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The Beginning of a Mathematical Simulation Theory

HANS-DETLEF GERHARDT

The paper contains the first step to a mathematical theory of simulation. The problem of the simulation of a first abstract mathematical system by a second with respect to the input-output behaviour as well as the input-output behaviour and the state is considered. A precise mathematical definition is given. An example is provided which gives evidence for the applicability of the theoretical development.

1. INTRODUCTION

In this paper the term "simulation" means the realization of experiments with the aid of mathematical models whereby the simulator is a computer (digital, analogue or hybrid computer). The aim of simulation is the analysis/synthesis/development of a system/study of a system.

An interaction between a man considering a simulation problem and a computer is advantageous. Today simulation methods solving different problems are used frequently.

Reading some of the articles available [3], [4], [6], [8], [9], [10], we see that there are important differences by the use of such notions as original, object, simulation, simulation model etc., and furthermore there exist many unsettled theoretical problems. Some theoretical problems of simulation have been presented in [5], [7], [11], [12], [14].

In this paper, we shall be concerned with the problem of the simulation of a first abstract mathematical system by a second. A precise mathematical definition will be given.

An example is provided which gives evidence for the applicability of the theoretical development.

Given an original, an important problem is the consideration of reactions of the original when we have actions in the environment of the original. Inversely, an important problem is to design an original with determined properties. An additional interesting question is the consideration of the possible effects on the original in order to obtain a determined reaction of the original.

Solving these problems, we replace the original by a suitable model which is similar (from a certain point of view) to the original and more convenient for the study than the original. The model reproduces all the pertinent properties and important interactions with the environment possessing the original under consideration. If the model exists in reality and it is investigated by real experiments, we come to the well known form of simulation. The model is signified as a simulator.

If the model is a mathematical one two possibilities exist to solve a given problem. First it is only a computation and experiments are not necessary, that means at least one practical mathematical algorithm exists for the computation of the solution of the given problem, e.g.

– solution of a linear equation system
– transport optimization
– quadrature by Monte-Carlo-method.

Second, if this is impossible or if the expense is very large for the application of such an algorithm, the simulation is a possible method to solve many problems given above.

In the following we consider the procedure of the simulation as illustrated in Fig. 1, where *O* is Original, *OS* Original system, *M* Mathematical model, *DM* Derived

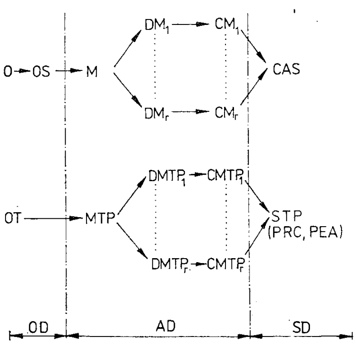

Fig. 1.

mathematical model, *CM* Computation model, *CAS* Computer as a simulator, *OT* Original test plan, *MTP* Mathematical model test plan, *DMTP* Derived mathematical model test plan, *CMTP* Computation model test plan, *STP* Simulation test plan, *PRC* Plan of result check, *PEA* Plan of experiments automation, *OD* Original domain, *AD* Abstract domain, *SD* Simulation domain.

Original systems are real or hypothetical systems that are studied (or developed) – regarding given aims – with the aid of experiments to be conducted with models of the original systems.

Simulation models are models of the original system with the following properties:

1. The model describes all the properties which are considered to be important to the original system.
2. The aim of simulation is described at the level of the model.
3. In order to obtain the aim of simulation it is possible to carry out manipulations with the model.
4. The results obtained for the model can be transferred to the original system.

If the simulation model is a mathematical model it is called "mathematical model of the original system", which will be abbreviated "mathematical model".

Sometimes it is impossible to map the mathematical model into a computation model at once. For example, the original system is described by means of partial differential equations and, the given resources (e.g. the available simulation language) do not allow to solve the partial differential equations.

Then either we extend the real resources or we map the mathematical model into a derived mathematical one. If it is possible we construct a derived mathematical model as a system of ordinary differential equations instead of the partial differential equations of the mathematical model. It is clear that we obtain only an approximating solution.

Now we have to map the derived mathematical model into a computation model suitable for running on a computer. As an example we consider the heat transmission in the insulating layer of an electric cable. If u_0 is the constant temperature of the electrical conductor, u_s the constant temperature of the environment of the cable where $u_s < u_0$ we have the equation of heat transmission

$$\frac{\partial u}{\partial t} = a(r)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

with the conditions

$$u(r, t) = \begin{cases} u_0 & \text{for} \quad r \leqq r_0 \\ u_s & \text{for} \quad r \geqq R\,, \end{cases}$$

$$u(r, 0) = \begin{cases} u_0 & \text{for} \quad r \leqq r_0 \\ u_s & \text{for} \quad r > r_0\,, \end{cases}$$

that means, at the beginning the insulating layer has the temperature of the environment. See Fig. 2.

If it is impossible to solve this problem in a direct way (e.g. if an analogue computer exists only), one tries to build a derived mathematical model. We replace the expressions $\partial u/\partial r$, $\partial^2 u/\partial r^2$ by difference quotients.

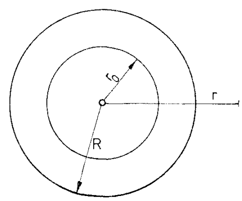

Fig. 2.

Therefore we obtain

$$\frac{du_i}{dt} = a_i\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \frac{1}{r_0 + i\,\Delta r}\,\frac{u_{i+1} - u_{i-1}}{2\Delta r}\right)$$

for $i = 1(1)\,n$ with the initial values

$$u_i(0) = \begin{cases} u_0 & \text{for} \quad i = 0 \\ u_s & \text{for} \quad i = 1(1)\,n\,. \end{cases}$$

The conditions yield u_0 and $u_{n+1} = u_s$. In order to solve this problem on an analogue computer a qualitative and a quantitative programming are necessary.

In order to solve this problem on a digital computer we formulate the computation model in a simulation language as CSMP/III. Thereby the scientist can select one of the given integration methods.

It is often difficult to build a mathematical model. Some methods which can be used are considered in [2].

If we have a computation model derived from a mathematical model in most cases, it is necessary to test by experiments with the computer whether or not the computation model simulates the original system, i.e. roughly speaking the results which are obtained by experiments with the computer and transferred to the original system are similar (from a certain point of view) to the results obtained by experiments with the original system.

If the computation model does not simulate the original system various reasons are possible.

1. The quality of the mathematical model is unsatisfactory.

2. The derived mathematical model does not simulate the mathematical model.
3. The computation model does not simulate the derived mathematical model.
4. The general principle that "similar models have similar properties" is not true for our example.

Clearly, if reasons 1–4 are not fulfilled it need not follow that the computation model simulates the original system.

In case 2 it may be either possible to build a derived mathematical model which is more convenient than the first or it is impossible to find a mathematical method convenient to solve the given problem. Then it is necessary either to construct another mathematical model or to extend the given resources.

In case 3 it can be helpful either to choose a new step length or to use another integration method.

These verbal views to be represented here are a first step. Thus it is necessary to define a number of terms exactly. The term "simulate" has central significance. Therefore in the next chapter a proposition for the mathematical definition of this term is given. Then if we could prove that the computation model simulates the mathematical one it is "only" necessary to show that the mathematical model simulates (from a certain point of view) the behaviour and/or the state of the original system with sufficient precision.

3. SIMULATION OF AN ABSTRACT MATHEMATICAL SYSTEM S' BY AN ABSTRACT MATHEMATICAL SYSTEM S

3.1. Definition of an abstract mathematical system S

Let R be the set of real numbers and I the set of integers. If T is a set such that $T \subseteq R$ or $T \subseteq I$, if $\leqq$ is an order relation between two arbitrary elements of the set T, and if t_0 is the minimum element of $(T, \leqq)$, then $(T, \leqq, t_0)$ is called "*time set*", which will be abbreviated "*time*".

Let time $= (T, \leqq, t_0)$, and $t_0 \leqq t_1 \leqq t_2$ for $t_1, t_2 \in T$. Consider the subsets T^{t_1}, T_{t_1} and T_{t_1,t_2} of T defined as follows:

$$T^{t_1} = \{\bar{t} : t \in T \wedge \bar{t} < t_1\},$$
$$T_{t_1} = \{\bar{t} : t \in T \wedge \bar{t} \geqq t_1\},$$
$$T_{t_1,t_2} = \{\bar{t} : t \in T \wedge t_1 \leqq \bar{t} \leqq t_2\}.$$

A function f defined on the set T is called *time function* (*signal*). Let M be the range of the time function f. The restriction of f to the subset T_{t_1,t_2} of T is denoted by f_{t_1,t_2}.

Let $f'_{t_1+\tau,t_2+\tau}$ be a time function such that

$$f'_{t_1+\tau,t_2+\tau} : T_{t_1+\tau,t_2+\tau} \to M \quad \text{for} \quad \tau \in T_0 .$$

Let f_{t_1,t_2} be a time function such that

$$f_{t_1,t_2} : T_{t_1,t_2} \to M .$$

If $f'_{t_1+\tau,t_2+\tau}(t) = f_{t_1,t_2}(t - \tau)$ for $\tau \in T_0$ and for all $t \in T_{t_1+\tau,t_2+\tau}$, then $f'_{t_1+\tau,t_2+\tau}$ is called a *translation* of f_{t_1,t_2} with respect to T.

Let be $T = R$. If $f_{t_1,t_2}, f'_{t_2,t_3}$ are time functions, and if $t_0 \leqq t_1 \leqq t_2 \leqq t_3$, then there exists $(ff')_{t_1,t_3}$. It is defined as follows:

$$(ff')_{t_1,t_3} = \begin{cases} f_{t_1,t_2}(t) & \text{for all} \quad t \in T_{t_1,t_2} \\ f'_{t_2,t_3}(t) & \text{for all} \quad t \in T_{t_2,t_3} - t_2 \end{cases}$$

if $t_1 \leqq t_2 < t_3$,

$$(ff')_{t_1,t_3} = f_{t_1,t_2}(t) \quad \text{for all} \quad t \in T_{t_1,t_3}$$

if $t_1 < t_2 = t_3$.

We can define $(ff')_{t_1,t_3}$ for $T = I$ in a similar way.

Now it is possible to define an abstract mathematical system S.

Definition 1. An *abstract mathematical system* S is a mathematical structure $S = (T, \leqq, t_0, \Omega, Q, H, \delta, \lambda)$, where:

$(T, \leqq, t_0)$ is a time set;

Ω is a set of input signals;

Q is a nonempty set of states;

H is a set of output signals;

and

$\delta : T \times Q^m \times \Omega^m \to Q^m$,

$\lambda : T \times Q^m \times \Omega^m \to H^m$

are the state transition function and output function, respectively.

m is the maximum value of j, n, k where:

j is the number of inputs,

n the number of states, and

k the number of outputs.

We use the following notation:

$$\bar{\omega} = (\omega_1, \ldots, \omega_j, 0, \ldots, 0) \in \Omega^m = \Omega \times \Omega \times \ldots \times \Omega ,$$

where $\omega_i \in \Omega$ for all $i = 1, \ldots, j$;

$$\bar{\eta} = (\eta_1, \ldots, \eta_k, 0, \ldots, 0) \in H^m ,$$

where $\eta_i \in H$ for all $i = 1, \ldots, k$;

$$\bar{q} = (q_1, \ldots, q_n, 0, \ldots, 0) \in Q^m ,$$

where $q_i \in Q$ for all $i = 1, \ldots, n$. (See Fig. 3 for illustration.)

The objects above must satisfy the following conditions:

Condition 1. Given a nonempty set X of the input signal values, for example $X = R$, then

$$\Omega \subseteq \{\omega : \omega : T_{t_1,t_2} \to X \text{ for all } t_1, t_2 \in T \wedge t_0 \leqq t_1 \leqq t_2\} .$$

Given a nonempty set Y of the output signal values, then

$$H \subseteq \{\eta : \eta : T_{t_1,t_2} \to Y \text{ for all } t_1, t_2 \in T \wedge t_0 \leqq t_1 \leqq t_2\} .$$

When no confusion is likely to arise, we use ω, η instead of ω_{t_1,t_2} and η_{t_1,t_2}, respectively.

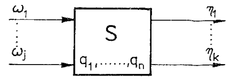

Fig. 3.

Condition 2. Ω and H are closed under translation. Ω and H are closed under concatenation.

Condition 3. δ has the following properties:

(3.1) $\delta(t, \bar{q}, \bar{\omega})(\tau)$ is defined for all $\tau \in T_t$;

(3.2) $\delta(t, \bar{q}, \bar{\omega})(t) = \bar{q}$,

$\delta(t_1, \bar{q}, \bar{\omega}_{t_1,t_2})$ will be defined by

$$\delta(t_1, \bar{q}, \bar{\omega}_{t_1,t_2}) = \delta(t_1, \bar{q}, \bar{\omega}_{t_1,t})(t_2) \quad \text{for all} \quad t \in T_{t_2} ;$$

(3.3) if $\bar{\omega}^1, \bar{\omega}^2 \in \Omega^m$, $t_1 < t_2$, $t_1, t_2 \in T$, and if

$\bar{\omega}^1_{t_1,t_2} = \bar{\omega}^2_{t_1,t_2}$, then

$$\delta(t_1, \bar{q}, \bar{\omega}^1)(\tau) = \delta(t_1, \bar{q}, \bar{\omega}^2)(\tau) \quad \text{for all} \quad \bar{q} \in Q^m \quad \text{and} \quad \tau \in T_{t_1,t_2} ;$$

(3.4) for all $\bar{\omega}^1_{t_1,t_2}, \bar{\omega}^2_{t_2,t_3} \in \Omega^m$, $\bar{q} \in Q^m$

$$\delta(t_1, \bar{q}, (\bar{\omega}^1\bar{\omega}^2)_{t_1,t_3}) = \delta(t_2, \delta(t_1, \bar{q}, \bar{\omega}^1_{t_1,t_2}), \bar{\omega}^2_{t_2,t_3}) .$$

Condition 4. λ has the following properties:

(4.1) $\lambda(t, \bar{q}, \bar{\omega})(\tau)$ is defined for all $\tau \in T_t$;

(4.2) if $\bar{\omega}^1, \bar{\omega}^2 \in \Omega^m$, $t_1 < t_2$, $t_1, t_2 \in T$, and if $\bar{\omega}^1_{t_1,t_2} = \bar{\omega}^2_{t_1,t_2}$, then

$$\lambda(t_1, \bar{q}, \bar{\omega}^1)(\tau) = \lambda(t_1, \bar{q}, \bar{\omega}^2)(\tau) \quad \text{for all } \bar{q} \in Q^m \text{ and } \tau \in T_{t_1,t_2} ;$$

(4.3) for all $\bar{\omega}^1_{t_1,t_2}, \bar{\omega}^2_{t_2,t_3} \in \Omega^m$, $\bar{q} \in Q^m$, $\tau \in T_{t_1,t_3}$

$$\lambda(t_1, \bar{q}, (\bar{\omega}^1\bar{\omega}^2)_{t_1,t_3})(\tau) = \lambda(t_2, \delta(t_1, \bar{q}, \bar{\omega}^1_{t_1,t_2}), \ \bar{\omega}^2_{t_2,t_3})(\tau) .$$

Definition 2. The *input-output behaviour* of a system $S = (T, \leqq, t_0, \Omega, Q, H, \delta, \lambda)$ with respect to a set of input signals $\bar{\omega} \in \Omega^m$ is the set $\{\beta_{t,\bar{q}} : \bar{q} \in Q^m,\ t \in T\}$ where for $\bar{q} \in Q^m$

$$\beta_{t,\bar{q}} : \Omega^m \to H^m$$

is given by $\beta_{t,\bar{q}}(\bar{\omega}) = \lambda(t, \bar{q}, \bar{\omega})$, for all $\bar{\omega} \in \Omega^m$, $t \in T$.

Consider an abstract mathematical system S'. Let R_1 be a relation between two elements of $(H')^{m'}$. The subset $(H')^{2m'}_{R_1'}$ of $(H')^{2m'}$ consists of the unique set of the elements of $(H')^{m'}$ for which the relation R_1' is true.

Definition 3. Let S, S' be two abstract mathematical systems. We say S *simulates the input-output behaviour* of S' with respect to

– the set Ω'

– the relation R_1'

if there exists (h_1, h_2, h_3), where

$$h_1 : (\Omega')^{m'} \to (\Omega)^m,$$
$$h_2 : T' \times (Q')^{m'} \to T \times (Q)^m,$$
$$h_3 : (H)^m \to (H')^{m'},$$

are such that

$$\left(\beta'_{t',\bar{q}'}(\bar{\omega}'),\quad h_3 \circ \beta_{h_2(t',\bar{q}')} \circ h_1(\bar{\omega}')\right) \in (H')^{2m'}_{R_1'}$$

for all $\bar{q}' \in (Q')^{m'}$, $t' \in T'$, $\bar{\omega}' \in (\Omega')^{m'}$.

The definition is illustrated in Fig. 4.

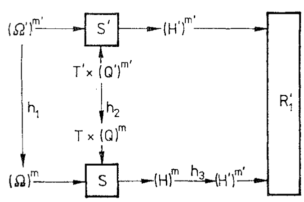

Fig. 4.

The 3-tuple $(t', \bar{q}', \bar{\omega}')$ of the system S' is mapped in a 3-tuple $(t, \bar{q}, \bar{\omega})$ of the system S. Then if the output $\bar{\eta}$ is computed, we have a map h_3 relating the output $\bar{\eta}$ of the system S to the output $\bar{\eta}'$ of the system S'. $\bar{\eta}'$ is compared with the element $\bar{\eta}'$ with respect to the relation R_1' where $\bar{\eta}'$ is computed in the system S'.

Definition 4. P' is a mapping such that $P'(t', \bar{q}') = \bar{q}'$ for all $t' \in T'$, $\bar{q}' \in Q'$.

3.3. Simulation of the input-output behaviour and of the state of a system S' by a system S

Consider an abstract mathematical system S'. Let R'_2 be a relation between two elements of $(H')^{m'}$. The subset $(H')^{2m'}_{R'_2}$ of $(H')^{2m'}$ consists of the unique set of the elements of $(H')^{m'}$ for which the relation R'_2 is true. Let R'_3 be a relation between two elements of $(Q')^{m'}$. The subset $(Q')^{2m'}_{R'_3}$ of $(Q')^{2m'}$ consists of the unique set of the elements of $(Q')^{m'}$ for which the relation R'_3 is true.

Definition 5. Let S, S' be two abstract mathematical systems. We say, S *simulates the input-output behaviour and the state* of S' with respect to

– the set Ω'
– the relation R'_2
– the relation R'_3

if there exists $(h_1, \bar{h}_2, h_3)$, where

$$h_1 : (\Omega')^{m'} \to (\Omega)^m ,$$
$$\bar{h}_2 : T \times (Q_1)^m \to T' \times (Q')^{m'} \text{ (onto)} ,$$
$$h_3 : (H)^m \to (H')^{m'} ,$$

are such that

1. $(Q_1)^m \subseteq (Q)^m$ is closed under $h_1((\Omega')^{m'})$, if $\bar{\omega}' \in (\Omega')^{m'}$, $t \in T$, $\bar{q} \in (Q_1)^m$ then $\delta(t, \bar{q}, h_1(\bar{\omega}')) \in (Q_1)^m$;
2. for all $(t', \bar{q}') \in T' \times (Q')^{m'}$ exists at least a tuple $(t, \bar{q}) \in T \times (Q_1)^m$ with $\bar{h}_2(t, \bar{q}) = (t', \bar{q}')$;
3. if $\bar{h}_2(t_1, \bar{q}_1) = (t'_1, \bar{q}'_1)$, $\bar{h}_2(t_2, \bar{q}_2) = (t'_2, \bar{q}'_2)$ and if $t_1 \neq t_2$ then $t'_1 \neq t'_2$, where $t_1, t_2 \in T$, $t'_1, t'_2 \in T'$, $\bar{q}_1, \bar{q}_2 \in (Q_1)^m$, $\bar{q}'_1, \bar{q}'_2 \in (Q')^{m'}$;
4. for all $\bar{q} \in (Q_1)^m$, $t \in T$, $\bar{\omega}' \in (\Omega')^{m'}$

$$(P'(\bar{h}_2(t, \delta(t, \bar{q}, h_1(\bar{\omega}')))) , \quad \delta'(\bar{h}_2(t, \bar{q}), \bar{\omega}')) \in (Q')^{2m'}_{R'_3}$$

and

$$(\beta'_{\bar{h}_2(t,\bar{q})}(\bar{\omega}'), \ h_3 \cdot \beta_{t,\bar{q}} \cdot h_1(\bar{\omega}')) \in (H')^{2m'}_{R'_2} .$$

The definition is illustrated in Fig. 5.

By assumption, there exists a tuple $(t, \bar{q}) \in T \times (Q)^m$ such that $h_2(t, \bar{q}) = (t', \bar{q}')$.

3.4. Example

Given the linear differential equation

$$A \frac{\mathrm{d}x}{\mathrm{d}t} + x(t) = 1(t) , \quad A \geqq 0{\cdot}5 ,$$

with the initial condition $x(0) = 0$. We wish to examine the possibility of finding a system S'. Setting

$$\begin{aligned} T' &= R\,, \quad t'_0 = 0\,, \\ \Omega' &= \{\omega' : \omega' : T'_{t_1,t_2} \to 1 \text{ for all } t_1, t_2 \in T'_0,\ 0 \leqq t_1 < t_2\}\,, \\ Q' &= R\,, \\ H' &= \{\eta' : \eta' : T'_{t_1,t_2} \to Y' \text{ for all } t_1, t_2 \in T'_0,\ t_1 < t_2,\ Y' = R\,, \\ &\qquad \eta'_{t_1,t_2}(t) \text{ is continuous for all } t \in T'_{t_1,t_2}\}\,, \\ \delta'(0, 0, \omega'_{0,t_2}) &= 1 - \mathrm{e}^{-t_2/A}\,, \\ \lambda'(0, 0, \omega'_{0,t_2})(t) &= 1 - \mathrm{e}^{-t/A} \quad \text{for all} \quad t \in T'_{0,t_2}\,, \end{aligned}$$

we can readily see that conditions 1–4 hold. Hence S' is an abstract mathematical system.

Let us consider the difference equation

$$(2A + 1)\,x(n + 1) - (2A - 1)\,x(n) = 2\,.\,1(n)\,, \quad A \geqq 0{\cdot}5\,,$$

with the initial condition $x(0) = 0$.

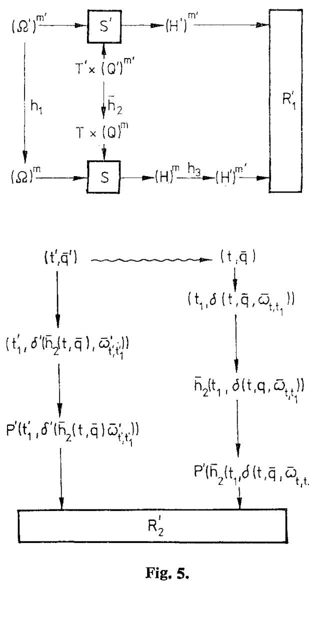

Fig. 5.

To obtain an abstract mathematical system S, let us set

$$T = I\,, \quad t_0 = 0\,,$$
$$\Omega = \{\omega : \omega : T_{i,i+j} \to 2 \text{ for all } i, j \in T_0\}\,,$$
$$Q = R\,,$$
$$H = \{\eta : \eta : T_{i,i+j} \to Y \text{ for all } i, j \in T_0,\ Y = R\}\,,$$
$$\delta(0, 0, \omega_{0,n_1}) = 1 - \left(\frac{2A - 1}{2A + 1}\right)^{n_1},$$
$$\lambda(0, 0, \omega_{0,n_1}) = 1 - \left(\frac{2A - 1}{2A + 1}\right)^{n} \quad \text{for all} \quad n = 0, 1, \ldots, n_1\,.$$

Conditions 1–4 are satisfied. Therefore, we have defined an abstract mathematical system S.

Now let us discuss the computation of magnitudes of A, for which the system S simulates the input-output behaviour of the system S' with respect to the relation R_1'. It is defined: $(\eta_1', \eta_2') \in (H')^2_{R_1'}$ if and only if

$$s_R = \tfrac{1}{5} \sum_{i=1}^{5} ((\eta_1')_{0,2}\,(0{\cdot}4\,.\,i) - (\eta_2')_{0,2}\,(0{\cdot}4\,.\,i)) \leqq 0{\cdot}025\,;$$

$h_1(\omega'_{t_1,t_2}) = \omega_{n_1,n_2}$, where n_1 is the least integer with $n_1 \geqq t_1$, n_2 the largest integer with $n_2 \leqq t_2$, and $\omega_{n_1,n_2}(n_1 + i) = 2\,.\,\omega'_{t_1,t_2}(n_1 + i)$ for all $i = 0(1)n_2 - n_1$; $h_2(t', q') = (n, q_n)$, where $n = n'$ and n' is the largest integer with $n' \leqq t'$ and $q_n = q'_{n'}$ (q_n is the state for $t = n$.)

There are many ways to define the mapping h_3. The answer of the question whether the system S simulates the input-output behaviour of the system S' depends on the definition of h_3, too.

Setting

$$a_{i+1} = \eta_{n_1,n_2}(n_1 + i + 1) - \eta_{n_1,n_2}(n_1 + i)\,,$$
$$b_{i+1} = (n_1 + i + 1)\,\eta_{n_1,n_2}(n_1 + i) - (n_1 + i)\,\eta_{n_1,n_2}(n_1 + i + 1)$$

for all $i = 0(1)n_2 - n_1 - 1$, we can define

$$\eta'_{t_1,t_2}(t) = h_3(\eta_{n_1,n_2})\,(t) = \begin{cases} a_1 t + b_1 & \text{for} \quad t \in T_{t_1,n_1+1}\,, \\ a_{i+1} t + b_{i+1} & \text{for} \quad t \in T_{n_1+i,n_1+i+1}\,, \\ a_{n_2-n_1} t + b_{n_2-n_1} & \text{for} \quad t \in T_{n_2-1,t_2}\,. \end{cases}$$

Now it is possible to determine the quantities of A for which the system S simulates the input-output behaviour of the system S'.

We have

$$\beta'_{t',q'}(\omega'_{0,2}) = \beta'_{0,0}(1(t)_{0,2}) \quad \text{and}$$

$$\eta'_1(t) = \beta'_{0,0}(1(t)_{0,2})(t) = \lambda'(0, 0, 1(t)_{0,2})(t) = 1 - e^{-t/A}$$

for $t \in T_{0,2}$, furthermore, if $\omega_{0,2}(t) = 1(t)_{0,2}$ we obtain

$$h_1(1(t)_{0,2}) = 2 \,.\, 1(n)_{0,2}\,, \quad \text{and}$$

$$\eta'_2(t) = \beta_{0,0}(2 \,.\, (1(n)_{0,2})(t) = \lambda(0, 0, 2 \,.\, 1(n)_{0,2})(t) =$$

$$= 1 - \left(\frac{2A - 1}{2A + 1}\right)^t \quad \text{for} \quad t = 0, 1, 2\,.$$

Here

$$a_1 = \eta_2(1) - \eta_2(0) = \frac{2}{2A + 1}\,,$$

$$b_1 = \eta_2(0) = 0\,,$$

$$a_2 = \eta_2(2) - \eta_2(1) = \frac{4A - 2}{(2A + 1)^2}\,,$$

$$b_2 = 2 \,.\, \eta_2(1) - \eta_2(2) = \frac{4}{(2A + 1)^2}$$

and

$$(\eta'_2)_{0,1}(t) = h_3((\eta_2)_{0,1})(t) = \frac{2}{2A + 1}\, t \quad \text{for} \quad t \in T'_{0,1}$$

$$(\eta'_2)_{1,2}(t) = h_3((\eta_2)_{1,2})(t) = \frac{4A - 2}{(2A + 1)^2}\, t + \frac{4}{(2A + 1)^2} \quad \text{for} \quad t \in T'_{1,2}\,.$$

Table 1.

A	$s(0{\cdot}4)$	$s(0{\cdot}8)$	$s(1{\cdot}2)$	$s(1{\cdot}6)$	$s(2{\cdot}0)$	s_R
0·6	0·12294	0·00913	0·06095	0·02814	0·02739	0·04921
1·0	0·06301	0·01734	0·01229	0·00190	0·02422	0·02375
2·0	0·02127	0·00968	0·00319	0·00667	0·00788	0·00974
5·0	0·00415	0·00240	0·00180	0·00277	0·00090	0·00240
10·0	0·00111	0·00069	0·00061	0·00092	0·00014	0·00069

$$s(t) = |\eta'_1(t) - \eta'_2(t)|$$

$$s_R = \tfrac{1}{5} \sum_{i=1}^{5} s(0{\cdot}4 \,.\, i)$$

It is possible to reexamine whether R_1 is true for certain $(\eta'_1(t), \eta'_2(t))$. Table 1 shows the results for various magnitudes of A. For example, the relation is true for $A = 1{\cdot}0, 2{\cdot}0$, and $10{\cdot}0$.

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