$M_t = N_t - A_t , \quad t \geqq 0 ,$$

is a martingale. For each t, $\sigma(t)$ is a stopping time, since

$$\{\sigma(t) \leqq s\} = \{A_s \geqq t\} \in \mathscr{F}_s, \; s \geqq 0.$$

According to Lemma 2, $N_{\sigma(t)}$ is $\mathscr{F}_{\sigma(t)}$-measurable, $t \geqq 0$. We conclude that ${}^\sigma N$ is a point process with respect to ${}^\sigma\mathscr{F}$. Let us verify $\mathsf{E}N_{\sigma(t)} < \infty$. From Lemma 3 follows for arbitrary $T < \infty$

$$\mathsf{E}(N_{\sigma(t) \wedge T} - A_{\sigma(t) \wedge T}) = \mathsf{E}M_{\sigma(t) \wedge T} = 0 .$$

Hence,

$$\mathsf{E}N_{\sigma(t) \wedge T} = \mathsf{E}A_{\sigma(t) \wedge T} .$$

Letting $T \to \infty$ we get

$$\mathsf{E} N_{\sigma(t)} = \mathsf{E} A_{\sigma(t)} = t .$$

With regard to Proposition 5 we have to show that $N_{\sigma(t)} - t$, $t \geqq 0$, is a martingale with respect to ${}^{\sigma}\mathscr{F}$. This amounts to verifying

$$\mathsf{E}\{N_{\sigma(t)} - N_{\sigma(s)} - (t - s) \mid \mathscr{F}_{\sigma(s)}\} = 0 , \quad 0 \leqq s < t .$$

Let t be fixed. By the corollary,

$$M_{\sigma(t) \wedge u} , \quad u \geqq 0 , \tag{23}$$

is a martingale with respect to $\mathscr{F}$. Moreover,

$$|M_{\sigma(t) \wedge u}| \leqq N_{\sigma(t) \wedge u} + A_{\sigma(t) \wedge u} \leqq N_{\sigma(t)} + A_{\sigma(t)} , \quad u \geqq 0 .$$

Thus, martingale (23) has an integrable majorant. Applying Lemma 4 to the stopping times $\sigma(s)$, $\sigma(t)$ we get

$$\mathsf{E}\{M_{\sigma(t) \wedge \sigma(t)} \mid \mathscr{F}_{\sigma(s)}\} = M_{\sigma(t) \wedge \sigma(s)} = M_{\sigma(s)} .$$

Consequently,

$$0 = \mathsf{E}\{M_{\sigma(t)} - M_{\sigma(s)} \mid \mathscr{F}_{\sigma(s)}\} = \mathsf{E}\{N_{\sigma(t)} - N_{\sigma(s)} - (A_{\sigma(t)} - A_{\sigma(s)}) \mid \mathscr{F}_{\sigma(s)}\} =$$
$$= \mathsf{E}\{N_{\sigma(t)} - N_{\sigma(s)} - (t - s) \mid \mathscr{F}_{\sigma(s)}\} . \qquad \square$$

As consequence of Proposition 6 we can write

$$N_t = \mathscr{N}_{A_t} , \quad t \geqq 0 , \quad A_t = \int_0^t Q_s \, \mathrm{d}s , \tag{24}$$

where $\mathscr{N} = \{\mathscr{N}_t, t \geqq 0\}$ is a Poisson process with unit rate. In fact the hypothesis $A_\infty = \infty$ can be dropped.

Example 8. Let the point process N have on $[0, T]$ a rate satisfying $Q_t \leqq q$, $t \in [0, T]$, where q is a constant. It is intuitively evident that N_T will be stochastically smaller than the corresponding quantity in a Poisson process with rate q. More explicitly, for any nondecreasing function $h(k)$, $k = 0, 1, 2, \ldots$, holds

$$\mathsf{E} h(N_T) \leqq \sum_{k=0}^{\infty} h(k) \frac{(qT)^k}{k!} \mathrm{e}^{-qT} .$$

The proof follows from (24) and from the inequality

$$h(N_T) = h(\mathscr{N}_{A_T}) \leqq h(\mathscr{N}_{qT})$$

implied by $A_T \leqq qT$. $\square$

Example 9. A Poisson process $\mathscr{N}$ with unit rate fulfils the law of the iterated logarithm

(25) $$\overline{\lim_{t\to\infty}} \pm \frac{\mathcal{N}_t - t}{\sqrt{(2t \log\log t)}} = 1 \qquad \text{a.s.}$$

A.s. means almost surely. Since $\mathcal{N}$ has independent increments, (25) is a consequence of the classical result for mutually independent random variables. Let N be a point process satisfying $A_\infty = \int_0^\infty Q_s \, ds = \infty$. From (24), (25) follows

$$\overline{\lim_{t\to\infty}} \pm \frac{N_t - A_t}{\sqrt{(2A_t \log\log A_t)}} = 1 \quad \text{a.s.} \qquad \square$$

II. PROBABILITY DENSITIES

5. Densities of point processes

Consider a renewal process $\tau = \{\tau_n, n = 1, 2, \ldots\}$, $\tau_1 = \sigma_0$, $\tau_n - \tau_{n-1} = \sigma_{n-1}$, $n = 2, 3, \ldots$. Let the probability density of the mutually independent random variables σ_n be $f(t, a)$ where a is unknown parameter. Let us wish to estimate the parameter from the observation of τ during the time interval $[0, T]$ by the maximum likelihood method. Let the observed points be

$$\tau_1 = s_1, \tau_2 = s_2, \ldots, \tau_k = s_k .$$

To get the likelihood function evaluate the probability

$$P(\tau_1 \in (s_1, s_1 + ds_1), \quad \tau_2 \in (s_2, s_2 + ds_2), \ldots, \tau_k \in (s_k, s_k + ds_k)) =$$
$$= f(s_1, a) \, ds_1 f(s_2 - s_1, a) \, ds_2 \ldots f(s_k - s_{k-1}, a) \, ds_k .$$

Moreover, the observation says that $\tau_{k+1} - \tau_k > T - s_k$. This event has probability

$$P(\tau_{k+1} - \tau_k > T - s_k) = \int_{T-s_k}^{\infty} f(z, a) \, dz .$$

Consequently, the likelihood function corresponding to the observation is

$$f(s_1, a) f(s_2 - s_1, a) \ldots f(s_k - s_{k-1}, a) \int_{T-s_k}^{\infty} f(z, a) \, dz .$$

Likelihood function is probability density (Radon-Nikodym derivative) with respect to a basic measure. In this section we shall investigate the densities of point processes in detail. First we shall deal with the situation, when besides of the realization of τ on $[0, T]$, $T < \infty$, no other random events are taken into account.

As in § 3, let the probability distribution of τ be defined by the conditional densities

$$f_1(t) = \frac{d}{dt} P(\tau_1 \leqq t), \quad f_n(t; t_1, \dots, t_{n-1}) = \frac{d}{dt} P(\tau_n \leqq t \mid \tau_1 = t_1, \dots, \tau_{n-1} = t_{n-1}), \quad n = 2, 3, \dots .$$

The records of the observation of τ during $[0, T]$ are sequences

$$\omega = (s_1, s_2, \dots, s_k)$$

satisfying $0 < s_1 < s_2 < \dots < s_k \leqq T$, and the void sequence $\omega = \emptyset$. Denote their totality by Ω_T. Further, set

$$\tau_j(\omega) = s_j, \quad j \leqq N_T(\omega) = k .$$

We have

$$P(N_T = 0) = \int_T^\infty f_1(z)\,dz ,$$

and for $k = 1, 2, \dots$

(26) $$P(N_T = k, \quad \tau_1 \in (s_1, s_1 + ds_1), \dots, \tau_k \in (s_k, s_k + ds_k)) =$$
$$= f_1(s_1)\,ds_1\, f_2(s_2; s_1)\,ds_2 \dots f_k(s_k; s_1, \dots, s_{k-1})\,ds_k \int_T^\infty f_{k+1}(z; s_1, \dots, s_k)\,dz .$$

In § 3 conditional rates were defined as

$$q_n(t; t_1, \dots, t_{n-1}) = f_n(t; t_1, \dots, t_{n-1}) / \int_t^\infty f_n\,dz .$$

Hence,

$$\int_t^\infty f_n\,dz = \exp\left\{-\int_0^t q_n\,dz\right\}, \quad f_n = q_n \exp\left\{-\int_0^t q_n\,dz\right\}.$$

(26) can therefore be rewritten as

(27) $$P(N_T = k, \tau_1 \in (s_1, s_1 + ds_1), \dots, \tau_k \in (s_k, s_k + ds_k)) =$$
$$= q_1(s_1) \exp\left\{-\int_0^{s_1} q_1\,dz\right\} ds_1\, q_2(s_2; s_1) \exp\left\{-\int_{s_1}^{s_2} q_2\,dz\right\} ds_2 \dots$$
$$\dots q_k(s_k; s_1, \dots, s_{k-1}) \exp\left\{-\int_{s_{k-1}}^{s_k} q_k\,dz\right\} ds_k \exp\left\{-\int_{s_k}^{T} q_{k+1}\,dz\right\}.$$

Recall the definition of rate Q (Fig. 4):

$$Q_t(\omega) = q_1(t), \quad 0 \leqq t \leqq s_1, \quad Q_t(\omega) = q_n(t; s_1, \dots, s_{n-1}), s_{n-1} < t \leqq s_n,$$
$$n = 2, \dots, k, \quad Q_t(\omega) = q_{k+1}(t; s_1, \dots, s_k), \quad s_k < t \leqq T .$$

The trajectories of $Q = \{Q_t, t \in [0, T]\}$ are assumed to be bounded and left-continuous. Introducing Q into (27) we get

$$P(N_T = k,\ \tau_1 \in (s_1, s_1 + ds_1), \dots, \tau_k \in (s_k, s_k + ds_k)) =$$

$$= \exp\left\{\int_0^T \log Q_z \, dN_z - \int_0^T Q_z \, dz\right\} ds_1 \dots ds_k .$$

For the Poisson process with unit rate holds

$$P_{\text{oiss.}}(N_T = k, \dots, \tau_k \in (s_k, s_k + ds_k)) = \frac{T^k}{k!} e^{-T} \frac{k!}{T^k} ds_1 \dots ds_k .$$

Hence, we obtain the probability density

$$\frac{dP}{dP_{\text{oiss.}}} = \exp\left\{\int_0^T \log Q_z \, dN_z + \int_0^T (1 - Q_z) \, dz\right\} . \tag{28}$$

The right-hand side of (28) depends only on ω and on $Q_z(\omega)$, $z \in [0, T]$. This proves the next statement.

Proposition 7. The rate Q determines the probability measure P on $(\Omega_T, \mathscr{F}_T^N)$ uniquely.

Next we shall generalize formula (28) slightly. Let P, P_0 denote probability distributions on $(\Omega_T, \mathscr{F}_T^N)$ associated with rates $Q = \{Q_t, t \in [0, T]\}$ and ${}^0Q = \{{}^0Q_t, t \in [0, T]\}$, respectively. Assume

$$Q_t = R_t \, {}^0Q_t , \quad t \in [0, T] , \tag{29}$$

where ${}^0Q$ as well as the ratio of the rates R have bounded left-continuous trajectories. The probability density L_T of P with respect to P_0 is then

$$L_T = \frac{dP}{dP_0} = \frac{dP}{dP_{\text{oiss.}}} \Big/ \frac{dP_0}{dP_{\text{oiss.}}} = \exp\left\{\int_0^T \log \frac{Q_z}{{}^0Q_z} \, dN_z + \int_0^T ({}^0Q_z - Q_z) \, dz\right\} =$$

$$= \exp\left\{\int_0^T \log R_z \, dN_z + \int_0^T (1 - R_z) \, {}^0Q_z \, dz\right\} =$$

$$= \left(\prod_{\tau_n \leqq T} R_{\tau_n}\right) . \exp\left\{\int_0^T (1 - R_z) \, {}^0Q_z \, dz\right\} .$$

The probability density on σ-algebra $\mathscr{F}_t^N$, $0 \leqq t \leqq T$, corresponding to the observation of N during $[0, t]$ is from the preceding

$$\left.\frac{dP}{dP_0}\right|_{\mathscr{F}_t^N} = \exp\left\{\int_0^t \log R_z \, dN_z + \int_0^t (1 - R_z) \, {}^0Q_z \, dz\right\} = L_t . \tag{30}$$

On the other hand

$$\left.\frac{\mathrm{d}P}{\mathrm{d}P_0}\right|_{\mathscr{F}_t^N} = \mathsf{E}_0\{L_T \mid \mathscr{F}_t^N\}.$$

Namely, from the definition of the conditional expectation we have for $A \in \mathscr{F}_t^N$

$$\int_A \mathsf{E}_0\{L_T \mid \mathscr{F}_t^N\}\,\mathrm{d}P_0 = \int_A L_T\,\mathrm{d}P_0 = P(A).$$

We conclude that $L = \{L_t, t \in [0, T]\}$ is a P_0-martingale.

In the general theory the following equation satisfied by L is of importance.

Proposition 8. It holds

$$L_t = 1 + \int_0^t L_{s-}(R_s - 1)\,\mathrm{d}^0M_s\,, \quad t \in [0, T]\,, \tag{31}$$

where

$${}^0M_t = N_t - \int_0^t {}^0Q_s\,\mathrm{d}s\,, \quad t \in [0, T]\,.$$

L is the unique solution of (31).

Proof. (31) means subsequently:

$$L_t = 1 + \int_0^t L_{s-}(1 - R_s)\,{}^0Q_s\,\mathrm{d}s\,, \qquad t \in [0, \tau_1)\,,$$

$$L_{\tau_1} = L_{\tau_1 -} + L_{\tau_1 -}(R_{\tau_1} - 1)\,,$$

$$L_t = L_{\tau_1} + \int_{\tau_1}^t L_{s-}(1 - R_s)\,{}^0Q_s\,\mathrm{d}s\,, \quad t \in [\tau_1, \tau_2), \ldots$$

First equation has unique solution $L_t = \exp\left\{\int_0^t (1 - R)\,{}^0Q\,\mathrm{d}z\right\}$. The second one implies

$$L_{\tau_1} = R_{\tau_1} \exp\left\{\int_0^{\tau_1} (1 - R)\,{}^0Q\,\mathrm{d}z\right\},$$

the third one

$$L_t = L_{\tau_1} \exp\left\{\int_{\tau_1}^t (1 - R)\,{}^0Q\,\mathrm{d}z\right\} = \exp\left\{\int_0^t \log R\,\mathrm{d}N + \int_0^t (1 - R)\,{}^0Q\,\mathrm{d}z\right\},$$

$$t \in [\tau_1, \tau_2)\ \text{etc.}$$

Thus we get stepwise that L defined by (30) is the unique solution of (31). □

The hypothesis that the trajectories of R are bounded implies the absolute continuity $P \prec P_0$ for $T < \infty$, because it guarantees that $L_T < \infty$ P_0-a.s. Let us pass to the case $T = \infty$.

Let $(\Omega, \mathscr{F}^N_\infty)$ be the basic space from the beginning of § 3. Let P, P_0 be probability distributions on $(\Omega, \mathscr{F}^N_\infty)$ satisfying (29) for all $T < \infty$. $L = \{L_t, t \geqq 0\}$ defined by (30) is under P_0 a nonnegative martingale with respect to $\mathscr{F}^N = \{\mathscr{F}^N_t, t \geqq 0\}$. Moreover, $\mathsf{E}_0 L_t = 1$, $t \geqq 0$, and hence, $\sup_{t \geqq 0} \mathsf{E}|L_t| < \infty$. This implies (see e.g. [3]) the existence of an L_∞ such that

$$\lim_{t\to\infty} L_t = L_\infty \quad \text{a.s.} \tag{32}$$

To establish $P \prec P_0$ we demonstrate

$$L_t = \mathsf{E}\{L_\infty \mid \mathscr{F}^N_t\}\,, \quad t \geqq 0\,. \tag{33}$$

Namely, (33) means for $t \geqq 0$

$$\int_A L_\infty \,\mathrm{d}P_0 = \int_A L_t \,\mathrm{d}P_0 = P(A)\,, \quad A \in \mathscr{F}^N_t\,. \tag{34}$$

Extending (34) to A from the σ-algebra $\mathscr{F}^N_\infty = \bigvee_{t \geqq 0} \mathscr{F}^N_t$, we get

$$\int_A L_\infty \mathrm{d}P_0 = P(A)\,, \quad A \in \mathscr{F}^N_\infty\,.$$

(34) follows by letting $t \to \infty$ in its second equality provided that

$$\lim_{t\to\infty} \mathsf{E}_0|L_t - L_\infty| = 0\,. \tag{35}$$

If L is uniformly integrable, (35) holds because of (32). To guarantee the uniform integrability we shall make assumptions derived from the Theorem of de la Valée-Poussin.

Lemma 5. Let $h(x)$, $x \geqq 0$, be a measurable function bounded from bellow and satisfying $\lim_{x\to\infty} h(x)/x = \infty$. If

$$\sup_{t \geqq 0} \mathsf{E}_0\, h(L_t) < \infty\,,$$

then L is uniformly integrable.

Proof. We have for K sufficiently large, $t \geqq 0$,

$$\int_{\{L_t \geqq K\}} L_t \,\mathrm{d}P_0 = \int_{\{L_t \geqq K\}} (L_t/h(L_t))\, h(L_t) \,\mathrm{d}P_0 \leqq$$

$$\leqq \sup_{x \geqq K} (x/h(x)) \sup_{s \geqq 0} \mathsf{E}_0(h(L_s) + \text{const.}).$$

The last term tends to 0 as $K \to \infty$, and does not depend on t. □

In our problem $h(x) = x \log x$ is a suitable choice. Assuming first

$$(36) \qquad 0 < \frac{1}{K} < R_z < K \,, \quad z \geqq 0 \,,$$

for a constant K we get

$$\mathsf{E}_0\, L_t \log L_t = \mathsf{E} \log L_t = \mathsf{E}\left(\int_0^t \log R_z \,\mathrm{d}N_z + \int_0^t (1 - R_z)\, {}^0Q_z \,\mathrm{d}z\right) =$$

$$= \mathsf{E}\left(\int_0^t \log R_z \mathrm{d}M_z + \int_0^t \log R_z Q_z \,\mathrm{d}z + \int_0^t (1 - R_z)\, {}^0Q_z \,\mathrm{d}z\right) =$$

$$= \mathsf{E}\int_0^t (1 + R_z \log R_z - R_z)\, {}^0Q_z \,\mathrm{d}z \,.$$

$\int_0^t \log R \,\mathrm{d}M$ has zero expectation, because it is an integral of a bounded left-continuous nonanticipative function with respect to a martingale. Note that $1 + x \log x - x \geqq 0$ for $x \geqq 0$.

In the general case, when (36) is dropped we shall limit ourselves to a short proof of

$$(37) \qquad \mathsf{E}_0\, L_t \log L_t \leqq \mathsf{E}\int_0^t (1 + R_z \log R_z - R_z)\, {}^0Q_z \,\mathrm{d}z \,, \quad t \geqq 0 \,.$$

But it can be shown that equality holds in (37). Introduce

$$\sigma_n = \tau_n \wedge \inf\{t : R_t \geqq n\} \,, \quad {}^mR_t = \max\left(R_t, \frac{1}{m}\right), \quad m, n = 1, 2, \ldots$$

Then as above

$$\mathsf{E}\left(\int_0^{t \wedge \sigma_n} \log {}^mR_z \,\mathrm{d}N_z + \int_0^{t \wedge \sigma_n} (1 - {}^mR_z)\, {}^0Q_z \,\mathrm{d}z\right) =$$

$$= \mathsf{E}\int_0^{t \wedge \sigma_n} (1 + {}^mR_z \log {}^mR_z - {}^mR_z)\, {}^0Q_z \,\mathrm{d}z + \mathsf{E}\int_0^{t \wedge \sigma_n} (R_z - {}^mR_z) \log {}^mR_z\, {}^0Q_z \,\mathrm{d}z \,.$$

Letting $m \to \infty$ one gets

$$\mathsf{E}_0\, L_{t \wedge \sigma_n} \log L_{t \wedge \sigma_n} = \mathsf{E}\int_0^{t \wedge \sigma_n} (1 + R_z \log R_z - R_z)\, {}^0Q_z \,\mathrm{d}z \,.$$

From here and from the Fatou Lemma, (37) follows as $n \to \infty$. Let us recapitulate the result.

Proposition 9. If

$$\mathsf{E}\int_0^\infty (1 + R_z \log R_z - R_z)\,{}^0Q_z\,\mathrm{d}z < \infty\,, \tag{38}$$

then $P \prec P_0$ on $\mathscr{F}^N_\infty$.

Example 10. Let P and P_0 be the probability distribution of the Poisson process with time-dependent rate $q(t)$, $t \geqq 0$, and constant rate q_0, respectively. (38) is in this case

$$\int_0^\infty (q_0 + q(z)(\log q(z) - \log q_0) - q(z))\,\mathrm{d}z < \infty\,. \tag{39}$$

Thus (39) implies $P \prec P_0$. Similarly,

$$\int_0^\infty (q(z) + q_0(\log q_0 - \log q(z)) - q_0)\,\mathrm{d}z < \infty\,, \tag{40}$$

implies $P_0 \prec P$. Adding (39) and (40) we get that

$$\int_0^\infty (q(z) - q_0)(\log q(z) - \log q_0)\,\mathrm{d}z < \infty$$

is sufficient for $P \sim P_0$. □

Example 11. Let P be the distribution of a pure birth Markov process with jump rates q_n, $n = 0, 1, \ldots$, and let P_0 be the distribution of the Poisson process with constant rate q. Recall that in the birth process the holding times

$$\sigma_0 = \tau_1\,, \quad \sigma_n = \tau_{n+1} - \tau_n\,, \quad n = 1, 2, \ldots,$$

are mutually independent and have exponential distribution with mean $1/q_n$, $n = 0, 1, \ldots$. Proposition 9 yields for $P \prec P_0$ the condition

$$\mathsf{E}\int_0^\infty (1 + R_z \log R_z - R_z)\,{}^0Q_z\,\mathrm{d}z = \mathsf{E}\sum_{n=0}^\infty \left(1 + \frac{q_n}{q}\log\frac{q_n}{q} - \frac{q_n}{q}\right) q\sigma_n =$$

$$= \sum_{n=0}^\infty \left(\frac{q}{q_n} + \log\frac{q_n}{q} - 1\right) < \infty\,.$$ □

6. Girsanov type theorems

In this section we shall deal with the problem of constructing a point process with a given rate. The results are an analogy of the Girsanov Theorem for the diffusion processes.

On a probability space $(\Omega, \mathscr{A}, P_0)$ let us have a point process $\tau = \{\tau_n, n = 1, 2, \ldots\} \sim N = \{N_t, t \geqq 0\}$ with respect to nondecreasing family of σ-algebras $\mathscr{F}$. Let τ have rate ${}^0Q = \{{}^0Q_t, t \geqq 0\}$ with left-continuous locally bounded trajectories. Further, let process $Q = \{Q_t, t \geqq 0\}$ be given by the relation

$$Q_t = R_t{}^0Q_t\,, \quad t \geqq 0\,,$$

where R is nonanticipative with respect to $\mathscr{F}$, and also has locally bounded left continuous trajectories. The aim is to define a probability measure P on $(\Omega, \mathscr{A})$ for which τ has rate Q. We shall mostly omit the words "with respect to $\mathscr{F}$" when speaking about rates, martingales etc.

In § 5 measure P was given. Here we proceed in the reversed direction. We define

$$L_t = \exp\left\{\int_0^t R_z\,\mathrm{d}N_z + \int_0^t (1 - R_z)\,{}^0Q_z\,\mathrm{d}z\right\}, \quad t \geqq 0.$$

According to Proposition 8, L satisfies (31).

Lemma 6. L is a supermartingale under P_0. Whenever

$$\mathsf{E}_0 L_t = 1\,, \quad t \geqq 0\,, \tag{41}$$

holds, then L is a martingale.

Proof. Set

$$\sigma_n = \inf\{s : |L_{s-}(R_s - 1)| \geqq n\}\,, \quad n = 1, 2, \ldots\,.$$

Then

$$L_{t\wedge\sigma_n} = 1 + \int_0^t \chi\{s \leqq \sigma_n\}\,L_{s-}(R_s - 1)\,\mathrm{d}^0M_s\,, \quad t \geqq 0. \tag{42}$$

The integrand in (42) is left-continuous, nonanticipative and uniformly bounded. Hence $\{L_{t\wedge\sigma_n}, t \geqq 0\}$, $n = 1, 2, \ldots$, are nonnegative martingales. Their limit as $n \to \infty$

$$L_t = \lim_{n\to\infty} L_{t\wedge\sigma_n}\,, \quad t \geqq 0\,,$$

is a supermartingale. Namely, from the Fatou Lemma applied to the conditional expectations we get for $0 \leqq s < t$

$$L_s = \lim_{n\to\infty} L_{s\wedge\sigma_n} = \lim_{n\to\infty} \mathsf{E}_0\{L_{t\wedge\sigma_n} \mid \mathscr{F}_s\} \geqq$$
$$\geqq \mathsf{E}_0\{\lim_{n\to\infty} L_{t\wedge\sigma_n} \mid \mathscr{F}_s\} = \mathsf{E}_0\{L_t \mid \mathscr{F}_s\}\,.$$

To prove the second assertion of Lemma 6 assume on the contrary that (41) holds, but L is not a martingale. Then, $\mathsf{E}_0\{L_t \mid \mathscr{F}_s\} < L_s$ with positive probability for some $t > s$. This implies $\mathsf{E}_0 L_t < \mathsf{E}_0 L_s \leqq 1$, which contradicts to (41). □

A sufficient condition for the validity of (41) is $\mathsf{E}_0 L_\infty = 1$, where

$$L_\infty = \lim_{t \to \infty} L_t \quad \text{a.s.}$$

exists according to the theorems on the convergence of supermartingales (see [3]).

The next proposition is an imitation of Girsanov's Theorem.

Proposition 10. Let $\mathsf{E}_0 L_\infty = 1$. Define probability measure P on $\mathscr{A}$ by the integral

$$P(B) = \int_B L_\infty \, \mathrm{d}P_0 \,, \quad B \in \mathscr{A} .$$

If

$$\mathsf{E} N_t = \mathsf{E}_0 N_t L_t < \infty \,, \quad t \geqq 0 \,, \tag{43}$$

then

$$M_t = N_t - \int_0^t Q_s \, \mathrm{d}s \,, \quad t \geqq 0 \,,$$

is a martingale with respect to $\mathscr{F}$ on $(\Omega, \mathscr{A}, P)$.

The proof is based on the following relationship between martingales under P and under P_0.

Lemma 7. If $ML = \{M_t L_t, \, t \geqq 0\}$ is a P_0-martingale, then M is a P-martingale.

Proof. Take $0 \leqq s < t$. Let us show first that

$$\mathsf{E}\{M_t \mid \mathscr{F}_s\} = \mathsf{E}_0\{M_t L_t \mid \mathscr{F}_s\} \, L_s^{-1} \,. \tag{44}$$

For $A \in \mathscr{F}_s$ we have using $\mathrm{d}P = L_\infty \, \mathrm{d}P_0$

$$\int_A \mathsf{E}_0\{M_t L_t \mid \mathscr{F}_s\} \, L_s^{-1} \, \mathrm{d}P = \int_A \mathsf{E}_0\{M_t L_t \mid \mathscr{F}_s\} \, L_s^{-1} \, \mathsf{E}_0\{L_\infty \mid \mathscr{F}_s\} \, \mathrm{d}P_0 =$$

$$= \int_A M_t L_t \, \mathrm{d}P_0 = \int_A M_t L_\infty \, \mathrm{d}P_0 = \int_A M_t \, \mathrm{d}P \,.$$

This proves (44). If ML is a P_0-martingale, then

$$\mathsf{E}_0\{M_t L_t \mid \mathscr{F}_s\} \, L_s^{-1} = M_s L_s L_s^{-1} = M_s \,.$$

From here and from (44) the martingale property of M follows. □

Proof of Proposition 10. Let the hypotheses be fulfilled. Let σ be a stopping time, $t \geqq 0$ arbitrary. We begin by demonstrating the relation

$$M_{t \wedge \sigma} L_t = \int_0^t M_{v \wedge \sigma -} \, \mathrm{d}L_v + \int_0^t \chi\{v \leqq \sigma\} \, L_{v-} \, \mathrm{d}^0 M_v + L_{t \wedge \sigma} - 1 \,, \quad t \geqq 0, \tag{45}$$

where

$$ {}^0M_t = N_t - \int_0^t {}^0Q_s\,\mathrm{d}s\,, \quad t \geqq 0\,. $$

We have

$$ M_{t\wedge\sigma}L_t = M_{t\wedge\sigma} + M_{t\wedge\sigma}(L_t - 1) = M_{t\wedge\sigma} + \iint_B \mathrm{d}M_u\,\mathrm{d}L_v, $$

where

(46) $$ B = \{0 \leqq u \leqq t \wedge \sigma,\ 0 \leqq v \leqq t\} = \{0 \leqq v \leqq t,\ 0 \leqq u < v \wedge \sigma\} \cup $$
$$ \cup \{0 \leqq v < u \leqq t \wedge \sigma\} \cup \{0 \leqq u = v \leqq t \wedge \sigma\}\,. $$

Decomposing the integral over B according to (46) we get

(47) $$ M_{t\wedge\sigma}L_t = \int_0^t M_{v\wedge\sigma-}\,\mathrm{d}L_v + \int_0^{t\wedge\sigma} L_{u-}\,\mathrm{d}M_u + \sum_{v\leqq t\wedge\sigma} (L_v - L_{v-})(M_v - M_{v-}) = $$
$$ = \int_0^t M_{v\wedge\sigma-}\,\mathrm{d}L_v + \int_0^{t\wedge\sigma} L_{v-}(\mathrm{d}^0M_v + (1 - R_v)\,{}^0Q_v\,\mathrm{d}v) + \sum_{v\leqq t\wedge\sigma} (L_v - L_{v-})(N_v - N_{v-}). $$

Further, from (31),

(48) $$ \int_0^{t\wedge\sigma} L_{v-}(1 - R_v)\,{}^0Q_v\,\mathrm{d}v = L_{t\wedge\sigma} - 1 + \int_0^{t\wedge\sigma} L_{v-}(1 - R_v)\,\mathrm{d}N_v = $$
$$ = L_{t\wedge\sigma} - 1 + \sum_{\tau_n\leqq t\wedge\sigma} L_{\tau_n-}(1 - R_{\tau_n}) = L_{t\wedge\sigma} - 1 - \sum_{v\leqq t\wedge\sigma} (L_v - L_{v-})(N_v - N_{v-})\,, $$

because of

$$ L_{\tau_n} = \big(\prod_{k\leqq n} R_{\tau_k}\big) \exp\left\{\int_0^{\tau_n} (1 - R)\,{}^0Q\,\mathrm{d}z\right\} = R_{\tau_n}L_{\tau_n-}\,. $$

Inserting (48) into (47) we get (45).

Expression (45) of $M_{t\wedge\sigma}L_t$ contains integrals of left-continuous nonanticipative functions with respect to P_0-martingales and a P_0-martingale. Consequently, if σ is chosen so that the integrands are bounded, then $M_{t\wedge\sigma}L_t$, $t \geqq 0$, is itself a P_0-martingale. This is the case for the stopping times

$$ \sigma_n = \inf\{v : |M_v| + |L_v| \geqq n\}\,, \quad n = 1, 2, \ldots\,. $$

Hence, by Lemma 7, $\{M_{t\wedge\sigma_n}, t \geqq 0\}$, $n = 1, 2, \ldots$, are P-martingales.

We have

$$ 0 = \mathsf{E}M_{t\wedge\sigma_n}\,, \quad \text{i.e.} \quad \mathsf{E}N_{t\wedge\sigma_n} = \mathsf{E}\int_0^{t\wedge\sigma_n} Q\,\mathrm{d}s\,. $$

From here with regard to (43) we obtain letting $n \to \infty$

$$ \infty > \mathsf{E}N_t = \mathsf{E}\int_0^t Q\,\mathrm{d}s\,, \quad t \geqq 0\,. $$

The sequence of martingales $\{M_{t\wedge\sigma_n}, t \geqq 0\}$, $n = 1, 2, \ldots$, has therefore an integrable majorant

$$N_T + \int_0^T Q \, \mathrm{d}s \geqq N_{t \wedge \sigma_n} + \int_0^{t \wedge \sigma_n} Q \, \mathrm{d}s \geqq |M_{t \wedge \sigma_n}| \,, \quad t \in [0, T] \,,$$

where $T < \infty$ is arbitrary. This implies that

$$M_t = \lim_{n \to \infty} M_{t \wedge \sigma_n} \,, \quad t \geqq 0 \,,$$

is a P-martingale, as Proposition 10 asserts. □

For the application of Proposition 10 we need to have conditions guaranteeing $\mathsf{E}_0 L_\infty = 1$. The condition which we introduce employs the results of § 5 on uniform integrability.

Proposition 11. Let

$$\int_0^\infty (1 + R_z \log R_z - R_z) \, {}^0Q_z \, \mathrm{d}z \leqq c \,, \quad P_0\text{-a.s.} \,, \tag{49}$$

where c is a constant. Then $\mathsf{E}_0 L_\infty = 1$.

Proof. We shall show that under (49) is L a uniformly integrable P_0-martingale. Recall relation (31),

$$L_t = 1 + \int_0^t L_{s-}(R_s - 1) \, \mathrm{d}^0 M_s \,, \quad t \geqq 0 \,.$$

Again, to get on the right-hand side the integral of a bounded function, and hence a martingale, we stop the trajectory. Let

$$\sigma_n = \inf \{ s : |L_{s-}(R_s - 1)| \geqq n \} \,, \quad n = 1, 2, \ldots \,.$$

Then

$$L_{t \wedge \sigma_n} = 1 + \int_0^t \chi\{s \leqq \sigma_n\} \, L_{s-}(R_s - 1) \, \mathrm{d}^0 M_s \,, \quad t \geqq 0 \,, \quad n = 1, 2, \ldots \,,$$

are P_0-martingales, $\mathsf{E}_0 L_{t \wedge \sigma_n} = \mathsf{E}_0 L_0 = 1$. An inequality analogous to (37) is

$$\mathsf{E}_0 L_{t \wedge \sigma_n} \log L_{t \wedge \sigma_n} \leqq \mathsf{E}_0 L_{t \wedge \sigma_n} \int_0^{t \wedge \sigma_n} (1 + R_z \log R_z - R_z) \, {}^0Q_z \, \mathrm{d}z \,.$$

Hence, with regard to (49)

$$\mathsf{E}_0 L_{t \wedge \sigma_n} \log L_{t \wedge \sigma_n} \leqq c \,, \quad t \geqq 0 \,, \quad n = 1, 2, \ldots \,. \tag{50}$$

By Lemma 5, the random variables

$$L_{t \wedge \sigma_n} \,, \quad t \geqq 0 \,, \quad n = 1, 2, \ldots \,,$$

form a uniformly integrable family. Letting $n \to \infty$ we obtain that L is a P_0-martingale. Moreover, from (50) and from the Fatou Lemma follows

$$\mathsf{E}_0 L_t \log L_t \leqq c \,, \quad t \geqq 0 \,.$$

Consequently, L is uniformly integrable by Lemma 5, and this implies

$$1 = \lim_{t \to \infty} \mathsf{E}_0 L_t = \mathsf{E}_0 L_\infty . \qquad \square$$

We concentrated on point processes on the infinite interval. Same statements are true for point processes on a bounded interval. They can be conceived as processes with infinite range such that $\tau_n > T$ implies $\tau_n = \infty$.

Example 12. For illustration consider again the Poisson process with a variable and a constant rate. In § 5 we proved the following: If $N = \{N_t, t \in [0, T]\}$ is the Poisson process with unit rate on the basic space $(\Omega_T, \mathscr{F}_T^N, P_0)$, then for

$$L_T = \exp\left\{\int_0^T \log q(z)\, \mathrm{d}N_z + \int_0^T (1 - q(z))\, \mathrm{d}z\right\},$$

$$P(B) = \int_B L_T \, \mathrm{d}P_0 , \quad B \in \mathscr{F}_T^N ,$$

N is the Poisson process with rate $q(t)$, $t \in [0, T]$, on $(\Omega_T, \mathscr{F}_T^N, P)$. The results of the present section say that the same is true, if N is a Poisson process with respect to $\mathscr{F} = \{\mathscr{F}_t, t \in [0, T]\}$ on an arbitrary basic space $(\Omega, \mathscr{A}, P_0)$.

Consider now the infinite process. Let $N = \{N_t, t \geqq 0\}$ be a Poisson process with respect to $\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$ on $(\Omega, \mathscr{A}, P_0)$ having unit rate. Let a given variable rate $q(t)$, $t \geqq 0$, satisfy

$$\int_0^\infty (1 + q(z) \log q(z) - q(z))\, \mathrm{d}z < \infty .$$

According to Propositions 10, 11 for

$$L_\infty = \lim_{t \to \infty} L_t \quad P_0\text{-a.s.} , \quad P(B) = \int_B L_\infty \, \mathrm{d}P_0 , \quad B \in \mathscr{A} ,$$

we have that N is a Poisson process with respect to $\mathscr{F}$ on $(\Omega, \mathscr{A}, P)$ having rate $q(t)$, $t \geqq 0$. $\square$

Let us present the extension of Proposition 10 to *labelled point processes* $\tau_\Lambda = \{(\tau_n, \lambda_n), n = 1, 2, \ldots\}$ with set $I = \{1, \ldots, m\}$ of admissible labels. We introduce the counting process $N = \{N_t = ({}^1N_t, \ldots, {}^mN_t), t \geqq 0\}$, where

$${}^iN_t = \sum_{n=1}^{\infty} \chi\{\tau_n \leqq t, \lambda_n = i\} , \quad t \geqq 0 , \quad i \in I .$$

Let the process be defined on the probability space $(\Omega, \mathscr{A}, P_0)$. Let the rate of ${}^iN$ with respect to nondecreasing family of σ-algebras $\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$ be ${}^{i0}Q = \{{}^{i0}Q_t, t \geqq 0\}$. We want to define on $(\Omega, \mathscr{A})$ a probability measure P such that, for $i = 1, \ldots, m$, ${}^iN$ has rate ${}^iQ$ defined as

$${}^iQ_t = {}^iR_t \, {}^{i0}Q_t , \quad t \geqq 0 .$$

As in the preceding we assume that ${}^{i0}Q$, ${}^{i}Q$, ${}^{i}R$, $i \in I$, have left-continuous locally bounded trajectories.

Proposition 12. Set

$$L_t = \exp\left\{\sum_{i=1}^{m}\left(\int_0^t \log {}^{i}R_z \, \mathrm{d}{}^{i}N_z + \int_0^t (1 - {}^{i}R_z)\, {}^{i0}Q_z \, \mathrm{d}z\right)\right\}, \quad t \geqq 0 .$$

L is a supermartingale with respect to $\mathscr{F}$ on $(\Omega, \mathscr{A}, P_0)$.

Let $\mathsf{E}_0 L_\infty = 1$. Then L is a martingale, and the integral

$$P(B) = \int_B L_\infty \, \mathrm{d}P_0 , \quad B \in \mathscr{A} ,$$

defines a probability measure on $(\Omega, \mathscr{A})$. If in addition

$$\mathsf{E}\, {}^{i}N_t = \mathsf{E}_0\, {}^{i}N_t L_t < \infty , \quad t \geqq 0 , \quad i \in I ,$$

then

$${}^{i}M_t = {}^{i}N_t - \int_0^t {}^{i}Q_s \, \mathrm{d}s , \quad t \geqq 0 , \quad i \in I ,$$

are martingales with respect to $\mathscr{F}$ *on* $(\Omega, \mathscr{A}, P)$.

The proof, which is the same as in the case $m = 1$, employs the relations

$$L_t = 1 + \sum_{i=1}^{m} \int_0^t L_{s-}({}^{i}R_s - 1)\, \mathrm{d}^{i0}M_s , \quad t \geqq 0 ,$$

$${}^{i}M_{t\wedge\sigma} L_t = \int_0^t {}^{i}M_{v\wedge\sigma-} \, \mathrm{d}L_v + \int_0^t \chi\{v \leqq \sigma\}\, L_{v-}\, \mathrm{d}^{i0}M_v +$$

$$+ \sum_{j \neq i} \int_0^t \chi\{v \leqq \sigma\}\, L_{v-}(1 - {}^{j}R_v)\, \mathrm{d}^{j0}M_v + Z_{t\wedge\sigma} - 1 , \quad t \geqq 0 , \quad i \in I .$$

7. Controlled Markov processes

In § 2 we dealt with Markov processes $X = \{X_t, t \geqq 0\}$ having finite state space I, say $I = \{1, \ldots, m\}$. X was supposed to be defined on the space Ω fo paths $\omega(t)$ as $X_t(\omega) = \omega(t)$. $\mathscr{F} = \{\mathscr{F}_t, t \geqq 0\}$ was the family generated by X itself. With a controlled Markov process we associated the transition rates

$$q(i, j, z) , \quad i, j \in I , \quad z \in J . \tag{51}$$

In (51) z denotes the control parameter. The initial distribution

$$P(X_0 = i) = p_i , \quad i \in I , \tag{52}$$

was fixed. We introduced the probability distribution of X under control $Z = \{Z_t, t \geqq 0\}$ as the probability measure P^Z on $(\Omega, \mathscr{F}_\infty)$ satisfying (52) and such that

$${}^{i}\overline{M}_t = \chi\{X_t = i\} - \int_0^t q(X_s^-, i; Z_s)\, \mathrm{d}s , \quad t \geqq 0 , \quad i \in I ,$$

are martingales with respect to $\mathscr{F}$. In what follows it is more convenient to insert the left-continuous version $\{X_t^-, t \geqq 0\}$ of X into the rate. The question of the existence and the unicity of P^Z was left open. To answer it, we shall use the results on labelled point processes stated in § 6.

Assume for the sake of simplicity that the process starts from a given state i_0, i.e. $P(X_0 = i_0) = 1$. Denote by τ_n, $n = 1, 2, \ldots$, the increasing sequence of times, in which the trajectory of X has a jump. There is a unique correspondence between the trajectory of X and the trajectory of the labelled point process $\tau_A = \{(\tau_n, X_{\tau_n}), n = 1, 2, \ldots\}$. Let the counting process of τ_A be $N = \{({}^1N_t, \ldots, {}^mN_t), t \geqq 0\}$. ${}^iN_t$ is the number of jumps into state i performed by X until time t,

$$ {}^iN_t = \sum_{s \leqq t} \chi\{X_s^- \neq X_s = i\}, \quad t \geqq 0, \quad i \in I. $$

In accordance with the intuitive meaning of the transition rates, the rate of occurrence of the jumps into i is

$$ {}^iQ_t = (1 - \chi\{X_t^- = i\})\, q(X_t^-, i; Z_t), \quad t \geqq 0. \tag{53} $$

The rate is taken with respect to $\mathscr{F}$. If $X_t^- = i$, the rate equals 0, since the transitions from i into i are not possible. Thus, under P^Z, by subtracting from ${}^iN$ the integral of ${}^iQ$ we expect to obtain a martingale

$$ {}^iM_t = {}^iN_t - \int_0^t (1 - \chi\{X_s^- = i\})\, q(X_s^-, i; Z_s)\, \mathrm{d}s, \quad t \geqq 0. $$

Next proposition shows that this is the case. Moreover, the two characterizations of P^Z are equivalent.

Proposition 13. Let Z be a control, P a probability measure on $(\Omega, \mathscr{F}_\infty)$. Then ${}^iM$, $i \in I$, are martingales on $(\Omega, \mathscr{F}_\infty, P)$ if and only if ${}^i\overline{M}$, $i \in I$, are martingales.

Proof. The proof follows from the relations

$$ {}^iM_t = \int_0^t (1 - \chi\{X_s^- = i\})\, \mathrm{d}\chi\{X_s = i\} - $$

$$ - \int_0^t (1 - \chi\{X_s^- = i\})\, q(X_s^-, i; Z_s)\, \mathrm{d}s = \int_0^t (1 - \chi\{X_s^- = i\})\, \mathrm{d}{}^i\overline{M}_s, \quad t \geqq 0, \quad i \in I, $$

$$ {}^i\overline{M}_t = {}^iN_t - \sum_{j \neq i} \int_0^t \chi\{X_s^- = i\}\, \mathrm{d}{}^jN_s - \int_0^t (1 - \chi\{X_s^- = i\})\, q(X_s^-, i; Z_s)\, \mathrm{d}s + $$

$$ + \int_0^t \chi\{X_s^- = i\} \sum_{j \neq i} q(X_s^-, j; Z_s)\, \mathrm{d}s = {}^iM_t - \sum_{j \neq i} \int_0^t \chi\{X_s^- = i\}\, \mathrm{d}{}^jM_s, \quad t \geqq 0, \quad i \in I, $$

and from the fact that the integral with respect to a martingale of a bounded left-continuous nonanticipative function is a martingale. □

In the sequel we shall assume that the control parameter set J is compact, and that the transition rates (51) are continuous in Z on J. We shall consider controlled process $\{X_t,\ t \in [0, T]\}$, whose distributions are probability measures P_T^Z on the σ-algebra $\mathscr{F}_T$. T is finite. As the probability measure P_0 with respect to which the density of P_T^Z will be calculated, we take the probability distribution of the Markov process with the transition rates matrix

$$\text{(54)} \qquad \begin{pmatrix} -(m-1), & 1 & 1, \dots, & 1 \\ 1, & -(m-1), & 1, \dots, & 1 \\ \dots & \dots & \dots & \dots \\ 1, & 1, & 1, \dots, & -(m-1) \end{pmatrix}$$

and initial distribution $P_0(X_0 = i_0) = 1$.

Recall from Proposition 12 the formula for the density of a point process

$$L_T = \exp\left\{\sum_{i=1}^{m}\left(\int_0^T \log {}^iR_s \, \mathrm{d}{}^iN_s + \int_0^T (1 - {}^iR_s)\, {}^{i0}Q_s \, \mathrm{d}s\right)\right\}.$$

Set, with regard to (54) and (53),

$${}^{i0}Q_s = 1 - \chi\{X_s^- = i\}\,, \quad {}^iR_s = (1 - \chi\{X_s^- = i\})\, q(X_s^-, i; Z_s)\,.$$

Then,

$$\sum_i \int_0^T \log {}^iR_s \, \mathrm{d}{}^iN_s = \int_0^T \log q(X_s^-, X_s; Z_s)\, \mathrm{d}\bar{N}_s\,,$$

where

$$\bar{N}_s = \sum_i {}^iN_s\,, \quad s \in [0, T]\,,$$

is the counting process of all jumps of X. Further,

$$- \sum_i {}^iR_s\, {}^{i0}Q_s = - \sum_i (1 - \chi\{X_s^- = i\})\, q(X_s^-, i; Z_s) =$$
$$= - \sum_{i \neq X_s^-} q(X_s^-, i; Z_s) = q(X_s^-, X_s^-; Z_s)\,.$$

It results that

$$\text{(55)} \quad L_T = \exp\left\{\int_0^T \log q(X_s^-, X_s; Z_s)\, \mathrm{d}\bar{N}_s + \int_0^T (m - 1 + q(X_s^-, X_s; Z_s))\, \mathrm{d}s\right\}.$$

Sufficient condition for $\mathsf{E}_0 L_T = 1$, namely

$$\sum_i \int_0^T (1 + {}^iR_s \log {}^iR_s - {}^iR_s)\, {}^{i0}Q_s \, \mathrm{d}s \leq \text{const.}$$

is fulfilled in virtue of the boundedness of ${}^iR$, ${}^{i0}Q$, $i \in I$. Probability distribution P_T^Z can therefore be defined as

$$P_T^Z(B) = \int_B L_T \, \mathrm{d}P_0\,, \quad B \in \mathscr{F}_T\,.$$

Its uniqueness follows from the fact that the probability distribution of a point process is uniquely determined by its rate. This was proved in § 5 for point processes with one type of events.

III. APPLICATIONS OF LIMIT THEOREMS FOR MARTINGALES

8. Limit theorems

We begin this chapter with recalling the law of large numbers and the central limit theorem for discrete parameter martingales. We denote by $M = \{M_n, n = 0, 1, \ldots\}$ a martingale with respect to a nondecreasing sequence of σ-algebras $\mathscr{F} = \{\mathscr{F}_n, n = 0, 1, \ldots\}$. Further, let

$$Y_n = M_{n+1} - M_n , \quad n = 0, 1, \ldots ,$$

be the martingale differences.

Proposition 14. If

$$\sum_{k=1}^{\infty} k^{-2} \mathsf{E} Y_k^2 < \infty ,$$

then

$$\lim_{n\to\infty} n^{-1} M_n = 0 \quad \text{a.s.}$$

Proof. Set

$$M_0' = 0 , \quad M_n' = \sum_{k=0}^{n-1} \frac{1}{k+1} Y_k , \quad n = 1, 2, \ldots$$

M' is martingale with respect to $\mathscr{F}$. Doob's submartingale inequality yields for $\varepsilon > 0$

$$P(\sup_{1\leq k\leq m} |M'_{n+k} - M'_n| \geqq \varepsilon) \leqq \frac{1}{\varepsilon^2} \mathsf{E}(M'_{n+m} - M'_n)^2 = \frac{1}{\varepsilon^2} \sum_{k=n}^{n+m-1} \frac{1}{(k+1)^2} \mathsf{E} Y_k^2 .$$

Letting $m \to \infty$ we obtain

$$P(\bigcap_{n=0}^{\infty} \{ \sup_{k=1,2,\ldots} |M'_{n+k} - M'_n| > \varepsilon\}) \leqq \lim_{n\to\infty} \frac{1}{\varepsilon^2} \sum_{k=n}^{\infty} \frac{1}{(k+1)^2} \mathsf{E} Y_k^2 = 0 .$$

Consequently, finite limit $\lim_{n\to\infty} M'_n = M'_\infty$ a.s. exists. We conclude that

$$\lim_{n\to\infty} n^{-1} M_n = \lim_{n\to\infty} (n^{-1} M_0 - n^{-1} \sum_{k=1}^{n-1} M'_k + M'_n) = -M'_\infty + M'_\infty = 0 \quad \text{a.s.} \qquad \square$$

Proposition 15. Let $\mathsf{E} Y_k^2 < \infty$, $k = 0, 1, \ldots$ Denote

$$S_n = \sum_{k=0}^{n-1} \mathsf{E}\{Y_k^2 \mid \mathscr{F}_k\} , \quad n = 1, 2, \ldots ,$$

and let the following hold:

$$\lim_{n\to\infty} n^{-1} S_n = \sigma^2 \quad \text{in probability,} \tag{56}$$

where $0 \leqq \sigma^2 < \infty$ is a constant;

(57) $$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathsf{E}\{Y_k^2\, \chi\{|Y_k| \geqq \varepsilon \sqrt{n}\} \mid \mathscr{F}_k\} = 0 \quad \text{in prob.},$$

for each $\varepsilon > 0$. Then $M_n/\sqrt{n}$ has asymptotically normal distribution $N(0, \sigma^2)$ as $n \to \infty$.

We shall give the proof of the asymptotic normality under the hypothesis that for a constant K

(58) $$|Y_k| \leqq K\,, \quad k = 0, 1, \ldots,$$

more restrictive than the Lindeberg type condition (57). We assume $M_0 = 0$ without losing generality. In terms of the characteristic functions we have to show that

(59) $$\lim_{n\to\infty} \mathsf{E} \exp\{\mathrm{i}u\, M_n/\sqrt{n}\} = \exp\{-\tfrac{1}{2}u^2\sigma^2\}\,,$$

where i is the imaginary unit and u a real parameter. To this purpose introduce

$$A_n = \exp\{(\mathrm{i}u\, M_n/\sqrt{n}) + (\tfrac{1}{2}u^2\, S_n/n)\} - 1\,, \quad n = 1, 2, \ldots$$

From (56), (58) it is seen that (59) is equivalent to

(60) $$\lim_{n\to\infty} \mathsf{E}A_n = 0\,.$$

Write

$$A_n = \sum_{k=1}^{n} B_{nk}\,, \quad n = 1, 2, \ldots,$$

where

$$B_{nk} = \exp\{(\mathrm{i}u\, M_k/\sqrt{n}) + (\tfrac{1}{2}u^2\, S_k/n)\} - \exp\{(\mathrm{i}u\, M_{k-1}/\sqrt{n}) + $$
$$+ (\tfrac{1}{2}u^2\, S_{k-1}/\sqrt{n})\} = \exp\{(\mathrm{i}u\, M_{k-1}/\sqrt{n}) + (\tfrac{1}{2}u^2\, S_k/n)\}\,.$$
$$.\, [\exp\{\mathrm{i}u\, Y_{k-1}/\sqrt{n}\} - \exp\{-\tfrac{1}{2}u^2\, \mathsf{E}\{Y_{k-1}^2 \mid \mathscr{F}_{k-1}\}/n\}]\,.$$

Further,

$$|\mathsf{E}A_n| = |\mathsf{E} \sum_{k=1}^{n} \mathsf{E}\{B_{nk} \mid \mathscr{F}_{k-1}\}| \leqq \mathsf{E} \sum_{k=1}^{n} |\mathsf{E}\{B_{nk} \mid \mathscr{F}_{k-1}\}| \leqq$$

(61) $$\leqq \exp\{\tfrac{1}{2}u^2K^2\} \sum_{k=1}^{n} \mathsf{E}|\mathsf{E}\{\exp\{\mathrm{i}u\, Y_{k-1}/\sqrt{n}\}| \mathscr{F}_{k-1}\} -$$
$$- \exp\{-\tfrac{1}{2}u^2 \mathsf{E}\{Y_{k-1}^2 \mid \mathscr{F}_{k-1}\}/n\}|\,.$$

Next we employ the relations

$$\mathrm{e}^{\mathrm{i}x} = 1 + \mathrm{i}x - \tfrac{1}{2}x^2 + h_1(x)\,, \quad |h_1(x)| \leqq |x^3|/6\,,$$
$$\mathrm{e}^{-x} = 1 - x + h_2(x)\,, \qquad |h_2(x)| \leqq x^2/2\,, \quad x \geqq 0\,.$$

From them it follows

$$\mathsf{E}\{\exp\{iu\,Y_{k-1}/\sqrt{n}\}\mid \mathscr{F}_{k-1}\} = 1 + \frac{iu}{\sqrt{n}}\,\mathsf{E}\{Y_{k-1}\mid \mathscr{F}_{k-1}\} - $$

$$- \frac{u^2}{2n}\,\mathsf{E}\{Y_k^2\mid \mathscr{F}_{k-1}\} + H_1\,,$$

where

$$|H_1| \leqq \frac{1}{6}\left(\frac{|u|\,K}{\sqrt{n}}\right)^3, \quad \mathsf{E}\{Y_{k-1}\mid \mathscr{F}_{k-1}\} = 0\,,$$

and

$$\exp\{-\tfrac{1}{2}u^2\mathsf{E}\{Y_{k-1}^2\mid \mathscr{F}_{k-1}\}/n\} = 1 - \frac{u^2}{2n}\,\mathsf{E}\{Y_{k-1}^2\mid \mathscr{F}_{k-1}\} + H_2\,,$$

with

$$|H_2| \leqq \tfrac{1}{2}\left(\frac{u^2K^2}{2n}\right)^2.$$

Thus, from (61) we obtain the estimate

$$|\mathsf{E}A_n| \leqq \mathrm{e}^{\frac{1}{2}(u^2K^2)}\,n\left(\frac{1}{6}\left(\frac{|u|\,K}{\sqrt{n}}\right)^3 + \frac{1}{2}\left(\frac{u^2K^2}{2n}\right)^2\right), \quad n = 1, 2, \ldots.$$

As $n \to \infty$ the right-hand side tends to 0. This proves (60). □

When investigating continuous time martingales $\{M_t,\, t \geqq 0\}$, to apply the limit theorems we establish first their validity for $\{M_n,\, n = 0, 1, \ldots\}$, and then prove the negligibility of the differences $M_t - M_{[t]}$, $t \geqq 0$. $[t]$ denotes the integer part of t.

9. Renewal processes with preventive replacements

Preventive replacements in renewal processes were mentioned in § 1. Here we return to them to illustrate the use of the limit theorems for martingales. As in § 1 we consider machine components whose life times have distribution function $F(t)$. One component is in operation. *Service replacements* after failure at cost c_1 or *preventive replacements* before failure at cost c_2 can be made. It is $c_1 > c_2 > 0$. We imagine an infinite stock of components with mutually independent life times. The replacements are instantaneous without causing delays. The objective is to minimize the average cost per unit operation time of the machine by means of an appropriate rule for making preventive replacements.

The age replacements policies (see Fig. 3) are specified by the replacement age x. If the operation time of the component reaches x, it is replaced by a new one. Let $\theta(x)$ denote the corresponding average cost per unit time. It is not difficult to calculate it. Namely,

$$\theta(x) = \frac{\text{average cost of replacement}}{\text{average time between replacements}} =$$

$$= \frac{c_1 F(x) + c_2(1 - F(x))}{\int_0^x y \, \mathrm{d}F(y) + x(1 - F(x))} = \frac{c_1 F(x) + c_2 \bar{F}(x)}{\int_0^x \bar{F}(y) \, \mathrm{d}y},$$

where $\bar{F}(x) = 1 - F(x)$. Let us denote by d the optimal replacement age. We make the following hypotheses:

1. There exists a $d \in (0, \infty)$ such that

$$\theta(d) \leqq \theta(x), \quad x \in (0, \infty).$$

I.e., we exclude $d = \infty$. We shall write briefly θ for $\theta(d)$.

2. The components have failure rate $q(t)$ nondecreasing and continuous on $[0, \infty)$. Hence,

$$\bar{F}(t) = \exp\left\{-\int_0^t q(y) \, \mathrm{d}y\right\}, \quad \frac{\mathrm{d}}{\mathrm{d}t} F(t) = f(t) = q(t) \exp\left\{-\int_0^t q(y) \, \mathrm{d}y\right\}.$$

The optimal value d can usually be obtained by equating to zero the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \theta(x) = ((c_1 - c_2) f(x) - \bar{F}(x) \theta(x)) \Big/ \int_0^x \bar{F}(y) \, \mathrm{d}y.$$

We get

(62) $$(c_1 - c_2) q(d) - \theta = 0.$$

This is a satisfactory solution of the problem. But only age replacement policies were taken into account. We have to show that the result cannot be improved by using other policies. Moreover, if the distribution of the life times is unknown, it is to be estimated during the replacement process. This leads to the self-optimizing policies ([1]). To proceed further we shall first define the concept of a replacement policy in generality.

We denote by $X = \{X_t, t \geqq 0\}$ the age of the component in operation. Let X be left-continuous, $X_0 = 0$. We identify the replacement process with a labelled point process $\tau_A = \{(\tau_n, \lambda_n), n = 1, 2, \ldots\}$ having labels 1, 2. Label 1 marks the service replacements, label 2 the preventive replacements. τ_A has bivariate counting process $N = \{N_t = ({}^1N_t, {}^2N_t), t \geqq 0\}$,

$${}^iN_t = \sum_{n=1}^{\infty} \chi\{\tau_n \leqq t, \lambda_n = i\}, \quad t \geqq 0, \quad i = 1, 2.$$

For the sake of definiteness, we shall assume that τ_A is defined on the space Ω of its trajectories. Thus, Ω has elements $\omega = \{(s_n, i_n), n = 1, 2, \ldots\}$, where $\{s_n, n = 1, 2, \ldots\}$ is a nondecreasing sequence of numbers, positive or ∞, satisfying

$\lim_{n\to\infty} s_n = \infty$, $s_n < s_{n+1}$ if $s_{n+1} \neq \infty$, and $i_n = 1$ or 2. We set

$$\tau_n(\omega) = s_n\,, \quad \lambda_n(\omega) = i_n\,, \quad n = 1, 2, \ldots .$$

The age process X fulfils

$$X_t = t\,, \quad 0 \leqq t \leqq \tau_1\,, \quad X_t = t - \tau_n\,, \quad \tau_n < t \leqq \tau_{n+1}\,, \quad n = 1, 2, \ldots .$$

The counting process N generates the family of σ-algebras

$$\mathscr{F} = \{\mathscr{F}_t = \sigma a(N_s, s \leqq t),\ t \geqq 0\}\,.$$

To the age replacement with replacement age d corresponds the probability measure P^d on $(\Omega, \mathscr{F}_\infty)$ with the following properties:

1. With probability 1 holds for all $t \geqq 0 : X_t = d$ if and only if ${}^2N_t - {}^2N_{t-} = 1$, i.e. $t = \tau_n$, $i_n = 2$ for some n.

2.
$$^1M_t = {}^1N_t - \int_0^t q(X_s)\,\mathrm{d}s\,, \quad t \geqq 0\,,$$
is a martingale with respect to $\mathscr{F}$.

Property 1 is obvious, Property 2 was stated in § 1. The failure rate depends only on the age of the component. The trajectory of ^1N together with the age replacement rule determines the trajectory of ^2N. The uniqueness of P^d having Properties 1, 2 follows therefore from Proposition 7.

Under a *general replacement policy* the replacement age is not fixed, but it is

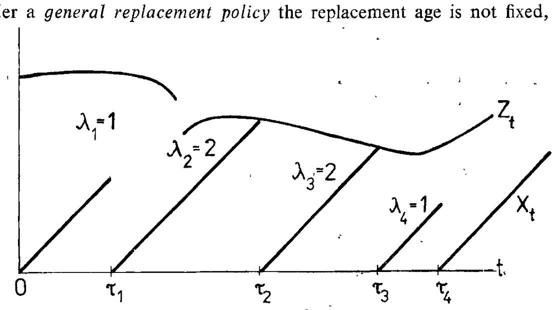

Fig. 6.

a nonanticipative random function $Z = \{Z_t, t \geqq 0\}$ taking on positive values including ∞, continuous piecewise and from the left. If $X_t = Z_t$, the preventive replacement is made. Z_t associates with the trajectory $\{N_s, s \in [0, t]\}$ the replacement age (Fig. 6).

The probability distribution of the replacement process under policy Z is the probability measure P^Z on $(\Omega, \mathscr{F}_\infty)$ having the properties:

1. With probability 1 holds for all $t \geqq 0 : X_t = Z_t$ if and only if ${}^2N_t - {}^2N_{t-} = 1$.

2.

$$ {}^1M_t = {}^1N_t - \int_0^t q(X_s)\,\mathrm{d}s\,, \quad t \geqq 0\,, $$

is a martingale with respect to $\mathscr{F}$.

We omit the proof that P^Z exists and is defined uniquely.

Let us now turn our attention to the statement that the average cost per unit time θ arising under the age replacement with replacement age d cannot be improved by using a general replacement policy. θ is the minimum in the class of the age replacement policies $Z_t = x$, $t \geqq 0$. Since c_1 is the cost of an after failure replacement, and c_2 the cost of a preventive replacement, the total cost incurred until time t is

$$ C_t = c_1\, {}^1N_t + c_2\, {}^2N_t\,. $$

If the replacements are made in age d, then

$$ \lim_{t\to\infty} t^{-1}C_t = \lim_{n\to\infty} \frac{C_{\tau_n}/n}{\tau_n/n} = \theta \quad \text{a.s.}\,, $$

according to the law of large numbers for the sums of mutually independent identically distributed random variables with finite expectation.

Let $Z = \{Z_t,\ t \geqq 0\}$ be an arbitrary replacement policy. We are going to demonstrate

$$ \varliminf_{t\to\infty} t^{-1}C_t \geqq \theta \quad P^Z\text{-a.s.} \tag{63} $$

(63) expresses the mentioned global optimality of θ. To prove it we shall look for a bounded function $w(x)$, $x \geqq 0$, such that

$$ S_t = C_t - t\theta + w(X_t^+)\,, \quad t \geqq 0\,, $$

is a submartingale. S is right continuous provided that $w(x)$ is right continuous. Note that X^+ is nonanticipative, since

$$ X_t^+ = X_t\,\chi\{N_t = N_{t-}\}\,, \quad t \geqq 0\,. $$

If we succeed in finding $w(x)$, then we have

$$ S_t = M_t + A_t\,, \quad t \geqq 0\,, $$

where M is a martingale, A a nonnegative nondecreasing process. Moreover, if M fulfils the law of large numbers, then (63) follows from

$$ 0 = \lim_{t\to\infty} t^{-1}M_t \leqq \varliminf_{t\to\infty} t^{-1}S_t = \varliminf_{t\to\infty} t^{-1}C_t - \theta \quad \text{a.s.} \tag{64} $$

To find $w(x)$ we shall use a heuristic argument. It has to hold

$$ \mathsf{E}\{S_{t+\Delta} - S_t \mid X_t^+ = x \text{ and no preventive replacement}\} \geqq 0\,. $$

Distinguishing whether there was a failure or not in the time interval $(t, t + \Delta)$, $\Delta \to 0+$, we obtain for the conditional probability the expression

$$q(x)\,\Delta(c_1 - \theta\Delta + w(0) - w(x)) +$$
$$+ (1 - q(x)\,\Delta)(-\theta\Delta + w(x + \Delta) - w(x)) + o(\Delta) =$$
$$= \Delta[-\theta + w'(x) + q(x)(c_1 + w(0) - w(x))] + o(\Delta) \geqq 0\,.$$

Similarly,

$$\mathsf{E}\{S_{t+\Delta} - S_t \mid X_t^+ = x \text{ and preventive replacement}\} =$$
$$= c_2 + w(0) - w(x) + o(1) \geqq 0\,.$$

From these inequalities we get

$$w'(x) + q(x)(c_1 + w(0) - w(x)) - \theta \geqq 0\,, \tag{65}$$

$$c_2 + w(0) - w(x) \geqq 0\,. \tag{66}$$

Let us set $w(0) = 0$. Suppose the equality to hold in (65), and solve the differential equation

$$w'(x) + q(x)(c_1 - w(x)) - \theta = 0\,, \quad w(0) = 0\,. \tag{67}$$

The solution is

$$w(x) = \left(-c_1 F(x) + \theta \int_0^x \bar{F}(y)\,\mathrm{d}y\right)/\bar{F}(x)\,. \tag{68}$$

(66) is also valid, since

$$c_2 - w(x) = \left(c_2 \bar{F}(x) + c_1 F(x) - \theta \int_0^x \bar{F}(y)\,\mathrm{d}y\right)/\bar{F}(x) \geqq \tag{69}$$
$$\geqq \left(c_2 \bar{F}(x) + c_1 F(x) - \theta(x) \int_0^x \bar{F}(y)\,\mathrm{d}y\right)/\bar{F}(x) = 0\,.$$

Note that (69) for $x = d$ implies $w(d) = c_2$. Further, using (62) we obtain from (67) that $w'(d) = 0$.

Define $w(x)$ by (68) for $x \in [0, d]$, and set

$$w(x) = c_2\,, \quad x \geqq d\,. \tag{70}$$

$w(x)$ has continuous derivative on $[0, \infty)$. (66) holds. Inserting from (70) into (65) we get the condition

$$q(x)(c_1 - c_2) - \theta \geqq 0\,, \quad x \geqq d\,.$$

Its fulfilment follows from (62) because $q(x)$ is assumed to be nondecreasing.

Proposition 16. It holds

$$C_t - t\theta + w(X_t^+) = \int_0^t (c_1 - w(X_s))(\mathrm{d}^1 N_s - q(X_s)\,\mathrm{d}s) + \tag{71}$$

$$+\int_0^t (c_2 - w(X_s))\,\mathrm{d}^2N_s + (c_1 - c_2)\int_0^t \chi\{d < X_s\}\,(q(X_s) - q(d))\,\mathrm{d}s\,, \quad t \geqq 0\,.$$

The last two terms are nonnegative and nondecreasing.

Proof. We have setting $\tau_0 = 0$

$$\int_0^t w'(X_s)\,\mathrm{d}s = \sum_{\tau_n < t} \int_{\tau_n}^{\tau_{n+1}\wedge t} w'(X_s)\,\mathrm{d}s = \sum_{\tau_n < t} w(X_{\tau_{n+1}\wedge t}) =$$

$$= \int_0^t w(X_s)\,\mathrm{d}(^1N_s + {}^2N_s) + w(X_t^+)\,.$$

Since $C_t = c_1\,{}^1N_t + c_2\,{}^2N_t$, (71) is equivalent to

$$\int_0^t (w'(X_s) - \theta)\,\mathrm{d}s = \int_0^t (w(X_s) - c_1)\,q(X_s)\,\mathrm{d}s + (c_1 - c_2)\int_0^t \chi\{d < X_s\}\,(q(X_s) - q(d))\,\mathrm{d}s\,.$$

This equality follows from (67), (62), (70). Namely,

$$\int_0^t (w'(X_s) - \theta)\,\mathrm{d}s = \int_0^t \chi\{X_s \leqq d\}\,(w(X_s) - c_1)\,q(X_s)\,\mathrm{d}s - \theta\int_0^t \chi\{d < X_s\}\,\mathrm{d}s =$$

$$= \int_0^t (w(X_s) - c_1)\,q(X_s)\,\mathrm{d}s + (c_1 - c_2)\int_0^t \chi\{d < X_s\}\,(q(X_s) - q(d))\,\mathrm{d}s\,.$$

The last two integrals in (71) have nonnegative integrands because of (66), and $q(x) - q(d) \geqq 0$, $x \geqq d$. Consequently, they are nonnegative and nondecreasing. □

The first term on the right in (71)

$$M_t = \int_0^t (c_1 - w(X_s))\,(d\,{}^1N_s - q(X_s)\,\mathrm{d}s)\,, \quad t \geqq 0\,,$$

is an integral of a bounded left-continuous nonanticipative function with respect to martingale under arbitrary replacement policy Z. Thus, M is a martingale (with respect to $\mathscr{F}$). Omitting the verification of the hypotheses of Proposition 14, let us state that M fulfils the law of large numbers. With regard to (64) we conclude that (63) holds.

Note that for the age replacement $Z_t = d$, $t \geqq 0$, the last two terms in (71) vanish, and equality holds in (64). This provides another proof of

$$\lim_{t\to\infty} t^{-1}C_t = \theta \quad P^d\text{-a.s.}$$

Let us now illustrate that decomposition (71) is a suitable tool for studying the average cost in the situation, when d is estimated during the renewal process (sequential improvement of the age replacement).

Proposition 17. Let Z be a replacement policy satisfying

(72) $$\lim_{t\to\infty} Z_t = d \quad P^Z\text{-a.s.}$$

Then

$$\lim_{t\to\infty} t^{-1} C_t = \theta \quad P^Z\text{-a.s.}$$

Proof. Let (72) hold. Decomposition (71) reduces the proof to the demonstration of

(73) $$\lim_{t\to\infty} t^{-1} \int_0^t (c_2 - w(X_s))\, \mathrm{d}^2 N_s = \lim_{t\to\infty} t^{-1} \int_0^t (c_2 - w(Z_s))\, \mathrm{d}^2 N_s = 0\,,$$

(74) $$\lim_{t\to\infty} t^{-1} \int_0^t \chi\{d < X_s \leqq Z_s\}\,(q(X_s) - q(d))\, \mathrm{d}s = 0 \quad \text{a.s.}$$

Since $w(d) = c_2$, the integrand on the right in (73) tends to 0 as $s \to \infty$. The time between two consecutive preventive replacements is in the limit at least d, i.e.

$$\overline{\lim_{t\to\infty}}\, {}^2N_t/t \leqq 1/d \quad \text{a.s.}$$

From here (73) follows. (74) holds, because the integrand there tends to 0 as well. □

A more elaborate investigation of $\{C_t, t \geqq 0\}$ under self-optimizing replacement policies can be made applying the law of the iterated logarithm to the martingale M.

10. Average cost in a Markov process

We turn our attention to controlled Markov processes $X = \{X_t, t \geqq 0\}$ having transition rates

$$q(i, j; z)\,, \quad i, j \in I\,, \quad z \in J\,,$$

as defined at the end of § 2. To be able to compare the controls we introduce an evaluation of the trajectory called here the cost. Under *the cost until time t* we understand the integral

$$C_t = \int_0^t c(X_s, Z_s)\, \mathrm{d}s\,, \quad t \geqq 0\,,$$

where $c(i, z)$ is a continuous function on $I \times J$. First we shall be looking for the minimal average cost per unit time, $\lim_{t\to\infty} t^{-1} C_t$, under homogeneous Markovian controls. Such controls are of the form

(75) $$Z_t = \bar{z}(X_t^-)\,, \quad t \geqq 0\,,$$

where $\bar{z}$ is a mapping from I to J. If (75) holds, then X is a homogeneous Markov process with transition rates

(76) $$q(i, j; \bar{z}(i)), \quad i, j \in I\,.$$

Its probability distribution will be denoted by $P^{\bar{z}}$.

Let us make the following hypothesis: For each $\bar{z}$, the rates (76) define a Markov process the states of which are recurrent and communicate with each other. (The indecomposability of the rate matrix.)

The hypothesis implies for each $\bar{z}$ the existence of the limit probabilities

$$\lim_{t\to\infty} P^{\bar{z}}(X_t = i) = p_i^\infty(\bar{z}) > 0 , \quad i \in I .$$

They are the unique solution of the system of equations

$$\sum_i p_i^\infty(\bar{z})\, q(i, j; \bar{z}(i)) = 0 , \quad j \in I , \quad \sum_i p_i^\infty(\bar{z}) = 1 . \tag{77}$$

Denote

$$\theta(\bar{z}) = \sum_i p_i^\infty(\bar{z})\, c(i, \bar{z}(i)) .$$

It is intuitively clear, and follows from the law of large numbers for Markov processes that

$$\lim_{t\to\infty} t^{-1} C_t = \theta(\bar{z}) \quad P^{\bar{z}}\text{-a.s.} \tag{78}$$

We shall in fact prove it in the sequel. The minimal average cost is

$$\min_{\bar{z}} \theta(\bar{z}) = \theta(\hat{z}) = \theta . \tag{79}$$

$\theta(\bar{z})$ as a continuous function on a compact set assumes its minimal value at an optimal control $\hat{z}$.

θ is the minimum in the class of homogeneous Markovian controls. As in § 9 we are first going to prove that under arbitrary control $Z = \{Z_t, t \geqq 0\}$ holds

$$\varliminf_{t\to\infty} t^{-1} C_t \geqq \theta \quad P^Z\text{-a.s.} \tag{80}$$

(80) follows in the same way as (63) if we find numbers $w(i)$, $i \in I$, such that

$$S_t = C_t - t\theta + w(X_t) , \quad t \geqq 0 ,$$

is a submartingale for each Z.

To define S recall that by definition of P^Z

$${}^jM_t = \chi\{X_t = j\} - \int_0^t q(X_s, j; Z_s)\, \mathrm{d}s , \quad t \geqq 0 , \quad j \in I ,$$

are martingales. Hence, for $w(j)$, $j \in I$, arbitrary

$$M_t = \sum_j w(j)\, {}^jM_t = w(X_t) - \int_0^t \sum_j q(X_s, j; Z_s)\, w(j)\, \mathrm{d}s , \quad t \geqq 0 ,$$

is a martingale with respect to $\mathscr{F}$ for each Z. It obeys to the law of large numbers, since the differences $M_{n+1} - M_n$, $n = 0, 1, \ldots$, are bounded. Setting

$$f(i, z) = c(i, z) + \sum_j q(i, j; z)\, w(j) - \theta , \quad i \in I , \quad z \in J ,$$

we have

$$C_t - t\theta + w(X_t) = M_t + \int_0^t f(X_s, Z_s)\,\mathrm{d}s\,, \quad t \geqq 0\,. \tag{81}$$

If we can choose $w(j)$, $j \in I$, so that

$$f(i, z) \geqq 0\,, \quad i \in I\,, \quad z \in J\,, \tag{82}$$

then (81) becomes the Doob-Meyer decomposition of submartingale S.

To this purpose let us solve the system of equations

$$c(i, \hat{z}(i)) + \sum_j q(i, j; \hat{z}(i))\, w(j) - \theta = 0\,, \quad i \in I\,, \tag{83}$$

for unknowns θ, $w(j)$, $j \in I$. Multiplying (83) by $p_i^\infty(\hat{z})$ and adding for $i \in I$, we obtain in virtue of (77)

$$\sum_i p_i^\infty(\hat{z}) c(i, \hat{z}(i)) = \theta\,.$$

Symbol θ is therefore consistent with that introduced by (79). The matrix of the system (83) is, up to the sign of the last column, transposed to the matrix of (77) for $\bar{z} = \hat{z}$. The latter has rank m, because (77) has unique solution. We conclude that the set of solutions of (83) is one-dimensional. The general solution of (83) has the form θ, $w(j)$ + const., $j \in I$.

Before proving (82), let us return to the law of large numbers for Markov processes. Let (83) hold, i.e.

$$f(i, \hat{z}(i)) = 0\,, \quad i \in I\,, \tag{84}$$

and let

$$Z_t = \hat{z}(X_t^-)\,, \quad t \geqq 0\,.$$

Then

$$C_t - t\theta + w(X_t) = M_t\,, \quad t \geqq 0\,.$$

Hence,

$$\lim_{t \to \infty} t^{-1} C_t = \theta \quad P^{\hat{z}}\text{-a.s.}$$

results from the law of large numbers for martingale M. The same proof applies to (78).

Lemma 8. If $w(j)$, $j \in I$, satisfy (83), then (82) holds.

Proof. Assume (83), and on the contrary $f(i_0, z_0) < 0$ for some $i_0 \in I$, $z_0 \in J$. Define

$$\bar{z}(i) = \hat{z}(i) \quad \text{for} \quad i \neq i_0\,, \quad \bar{z}(i_0) = z_0\,.$$

Then for

$$Z_t = \bar{z}(X_t^-)\,, \quad t \geqq 0\,,$$

follows from (81) and from (78)

$$\theta(\bar{z}) - \theta = \lim_{t\to\infty} t^{-1} \int_0^t f(X_s, z(X_s))\, ds = p_{i_0}^{\infty}(\bar{z}) f(i_0, z_0) < 0 \quad P^{\bar{z}}\text{-a.s.}$$

This contradicts to the minimality of θ. □

From (81), (82), (84), and from the law of large numbers applied to M one gets immediately the subsequent propositions.

Propositions 18. Under arbitrary control Z,

$$\varliminf_{t\to\infty} t^{-1} C_t \geqq \theta \quad P^Z\text{-a.s.}$$

Proposition 19. Let Z be such that

$$\lim_{t\to\infty} \varrho(Z_t, \hat{z}(X_t^-)) = 0 \begin{cases} P^Z\text{-a.s.} \\ P^Z\text{-in probability.} \end{cases} \tag{85}$$

ϱ denotes the distance. Then

$$\lim_{t\to\infty} t^{-1} C_t = \theta \begin{cases} P^Z\text{-a.s.} \\ P^Z\text{-in probability.} \end{cases} \tag{86}$$

Proof. We have

$$\lim_{t\to\infty} \left(t^{-1} C_t - \theta - t^{-1} \int_0^t f(X_s, Z_s)\, ds \right) = 0 \quad P^Z\text{-a.s.} \tag{87}$$

The a.s. convergence in (85) implies $\lim_{t\to\infty} f(X_t, Z_t) = 0$ a.s., and hence

$$\lim_{t\to\infty} t^{-1} \int_0^t f(X_s, Z_s)\, ds = 0 \quad P^Z\text{-a.s.}$$

The convergence in probability in (85) implies $\lim_{t\to\infty} \mathsf{E} f(X_t, Z_t) = 0$, and

$$\lim_{t\to\infty} t^{-1}\, \mathsf{E} \int_0^t f(X_s, Z_s)\, ds = 0 \quad \text{or} \quad P^Z\text{-}\lim_{t\to\infty} t^{-1} \int_0^t f(X_s, Z_s)\, ds = 0\,.$$

From here and from (87) follows (86). □

Proposition 19 refers to the case, when the optimal control $\hat{z}$ is unknown, and is estimated from the observed trajectory.

Let us now use decomposition (81) to investigate the asymptotic distribution of $(C_t - t\theta)/\sqrt{t}$ as $t \to \infty$. The control Z is supposed to fulfil

$$P^Z\text{-}\lim_{t\to\infty} \varrho(Z_t, \hat{z}(X_t^-)) = 0\,. \tag{88}$$

As we shall see, (88) implies that $M_t/\sqrt{t}$ has asymptotically normal distribution $N(0, \sigma^2)$ as $t \to \infty$. We shall determine the asymptotic variance σ^2. From (81) it is then seen that a necessary and sufficient condition for $(C_t - t\theta)/\sqrt{t}$ to have also the asymptotic distribution $N(0, \sigma^2)$ is

(89)
$$P^Z\text{-}\lim_{t\to\infty} \frac{1}{\sqrt{t}} \int_0^t f(X_s, Z_s)\, \mathrm{d}s = 0 .$$

Introduce the martingale differences

$$Y_n = M_{n+1} - M_n = w(X_{n+1}) - w(X_n) - \int_n^{n+1} \sum_j q(X_s, j; Z_s)\, w(j)\, \mathrm{d}s, \quad n = 0, 1, \ldots .$$

Since

$$|Y_n| \leqq \text{const.}, \quad n = 0, 1, \ldots ,$$

the Lindeberg condition (57) holds. Thus, to verify the hypotheses of Proposition 15 it remains to demonstrate

(90)
$$P^Z\text{-}\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathsf{E}\{Y_k^2 \mid \mathscr{F}_k\} = \sigma^2 .$$

Lemma 9. For $k = 0, 1, \ldots$

(91)
$$\mathsf{E}\{Y_k^2 \mid \mathscr{F}_k\} = \mathsf{E}\left\{\int_k^{k+1} g(X_s, Z_s)\, \mathrm{d}s \mid \mathscr{F}_k\right\},$$

where

$$g(i, z) = \sum_{j \neq i} q(i, j; z)\, (w(j) - w(i))^2 , \quad i \in I , \quad z \in J .$$

Proof. Divide the interval $[k, k + 1]$ into subintervals of length $\Delta = 1/n$, and let $n \to \infty$. This yields

(92)
$$\mathsf{E}\{Y_k^2 \mid \mathscr{F}_k\} = \mathsf{E}\{\sum_{j=0}^{n-1} (M_{k+(j+1)\Delta} - M_{k+j\Delta})^2 \mid \mathscr{F}_k\} = $$
$$= \mathsf{E}\{\sum_{k<s\leqq k+1} (w(X_s) - w(X_s^-))^2 \mid \mathscr{F}_k\} ,$$

since

$$(M_{t+\Delta} - M_t)^2 = (w(X_{t+\Delta}) - w(X_t))^2 +$$
$$+ 2(w(X_{t+\Delta}) - w(X_t)) \int_t^{t+\Delta} \sum qw\, \mathrm{d}s + \left(\int_t^{t+\Delta} \sum qw\, \mathrm{d}s\right)^2 ,$$

and the sum of the last two terms for $t = k + j\Delta$, $j = 0, 1, \ldots, n - 1$, is negligible as $n \to \infty$.

In § 7 we showed that the point process of the jumps into state j, ${}^jN = \{{}^jN_t, t \geqq 0\}$, has compensator

$$\int_0^t (1 - \chi\{X_s^- = j\})\, q(X_s^-, j; Z_s)\, \mathrm{d}s , \quad t \geqq 0 , \quad j \in I .$$

Consequently,

$$\sum_{s\leqq t} (w(X_s) - w(X_s^-))^2 = \sum_j \int_0^t (w(j) - w(X_s))^2\, \mathrm{d}^jN_s , \quad t \geqq 0 ,$$

has compensator

$$\sum_j \int_0^t (w(j) - w(X_s))^2 (1 - \chi\{X_s^- = j\}) \, q(X_s^-, j; Z_s) \, ds = \int_0^t g(X_s^-, Z_s) \, ds , \quad t \geqq 0 .$$

We conclude that

$$\sum_{s \leqq t} (w(X_s) - w(X_s^-))^2 - \int_0^t g(X_s, Z_s) \, ds , \quad t \geqq 0 ,$$

is a martingale with respect to $\mathscr{F}$. This and (92) yield (91). □

According to Proposition 19, (88) implies

$$(93) \qquad P^Z\text{-}\lim_{t \to \infty} t^{-1} \int_0^t g(X_s, Z_s) \, ds = \sum_j p_j^\infty(\hat{z}) \, g(j, \hat{z}(j)) = \sigma^2 .$$

Further,

$$L_n = \int_0^n g \, ds - \sum_{k=0}^{n-1} \mathsf{E}\{Y_k^2 \mid \mathscr{F}_k\} = \int_0^n g \, ds - \sum_{k=0}^{n-1} \mathsf{E}\left\{\int_k^{k+1} g \, ds \middle| \mathscr{F}_k\right\}, \quad n = 0, 1, \ldots ,$$

is a martingale with respect to $\{\mathscr{F}_n, \ n = 0, 1, \ldots\}$. Its differences are bounded. Hence, from Proposition 14,

$$\lim_{n \to \infty} n^{-1} L_n = 0 \quad P^Z\text{-a.s.} ,$$

which together with (93) gives (90).

Let us restate what was demonstrated.

Proposition 20. Let the control Z satisfy (88). Then $(C_t - t\theta)/\sqrt{t}$ has asymptotically normal distribution $N(0, \sigma^2)$ as $t \to \infty$ if and only if (89) holds.

Particularly, the optimal control

$$(94) \qquad Z_t = \hat{z}(X_t^-) , \quad t \geqq 0 ,$$

satisfies the hypotheses of Proposition 20. (89) states, how fast must a self-optimizing control converge to (94) to provide C with the same asymptotic distribution as (94). Self-optimizing controls based on maximum likelihood estimation of the unknown parameters fulfil (89) under certain regularity assumptions.

IV. FILTERING OF RATES

11. Γ-distributed rate

Poisson processes $N = \{N_t, t \geqq 0\}$ whose rate $Q = \{Q_t, t \geqq 0\}$ is a random process are called *doubly stochastic Poisson processes*. The observation of N gives us information about the rate Q. In this chapter we shall deal with the estimation of Q from N. The estimate of Q_t from N_s, $s \in [0, t]$, minimizing the expected quadratic error is the conditional expectation

$$(95) \qquad \hat{Q}_t = \mathsf{E}\{Q_t \mid \mathscr{F}_t^N\} , \quad t \geqq 0 ,$$

where $\mathscr{F}_t^N = \sigma a\{N_s, s \in [0, t]\}$. We consider t variable, and conceive (95) as a problem to design a filter having the observed process as the input and the estimate as the output. The mathematical solution is a differential equation for $\hat{Q}$. A filter is a device for materializing this equation.

We begin with a simple case. Let N be the Poisson process with rate

$$Q_t = \xi \exp\left\{\int_0^t a(s)\,\mathrm{d}s\right\}, \quad t \geqq 0.$$

ξ is a Γ-distributed random variable with probability density

$$(b^m/\Gamma(m))\, x^{m-1}\, \mathrm{e}^{-bx}, \quad x \geqq 0. \tag{96}$$

$a(t)$, $t \geqq 0$, is supposed to be a piecewise continuous function.

In § 5 we derived that the probability distribution $\bar{P}$ of a point process $\{\bar{N}_s, s \in [0, t]\}$ with rate $\{\bar{Q}_s, s \in [0, t]\}$ has with respect to the Poisson process with unit rate the density

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}P_{oiss.}} = \exp\left\{\int_0^t \log \bar{Q}_s\, \mathrm{d}\bar{N}_s + \int_0^t (1 - \bar{Q}_s)\, \mathrm{d}s\right\}. \tag{97}$$

Given $\xi = x$, we have $Q_s = x \exp\{\int_0^s a\,\mathrm{d}z\}$. Substituting for Q into (97) and multiplying by (96) we get the joint density of ξ and $\{N_s, s \in [0, t]\}$

$$\frac{\mathrm{d}P_{x,N}}{\mathrm{d}x \times \mathrm{d}P_{oiss.}} = f(x, N) = (b^m/\Gamma(m))\, x^{m-1}\, \mathrm{e}^{-bx} \exp\left\{\int_0^t \left(\log x + \int_0^s a\,\mathrm{d}z\right) \mathrm{d}N_s + \right.$$

$$\left. + \int_0^t \left(1 - x \exp\left\{\int_0^s a\,\mathrm{d}z\right\}\right) \mathrm{d}s\right\} = c x^{(m+N_t-1)} \exp\left\{-x\left(b + \int_0^t \exp\left\{\int_0^s a\,\mathrm{d}z\right\} \mathrm{d}s\right)\right\},$$

where c is independent of x. The conditional density is

$$f(x \mid N) = f(x, N) \Big/ \int_0^\infty f(y, N)\,\mathrm{d}y = \left(b + \int_0^t \exp\left\{\int_0^s a\,\mathrm{d}z\right\} \mathrm{d}s\right)^{m+N_t} .$$

$$. \, x^{m+N_t-1} \exp\left\{-x\left(b + \int_0^t \exp\left\{\int_0^s a\,\mathrm{d}z\right\} \mathrm{d}s\right)\right\} \Big/ \Gamma(m + N_t).$$

From here we conclude that

$$\hat{Q}_t = \exp\left\{\int_0^t a\,\mathrm{d}s\right\} \int_0^\infty x f(x \mid N)\,\mathrm{d}x =$$

$$= \exp\left\{\int_0^t a\,\mathrm{d}s\right\} \left(b + \int_0^t \exp\left\{\int_0^s a\,\mathrm{d}z\right\} \mathrm{d}s\right)^{-1} (m + N_t) = k(t)\,(m + N_t), \quad t \geqq 0.$$

Although the estimate is simple, let us present the differential equation for it in the form familiar in filtering theory. Computing the differential we get

(98) $$\mathrm{d}\hat{Q}_t = a(t)\, k(t)\, \mathrm{d}t(m + N_t) - k(t)^2\, \mathrm{d}t(m + N_t) + k(t)\, \mathrm{d}N_t = \\ = a(t)\, \hat{Q}_t\, \mathrm{d}t + k(t)\,(\mathrm{d}N_t - \hat{Q}_t\, \mathrm{d}t)\,, \quad t \geqq 0\,, \\ \hat{Q}(0) = \mathsf{E}\xi = m/b\,.$$

Further,

$$\frac{\mathrm{d}}{\mathrm{d}t}\, k(t) = a(t)\, k(t) - k(t)^2\,, \quad t \geqq 0\,, \quad k(0) = 1/b\,.$$

In § 3 we proved that $\hat{Q}$ is the rate of N with respect to $\mathscr{F}^N$. Hence, the last differential in (98) belongs to a martingale with respect to $\mathscr{F}^N$

$$\overline{M}_t = N_t - \int_0^t \hat{Q}_s\, \mathrm{d}s\,, \quad t \geqq 0\,.$$

Let us determine the mean quadratic error. We have

$$\mathsf{E}(\hat{Q}_t - Q_t)^2 = \mathsf{EE}\left\{\left[k(t)\,(m + N_t) - \xi\, k(t)\left(b + \int_0^t \exp\left\{\int_0^s a\, \mathrm{d}z\right\} \mathrm{d}s\right)\right]^2 \Big| \xi\right\} = \\ = \mathsf{E}\, k(t)^2 \left(\xi \int_0^t \exp\left\{\int_0^s a\, \mathrm{d}z\right\} \mathrm{d}s + b^2\left(\frac{m}{b} - \xi\right)^2\right) = \\ = k(t)^2 \left(\frac{m}{b} \int_0^t \exp\left\{\int_0^s a\, \mathrm{d}z\right\} \mathrm{d}s + b^2\, \frac{m}{b^2}\right) = k(t)\, \frac{m}{b} \exp\left\{\int_0^t a\, \mathrm{d}s\right\}, \quad t \geqq 0\,.$$

12. Markovian rate

Let $X = \{X_t, t \geqq 0\}$ be a Markov process with state space $I = \{1, \ldots, m\}$ having initial distribution

(99) $$P(X_0) = i) = p_i\,, \quad i \in I\,,$$

and transition rates

$$q(i, j, t)\,, \quad i, j \in I\,, \quad t \geqq 0\,,$$

continuous piecewise and from the left. Using (55) we can find the probability density of $\{X_s, s \in [0, t]\}$, since the difference between a controlled Markov process and a time-inhomogeneous one is for this purpose unsubstantial. The density is taken with respect to the distribution P_0 of the Markov process on $[0, t]$ with transition rate matrix (54) and initial distribution (99). We have

(100) $$\frac{\mathrm{d}P}{\mathrm{d}P_0} = \exp\left\{\int_0^t \log q(X_s^-, X_s, s)\, \mathrm{d}^X N_s + \int_0^t (m - 1 + q(X_s^-, X_s, s))\, \mathrm{d}s\right\}.$$

${}^X N$ is the counting process of the state changes in X.

Suppose that X is observed indirectly by means of the observation of a point process N with the following properties: For each $t > 0$, given the trajectory $\{X_s, s \in [0, t]\}$, the process $\{N_s, s \in [0, t]\}$ is Poissonian with rate

$$\varrho_s = a(s, X_s), \quad s \in [0, t].$$

The nonnegative function $a(s, i)$, $s \geqq 0$, $i \in I$, is assumed to be continuous piecewise and from the left. According to (28) the conditional distribution of N has with respect to the Poisson process with unit rate the density

$$\exp\left\{\int_0^t \log a(s, X_s)\, \mathrm{d}N_s + \int_0^t (1 - a(s, X_s))\, \mathrm{d}s\right\}.$$

Combining this with (100) we obtain the density of the joint distribution

$$L_t(X, N) = \frac{\mathrm{d}P}{\mathrm{d}P_0 \times \mathrm{d}P_{oiss.}} = \exp\left\{\int_0^t \log q(X_s^-, X_s, s)\, \mathrm{d}^X N_s + \right. \tag{101}$$

$$\left. + \int_0^t (m - 1 + q(X_s^-, X_s, s))\, \mathrm{d}s + \int_0^t \log a(s, X_s)\, \mathrm{d}N_s + \int_0^t (1 - a(s, X_s))\, \mathrm{d}s\right\}.$$

We intend to derive equations satisfied by the aposteriori distribution of X_t given the observation of $\{N_s, s \in [0, t]\}$, i.e. by the probabilities

$$^i\pi_t = P(X_t = i \mid \mathscr{F}_t^N) = \mathsf{E}\{\chi\{X_t = i\} \mid \mathscr{F}_t^N\}, \quad t \geqq 0, \quad i \in I.$$

We have

$$^i\pi_t(N) = \int_{\{X_t = i\}} L_t(X, N)\, \mathrm{d}P_0(X) \Big/ \int L_t(Y, N)\, \mathrm{d}P_0(Y) = {}^i\varrho_t \Big/ \sum_j {}^j\varrho_t,$$

where

$$^i\varrho_t = \int_{\{X_t = i\}} L_t\, \mathrm{d}P_0 = \mathsf{E}_0\, \chi\{X_t = i\}\, L_t. \tag{102}$$

E_0 denotes the expectation in X under P_0 for N fixed.

Let us calculate the differential $\mathrm{d}\,{}^i\varrho_t$. From (101) follows that $^i\varrho$ has discontinuities only in the jump points of N. In such points is $N_t - N_{t-} = 1$, and the differential becomes the difference (points of positive measure). Hence,

$$\mathrm{d}\,{}^i\varrho_t = {}^i\varrho_t - {}^i\varrho_{t-} = {}^i\varrho_{t-}(\exp\{\log a(N_t - N_{t-})\} - 1) = \tag{103}$$

$$= {}^i\varrho_{t-}(a(t, i) - 1)\, \mathrm{d}N_t,$$

because the integral in (102) extends over $\{X_t = i\}$.

Outside the discontinuity points of N is $^i\varrho$ absolutely continuous. To obtain its differential we shall investigate $^i\varrho_{t+\Delta}$ as $\Delta \to 0+$, employing the following property of Markov processes: For X_t given, $\{X_s, 0 \leqq s \leqq t\}$ and $\{X_s, s \geqq t\}$ are independent.

We have

(104) $$ {}^{i}\varrho_{t+\Delta} = \sum_j P_0(X_t = j)\, \mathsf{E}_0\{L_t \mid X_t = j\}\, \mathsf{E}_0\{\chi\{X_{t+\Delta} = i\}\, L_{t+\Delta} L_t^{-1} \mid X_t = j\} = $$
$$ = \sum_j {}^{j}\varrho_t\, \mathsf{E}_0\{\chi\{X_{t+\Delta} = i\}\, L_{t+\Delta} L_t^{-1} \mid X_t = j\}\,. $$

To estimate the conditional probability in (104) we shall use an argument common in the theory of Markov processes.

Let $X_t = j \neq i$, $N_t = N_{t-}$, and let t be a continuity point of q and of a. Consider three possibilities:

1) With probability $\Delta + o(\Delta)$ as $\Delta \to 0+$ is $X_{t+\Delta} = i$, and the jump from j to i is the only state change of X on $[t, t + \Delta]$. Then

$$ \chi\{X_{t+\Delta} = i\}\, L_{t+\Delta} L_t^{-1} = \exp\{\log q(j, i, t)\} + o(1) = q(j, i, t) + o(1)\,. $$

2) With probability of order $o(\Delta)$ is $X_{t+\Delta} = i$, and there are at least two state changes on $[t, t + \Delta]$.

3) With probability $1 - \Delta + o(\Delta)$ is $X_{t+\Delta} \neq i$, and therefore $\chi\{X_{t+\Delta} = i\} = 0$. It results that

(105) $$ \mathsf{E}_0\{\chi\{X_{t+\Delta} = i\}\, L_{t+\Delta} L_t^{-1} \mid X_t = j\} = q(j, i, t)\, \Delta + o(\Delta) \quad \text{as} \quad \Delta \to 0+\,. $$

Let $X_t = i$, $N_t = N_{t-}$, and let t be a continuity point of q and of a. Distinguish again three cases:

1) With probability $1 - (m - 1)\,\Delta + o(\Delta)$ is $X_s = i$, for all $s \in [t, t + \Delta]$. Then

$$ \chi\{X_{t+\Delta} = i\}\, L_{t+\Delta} L_t^{-1} = \exp\{(m - 1 + q(i, i, t))\,\Delta + (1 - a(t, i))\,\Delta + o(\Delta)\} = $$
$$ = 1 + (m - 1 + q(i, i, t) + 1 - a(t, i))\,\Delta + o(\Delta)\,. $$

2) With probability of order $o(\Delta)$ is $X_{t+\Delta} = i$, and there are at least two state changes of X on $[t, t + \Delta]$.

3) With probability $(m-1)\,\Delta + o(\Delta)$ is $X_{t+\Delta} \neq i$, and consequently, $\chi\{X_{t+\Delta} = i\} = 0$.

The result is

(106) $$ \mathsf{E}\{\chi\{X_{t+\Delta} = i\}\, L_{t+\Delta} L_t^{-1} \mid X_t = i\} = $$
$$ = 1 + (q(i, i, t) + 1 - a(t, i))\,\Delta + o(\Delta) \quad \text{as} \quad \Delta \to 0+\,. $$

From (104), (105), (106) we conclude that

$$ {}^{i}\varrho_{t+\Delta} - {}^{i}\varrho_t = \sum_{j \neq i} {}^{j}\varrho_t\, q(j, i, t)\,\Delta + {}^{i}\varrho_t(q(i, i, t) + 1 - a(t, i))\,\Delta + o(\Delta)\,. $$

Hence, for $N_t = N_{t-}$ we get the differential

(107) $$ \mathrm{d}\,{}^{i}\varrho_t = \sum_j {}^{j}\varrho_t\, q(j, i, t)\,\mathrm{d}t + {}^{i}\varrho_t(1 - a(t, i))\,\mathrm{d}t\,. $$

(103) and (107) can be combined into

$$\text{(108)} \qquad \mathrm{d}\,{}^i\varrho_t = \sum_j {}^j\varrho_t\, q(j, i, t)\,\mathrm{d}t + {}^i\varrho_{t-}(a(t, i) - 1)(\mathrm{d}N_t - \mathrm{d}t), \quad i \in I .$$

The initial conditions are ${}^i\varrho_0 = p_i$, $i \in I$. System (108) for *the unnormed probabilities* ${}^i\varrho$, $i \in I$, is more suitable for computational purposes than the system which we shall obtain for ${}^i\pi$, $i \in I$. (108) is linear, its solution has jumps when $N_t - N_{t-} = 1$, $a(t, i) \neq 1$.

It is not difficult to compute now the differential $\mathrm{d}\,{}^i\pi_t$. First we determine its purely discontinuous component. Let $N_t - N_{t-} = 1$, and consequently (103), hold. Then

$$\begin{aligned}\mathrm{d}\,{}^i\pi_t = {}^i\pi_t - {}^i\pi_{t-} &= \mathrm{d}\,{}^i\varrho_t(\sum_j {}^j\varrho_{t-})^{-1} - \\ &- {}^i\varrho_t \sum_j \mathrm{d}\,{}^j\varrho_t(\sum_j {}^j\varrho_{t-} \sum_j {}^j\varrho_t)^{-1} = \\ &= {}^i\pi_{t-}(a(t, i) - 1) - {}^i\pi_t \sum_j {}^j\pi_{t-}(a(t, j) - 1) .\end{aligned}$$

Substituting

$${}^i\pi_t = {}^i\pi_{t-} + \mathrm{d}\,{}^i\pi_t ,$$

we get from here

$$\mathrm{d}\,{}^i\pi_t = (\sum_j {}^j\pi_{t-}\, a(t, j))^{-1}\, {}^i\pi_{t-}(a(t, i) - \sum_j {}^j\pi_{t-}\, a(t, j))\,\mathrm{d}N_t .$$

The differential of the absolutely continuous component is for $N_t = N_{t-}$

$$\begin{aligned}\mathrm{d}\,{}^i\pi_t &= \mathrm{d}\,{}^i\varrho_t(\sum_j {}^j\varrho_t)^{-1} - {}^i\varrho_t \sum_j \mathrm{d}\,{}^j\varrho_t(\sum_j {}^j\varrho_t)^{-2} = \\ \text{(109)} \qquad &= \sum_j {}^j\pi_t\, q(j, i, t)\,\mathrm{d}t + {}^i\pi_t(1 - a(t, i))\,\mathrm{d}t - \\ &- {}^i\pi_t(\sum_j(\sum_k {}^k\pi_t\, q(k, j, t)\,\mathrm{d}t + {}^j\pi_t(1 - a(t, j))\,\mathrm{d}t) = \\ &= [\sum_j {}^j\pi_t\, q(j, i, t) - {}^i\pi_t(a(t, i) - \sum_j {}^j\pi_t\, a(t, j))]\,\mathrm{d}t .\end{aligned}$$

The sought system of filtering equations is obtained by putting both components together. In (109) ${}^j\pi_t$ can be replaced by the left limit ${}^j\pi_{t-}$. We get

$$\begin{aligned}\text{(110)} \qquad \mathrm{d}\,{}^i\pi_t &= \sum_j {}^j\pi_t\, q(j, i, t)\,\mathrm{d}t + {}^i\pi_{t-}(a(t, i) - \sum_j {}^j\pi_{t-}\, a(t, j)) \,. \\ &. (\sum_j {}^j\pi_{t-}\, a(t, j))^{-1}(\mathrm{d}N_t - \sum_j {}^j\pi_t\, a(t, j)\,\mathrm{d}t), \quad t \geqq 0, \quad j \in I .\end{aligned}$$

Integrating the last differential in (110) we obtain

$$M_t = N_t - \int_0^t \sum_j {}^j\pi_s\, a(s, j)\,\mathrm{d}s , \quad t \geqq 0 .$$

M is a martingale with respect to $\mathscr{F}^N$ since according to § 3 the rate of N with respect to $\mathscr{F}^N$ equals

$$\mathsf{E}\{a(t, X_t) \mid \mathscr{F}_t^N\} = \sum_j {}^j\pi_t\, a(j, t)\,, \quad t \geqq 0\,.$$

If $a(t, i) = a(t)$, $i \in I$, $t \geqq 0$, the observation does not depend on the trajectory of X. (110) turns into the forward system of Kolmogorov differential equations.

Example 13. Let X have two states 1, 2, and the transition rate matrix

$$\begin{pmatrix} -1, & 1 \\ 1, & -1 \end{pmatrix}.$$

Let the rate of the observed process be

$$a(t, 1) = a_1\,, \quad a(t, 2) = a_2\,, \quad t \geqq 0\,, \quad a_2 \neq a_1\,.$$

System (108) for the unnormed probabilities ${}^i\varrho$, $i = 1, 2$, is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}^1\varrho_t = -a_1\,{}^1\varrho_t + {}^2\varrho_t\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}^2\varrho_t = {}^1\varrho_t - a_2\,{}^2\varrho_t \quad \text{when} \quad N_t = N_{t-}\,,$$

$${}^1\varrho_t = {}^1\varrho_{t-}a_1,\ {}^2\varrho_t = {}^2\varrho_{t-}a_2 \quad \text{when} \quad N_t - N_{t-} = 1\,,$$

$${}^1\varrho_0 = p_1\,, \quad {}^2\varrho_0 = p_2\,.$$

□

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