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ALTERNATIVE POLYNOMIAL EQUATION APPROACH TO LQ DISCRETE-TIME OPEN-LOOP CONTROL

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Like [4] for the feedback control this contribution brings the modification of the polynomial equation way of solving LQ discrete-time SISO control problem in the open-loop structure. Using this approach the conditions are found under which the only implied equation minimum solution is the LQ optimal one.

1. INTRODUCTION

A single-input, single-output (SISO) open-loop control problem is considered according to Figure 1. A controlled process output Y , load disturbance V (referred to the output), possible nonzero starting conditions Y_0, as well as the model P of a controlled process, are assumed to be described in the discrete-time form.

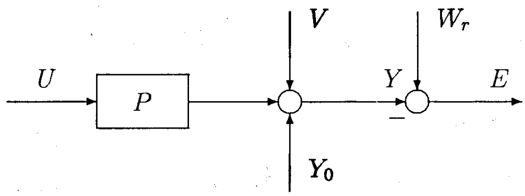

Fig. 1.

The error signal

$$E = W_r - Y = W - PU\,, \quad \text{where} \quad W = W_r - V - Y_0 \tag{1}$$

represents the only equivalent reference input.

Such a control sequence U is to be determined in LQ open-loop control, which minimizes the performance index

$$\vartheta = \sum_{k=0}^{\infty} \left[\psi e^2(kT) + \phi u^2(kT)\right], \tag{2}$$

where $e(kT)$ and/or $u(kT)$ are the error and/or control signal values at time kT, $k = 0, 1, \dots$; $\psi > 0$ and $\phi > 0$ are chosen weighting scalars.

Quadratic or the least squares control strategy is widely applied in both state-space as well as transfer function methods of the control design foı a long time. Many contributions concerning polynomial and polynomial matrix input-output methods in LQ and LQG control have been written following the fundamental book [2] in this field. Feedback SISO LQG control problems are treated in [1]. Based on the general results contained in [2], open-loop SISO LQ control solution using coprime polynomials for a system and signal description, has been presented in [3]. The same approach is used in this work.

Polynomials and sequences in d (one step delay in the time domain or the inverse Z-transform complex variable in the complex frequency domain) as well as usual symbols of polynomial theory [2] are used in the paper. Namely, $\deg a$, $a_* = a(d^{-1}), a^\sim = d^{\deg a} a_*$, $a = a^+ a^0 a^=$, where all zeros d_i of $a^+(d), a^0(d)$ and $a^=(d)$ have the property $|d_i| > 1$, $|d_i| = 1$ and $d_i < 1$, respectively, a^c denotes a polynomial for which $(a^c)^{-1}$ is a causal sequence. For two polynomials (a, b) is the greatest common divisor of a, b , $b|a$ means that $a = bc$ and $b \sim a$ denotes $a = bc$ with $\deg c = 0$. The sequence $F_*(d) = F(d^{-1})$ and $\langle F \rangle = \phi_0$ for a sequence $F = \dots + \phi_{-1} d^{-1} + \phi_0 + \phi_1 d + \dots$.

Following this Introduction the standard open-loop LQ control solution is described briefly in Section 2. The alternative possibility starting with the so-called "implied" equation is explained in Section 3. The part dealing with LQ optimality of the implied equation minimum solution follows in the fourth section. In Section 5 the respective conditions, which make this simpler solution possible, are compared with the similar ones beeing derived in [4] for the closed-loop control structure. One illustrative example is given at the end.

2. USUAL SOLUTION OF LQ OPEN-LOOP DISCRETE-TIME CONTROL

Considering the structure in Figure 1 with

$$P = \frac{b}{a}, \quad a,\, b \text{ coprime}, \; a \text{ causal}, \; b = d^\beta b^c \,, \; \beta \geq 0 \,, \tag{3}$$

and

$$W = W_r - V - Y_0 = \frac{f}{h}; \quad h, f \text{ coprime}, \; h = h^c, \tag{4}$$

the LQ optimal control and the corresponding error sequences are

$$U = \frac{a_h y}{h_a s} \quad \text{and} \quad E = \frac{(b, f)\, x}{h_a s}\,, \tag{5}$$

where

$$a_h = \frac{a}{(a, h)} \quad \text{and} \quad h_a = \frac{h}{(a, h)} \tag{6}$$

and $s = s^+$ follows from

$$s s_* = \phi a a_* + \psi b b_* \,. \tag{7}$$

The polynomials x and y in (5) along with z solve the couple of the equations

$$d^\rho s_* y + hz = d^\rho b_* \psi f \tag{8}$$

and

$$d^\rho s_* x - b_f h_a z = d^\rho a_* \phi a_h f_b \tag{9}$$

with the minimum $\deg z < \rho$, where

$$\rho = \max(\deg a, \deg b), \quad b_f = \frac{b}{(b,f)} \quad \text{and} \quad f_b = \frac{f}{(b,f)}. \tag{10}$$

The optimal solution exists if and only if $h_a = h_a^+$ and is unique.

The only equation (8) gives the optimal solution y, z with $\deg z < \rho$, if $\deg(d^\rho s_*, h) = 0$. The remaining x then follows from (9).

3. ALTERNATIVE SOLUTION OF LQ OPEN-LOOP DISCRETE-TIME CONTROL

The further, third equation implies from (8) and (9). Multiplying (8) by b and (9) by $(a,h)(b,f)$ and adding them mutually yields

$$d^\rho s_* [(a,h)(b,f)x + by] = d^\rho s_* sf$$

and hence

$$(a,h)x + b_f y = sf_b . \tag{11}$$

Using this, so-called "implied" open-loop equation (11), the alternative way to solve LQ control can be presented and proved.

Claim 1. LQ discrete-time, open-loop control, defined by the relations (1) to (7) and (10), is solved by

$$y = y_p + (a,h)t \quad \text{and} \quad x = x_p - b_f t , \tag{12}$$

where x_p, y_p is any arbitrary particular solution of equation (11) and t belongs to the minimum $\deg z$ solution t, z, $\deg z < \rho$, of the polynomial equation

$$d^\rho s_* t + h_a z = r , \tag{13}$$

where introducing

$$q = \psi b(b,f)_* x_{p*} - \phi a a_{h*} y_{p*} \tag{14}$$

yields

$$r = \frac{d^\rho q_*}{s} .$$

The optimal solution exists if and only if $h_a = h_a^+$ and is unique.

Proof. Substituting (12) into equations (8) and (9) yields

$$d^\rho s_* y_p + d^\rho s_* (a,h)t + hz = d^\rho b_* \psi f \tag{15}$$

and

$$d^\rho s_* x_p - d^\rho s_* b_f t - b_f h_a z = d^\rho a_* \phi a_h f_b \,. \tag{16}$$

If (15) multiplied by x_p and (16) by y_p are mutually subtracted, we obtain

$$[(a,h)\,x_p + b_f y_p]\,(d^\rho s_* t + h_a z) = d^\rho f_b [\psi b_* (b,f)\,x_p - \phi a_* a_h y_p] \,.$$

Since (11) is true for any x_p and y_p then using (13) and (14)

$$s(d^\rho s_* t + h_a z) = d^\rho q_* \quad \text{or} \quad sr = d^\rho q_* \,, \tag{17}$$

□

4. OPTIMAL LQ OPEN–LOOP CONTROL SOLUTION VIA THE IMPLIED EQUATION ONLY

Using the relations derived above the sufficient conditions can be found under which the minimum solution of the implied equation (11) is LQ optimal. The following claim gives the result.

Claim 2. LQ discrete-time,open-loop control problem, described by the relations (1) to (7) and (10), is solved uniquely by the minimum $\deg y$ solution x, y, $\deg y < \deg (a,h)$, of the equation (11), if simultaneously

$$\deg h_a = 0 \tag{18}$$

and

$$\deg (a,h) + \beta > \deg f \,. \tag{19}$$

Proof. If x_p, y_p is the minimum $\deg y$ solution of (11), then $t = 0$ in (12) as well as (13) and hence $h_a z = r$. Since generally h_a does not divide r, $z \sim r$ must be supposed. Therefore (18) and

$$\deg z = \deg r < \rho \tag{20}$$

are the necessary general conditions for x_p, y_p as the minimum $\deg y$ solution of (11) can be LQ optimal at all.

The following relations introduced in [4] are valid:

$$\deg (d^\rho s_*) = \deg (d^\rho a_*) = \rho \quad \text{and} \quad \deg (d^\rho b_*) = \rho - \beta \,;$$

i)	if	$\deg a > \deg b^c$	then	$\deg s = \deg a\,;$	(21)
ii)	if	$\deg a = \deg b^c$	then either	$\deg s = \deg a$	(22)
			or	$\deg s < \deg a\,;$	(23)
iii)	if	$\deg a < \deg b^c$	then	$\deg s = \deg b^c\,.$	(24)

The minimum $\deg y$ solution x_p, y_p of (11) has the following properties:

$$\deg y_p < \deg(a,h) \tag{25}$$

and

$$\deg x_p < \deg b_f \tag{26}$$

if

$$\deg(a,h) + \deg b_f > \deg s + \deg f_b \tag{27}$$

or

$$\deg x_p < \deg s + \deg f_b - \deg(a,h) + 1 \tag{28}$$

if

$$\deg(a,h) + \deg b_f \le \deg s + \deg f_b \,. \tag{29}$$

Using the presented relations along with (14) and (17) we can write

$$\begin{aligned} \deg r &= \deg(d^\rho q_*) - \deg s \\ &\le \max[\rho - \beta + \deg(b,f) + \deg x_p \,,\ \rho + \deg a_h + \deg y_p] - \deg s \\ &= \rho - \deg s + \max[\deg(b,f) - \beta + \deg x_p, \deg a_h + \deg y_p] \,. \end{aligned} \tag{30}$$

If (27) is true the relation (30) obtains the form

$$\deg r < \rho - \deg s + \max(\deg a, \deg b^c) \,.$$

Hence one can see that (20) will be valid if (21) or (22) or (24) holds. Assuming (23) we can write

$$\begin{aligned} (a,h)\, b_f d^\rho q_* &= (a,h)\, b_f [d^\rho b_* \psi (b,f)\, x_p - d^\rho a_* \phi a_h y_p] \\ &= d^\rho b_* \psi b(a,h)\, x_p - d^\rho s_* s b_f y_p + d^\rho b_* \psi b b_f y_p \\ &= d^\rho b_* \psi b s f_b - d^\rho s_* s b_f y_p = s b_f (d^\rho b_* \psi f - d^\rho s_* y_p) \end{aligned}$$

and hence

$$\begin{aligned} \deg r &= \deg(d^\rho b_* \psi f - d^\rho s_* y_p) - \deg(a,h) \\ &\le \max[\rho - \beta + \deg f,\ \rho + \deg(a,h) - 1] - \deg(a,h) \\ &= \rho + \max[\deg f - \beta - \deg(a,h), -1] \,. \end{aligned}$$

Therefore the condition (19) must be valid to secure (20) in this case. It is satisfied in the previous case too.

In the second case, when (29) holds, we obtain from (30)

$$\begin{aligned} \deg r < \rho - \deg s + \max[\deg(b,f) - \beta + \deg s + \deg f_b - \deg(a,h) + 1, \\ \deg a_h + \deg(a,h)] \ = \ \rho + \max[\deg f - \deg(a,h) - \beta + 1, \deg a - \deg s] \,. \end{aligned}$$

Hence provided (19) is true, (20) is satisfied in the cases (21) or (22). Considering (23) and (24) then (20) cannot be ensured since the contradictory relation $\deg(a,h) + \beta \le \deg f$ follows from (29). Moreover in the case (23) the requirement (20) can be broken by $\deg a - \deg s > 0$ too.

Thus the conditions (18) with (19) are found to be the sufficient ones for LQ optimality of the minimum $\deg y$ solution of (11). □

The condition (19)

— is always valid if either (21) or (22) or (24) along with (27) hold

— can be true if (23) with (27) or (21) or (22) with (29) are valid

— can never be true if (23) or (24) along with (29) hold.

5. COMPARISONS

The conditions (18) and (19) are very similar to the ones which have been derived in [4] for the minimum solution of the respective implied equation in feedback LQ optimal control.

Let us introduce the basic results concerning this closed-loop LQ problem treated in [4]. The structure under consideration is shown in Figure 2.

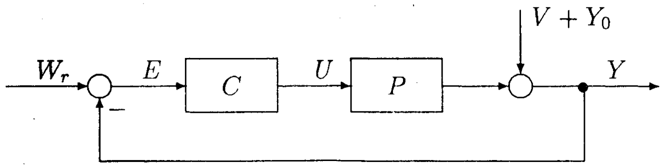

Fig. 2.

The relations (1) to (4) as well as (6) and (7) are valid in the same way (the only $\beta = 0$ must be excluded from (3)), ρ stands in (10) and a feedback controller is supposed to be described by

$$C = \frac{m}{n}, \quad n^-, m^- \text{ coprime}, \quad n = n^c.$$

Then LQ optimal controller is determined as

$$n = n_p - bt \quad \text{and} \quad m = m_p + at,$$

where n_p, m_p is any particular solution of the equation

$$an + bm = sp \tag{31}$$

with p following from $pp_* = a_h a_{h*} f f_*$, and t belongs to the minimum deg z solution z, t, deg $z < \rho$, of the equation

$$d^\rho s_* t + h_a z = l$$

with

$$sl = d^\rho(\psi b_* n_p - \phi a_* m_p).$$

The feedback LQ optimal solution exists if and only if $h_a = h_a^+$ and $p = p^+$ $(p^0 \sim 1)$. Provided

$$\deg h_a = 0$$

and

$$\deg a + \beta > \deg p \tag{32}$$

the only equation (31) may be solved for minimum $\deg m$, $\deg m < \deg a$.

Comparing now the conditions for the simplified solution of the closed-loop and open-loop LQ control, we can see that the first condition (18) is identical in both the cases. Provided it is valid and the feedback problem solvability is guaranteed, then

$$\deg (a,h) = \deg h \quad \text{and} \quad \deg a = \deg h + \deg a_h \,. \tag{33}$$

Using (33) we can find

$$\deg a_h^{=\sim} = \deg a_h^{=} \,, \quad f^{=} = d^{\nu} f^{=c} \,, \quad \nu \geq 0 \,,$$

and hence

$$\deg f^{=} = \nu + \deg f^{=c} \quad \text{but} \quad \deg f^{=\sim} = \deg f^{=c} \,,$$

and the second open-loop condition (19) obtains the form

$$\deg h + \beta > \deg f^{+} + f^{=c} + \nu \,. \tag{34}$$

In a similar way the closed-loop condition (32) can be rewritten as

$$\deg h + \beta > \deg f^{+} + \deg f^{=c} \,. \tag{35}$$

Comparing (34) and (35) they are found to be identical if $\nu = 0$. Provided $\nu > 0$ such a case can occur when LQ optimal feedback controller may be found through the implied equation while the open-loop control may not. For example, if $P = d/(1-d)$ and $W = d(1+0.5d)/(1-d)$, then (35) is satisfied while (34) is not.

Finally we shall return to the general case of LQ control when the conditions (18) and (19) or (32) play no role. The question can arise, when the LQ optimal feedback controller can simply be designed as the ratio $C = U/E$ where U and E are LQ optimal open-loop signals standing in (5).

Using these relations

$$C = \frac{m}{n} = \frac{a_h y}{(b,f)\, x}$$

and substituting it into the corresponding LQ optimal closed-loop equation (31) yields

$$a(b,f)\, x + b a_h y = s f^{+} f^{=\sim} a_h^{+} a_h^{=\sim} \,. \tag{36}$$

Hence

$$a_h(b,f)[(a,h)\, x + b_f y] = a_h(b,f)\, s f_b = s f a_h \,, \tag{37}$$

where the open-loop implied equation (11) has been applied.

Comparing right sides of (36) and (37) closed-loop and open-loop LQ optimal signals are found to be identical if and only if $a_h = a_h^{+}$ and $f = f^{+}$. Then $m = a_h^{+} y$ and $n = (b,f)^{+} x$ and the closed-loop coupled equations used in the standard design [2,4]

$$d^{\rho} s_* m + a h_a z = d^{\rho} b_* \psi p$$

and

$$d^{\rho} s_* n - b h_a z = d^{\rho} a_* \phi p$$

obtain the open-loop form (8) and (9).

6. EXAMPLE

Let us solve the LQ open-loop control problem for

$$P = \frac{b}{a} = \frac{d}{1-d}, \quad W = \frac{f}{h} = \frac{1+0.5d}{1-d} \quad \text{and} \quad \psi = \phi = 1 .$$

We have

$$h_a = a_h = 1 , \quad b_f = d , \quad f_b = f = 1 + 0.5d , \quad (a,h) = 1 - d ,$$

$$\beta = 1 , \quad \rho = 1 \quad \text{and} \quad s = 1.618 - 0.618d .$$

Since $\deg h_a = 0$ and $\deg (a,h) + \beta = 2 > 1 = \deg f$, the conditions (18) and (19) are satisfied and the simple solution according to Claim 2 can be applied.

Then the equation (11)

$$(1-d)\,x + dy = (1.618 - 0.618d)\,(1 + 0.5d)$$

has the minimum $\deg y$ solution $x = 1.618 + 0.309d$ and $y = 1.5$, which is just optimal one.

The resulting optimal signals according to (5) are

$$U = \frac{1.5}{1.618 - 0.618d} \quad \text{and} \quad E = \frac{1.618 + 0.309d}{1.618 - 0.618d} .$$

Using the way of Claim 1 we write the general solution of (11)

$$x = 1.618 + 0.309d - dt \quad \text{and} \quad y = 1.5 + (1-d)\,t ,$$

$$q = -1.191 + 3.118d \quad \text{and} \quad r = 1.927 .$$

The equation (13)

$$(-0.618 + 1.618d)\,t + z = 1.927$$

is solved for minimum $\deg z < \rho$ by $t = 0$, $z = r = 1.927$.

Since $a_h = a_h^+ = 1$ as well as $f = f^+ = 1 + 0.5d$, the closed-loop LQ optimal controller can be determined as the ratio U/E of the open-loop optimal signals and

$$C = \frac{1.5}{1.618 + 0.309d} .$$

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