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Necessary Optimality Conditions for N-Player Nonzero-Sum Multistage Games

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A general class of N-player nonzero-sum multistage games with state-dependent regions of admissible controls of each player is studied from the point of view of equilibrium, minimax and noninferior solution type. For all these solutions the necessary optimality conditions are obtained following the general author's scheme in [18]. Also some problems arising in connection with open-loop and closed-loop strategy classes are pointed out.

A special case of linear multistage games with quadratic cost functionals is treated separately. This effort resulted in analytic form of all studied solution types. To illustrate the presented theory, several simple examples are solved in detail, and also some areas of its possible application are briefly reviewed.

1. INTRODUCTION

The subject of the so-called multistage (discrete time) games is a relatively new one. In a certain sense, the multistage games can be considered as an approximation of more often and more in detail studied differential games. Especially in the last decade a number of interesting results and properties concerning the differential games was published. Let us mention at least the references [1] – [8] in this respect. To the author's knowledge, the literature dealing with the multistage games is not so rich.

The first systematic study in the field of multistage games is probably due to Blaquière, Leitman et al. in [9] – [11] and Propoj in [12] and [13]. Some problems arising in connection with the multistage games including delays were explored later by Rudnianski [14]. But all these authors study only the so-called two-player zero-sum multistage games. In fact, very little was done concerning the theory of general N-player nonzero-sum multistage games in comparison with the existing results in the theory of differential games, e.g., see [1] – [5].

Results described in this contribution are based on the author's thesis [15] and were partially presented in [16]. We study here, in principle, a certain class of N-player nonzero-sum multistage games with state-dependent regions of admissible controls for each player. For this class of multistage games we derive necessary optimality conditions for the equilibrium, minimax and noninferior solution types using the results of the author for the optimization of discrete systems with a state-dependent region of admissible controls [17] and [18].

The structure of the paper is as follows. In the next section we summarize the necessary foundations of the classical game theory. Third section contains the precise formulation of an N-player nonzero-sum multistage game with state-dependent regions of admissible controls. The following two sections are devoted to the derivation of necessary optimality conditions for the above mentioned solution types. We point out also some interesting aspects concerning the open-loop and the closed-loop strategy classes.

In the final sections, the obtained results are applied to the class of linear N-player nonzero-sum multistage games with quadratic cost functionals. For every studied solution type it is then possible to compute its analytic form in terms of the solution of discrete matrix Riccati-like equations. As an illustration of the presented results, also simple examples of multistage games are included, which are solved on applying the developed theory.

2. BASIC CONCEPTS FROM THE GAME THEORY

In this section we recall, for convenience, some basic concepts concerning the classical game theory. The reader unfamiliar with this theory can consult any textbook dealing with this subject, e.g. the monographs of Luce and Raiffa [19] and Owen [20] were consulted by the author in this respect. The following simple definition of N-player nonzero-sum game will be quite sufficient for our purposes.

Definition 1. Denote by Ω_i the so-called set of admissible strategies s_i of player i, $i = 1, \ldots, N$. Further let $J_i(s_1, \ldots, s_N)$, $i = 1, \ldots, N$ be real-valued functions defined on $\Omega = \Omega_1 \times \Omega_2 \times \ldots \times \Omega_N$ and let these functions describe the cost functional (costs) of player i, $i = 1, \ldots, N$ for an admissible strategy N-tuple $(s_1, \ldots, s_N) \in \Omega$. Then the triple

$$[N, (\Omega_1, \ldots, \Omega_N), (J_1, \ldots, J_N)] \tag{2.1}$$

is called the N-player nonzero-sum game.

We see that for $N = 2$ and $J_1 = -J_2$ we obtain the frequently studied two-player zero-sum game. We are interested in the following three solution types for such game: equilibrium, minimax and noninferior.

Definition 2. An admissible strategy N-tuple $(s_1^*, \ldots, s_N^*)$ is said the equilibrium strategy N-tuple if, for $i = 1, \ldots, N$,

$$J_i^* = J_i(s_1^*, \ldots, s_N^*) \leqq J_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_N^*)\,, \tag{2.2}$$

where $s_i \in \Omega_i$ is any admissible strategy of player i.

Otherwise speaking, the equilibrium strategy will be the optimal strategy for each of the players, provided that all of the other players are playing their corresponding equilibrium strategies. If the game in question is two-player and zero-sum, the equilibrium solution (strategy pair) is the well-known saddle-point solution. Let us also note that, in general, there can exist more equilibrium solutions (strategy N-tuples), which give different outcomes to the same player. For the more detailed discussion of this problem the reader is referred to [19].

Definition 3. An admissible strategy $\bar{s}_i \in \Omega_i$ is the minimax strategy of player i if, for all admissible strategy N-tuples $(s_1, \ldots, s_N)$,

$$\bar{J}_i = \max_{s_j \in \Omega_j, j \neq i} J_i(s_1, \ldots, \bar{s}_i, \ldots, s_N) \leqq \max_{s_j \in \Omega_j, j \neq i} J_i(s_1, \ldots, s_i, \ldots, s_N)\,. \tag{2.3}$$

The minimax strategy guarantees to the corresponding player that his costs does not exceed the security level given by Definition 3. This solution is extremely pessimistic, because it is assumed that all remaining players have the only objective to "destroy" the i-th player playing against him, i.e. maximizing his costs as much as possible. Clearly such solution is then hardly probable, especially in connection with nonzero-sum games. Moreover, in some realistic and well-posed problems we can have $\bar{J}_i = +\infty$, and thus the player i is forced to construct his strategy from other reasons.

Definition 4. The admissible strategy N-tuple $(\hat{s}_1, \ldots, \hat{s}_N)$ belongs to the noninferior set if, for any other admissible strategy N-tuple $(s_1, \ldots, s_N)$, from

$$J_i(s_1, \ldots, s_N) \leqq J_i(\hat{s}_1, \ldots, \hat{s}_N) = \hat{J}_i\,, \quad i = 1, \ldots, N \tag{2.4}$$

follows the relation

$$J_i(s_1, \ldots, s_N) = J_i(\hat{s}_1, \ldots, \hat{s}_N) = \hat{J}_i\,, \quad i = 1, \ldots, N\,. \tag{2.5}$$

In other words, to find the noninferior set, i.e. the set of all noninferior solutions, is equivalent to the solution of an optimization problem with the vector cost functional $J = (J_1, \ldots, J_N)$. Sometimes such solution is denoted also as a negotiated one. It is interesting to note that there exist certain cases, where one or more noninferior solutions strictly dominate the existing equilibrium solution, i.e. the equilibrium solution is not necessarily also noninferior – the so-called "prisoners' dilemma" situation. To these questions we shall return later when studying multistage games.

Of course, the above mentioned solution types do not exhaust all possibilities how to define some other solution types for N-player nonzero-sum game. For example, in the author's paper [21] the question of the so-called hierarchical (Stackelberg) solution for two-player nonzero-sum multistage game with a certain information structure is studied and necessary optimality conditions derived. Finally, let us point out that we consider only the "pure" strategies, i.e. the "mixed" strategies (regular Borel measures on the set of all pure strategies) approach is not explored in this contribution.

3. FORMULATION OF A MULTISTAGE GAME

Multistage (discrete) optimal control problems are studied for example in [18]. A multistage game can be defined in a quite analogous way as a discrete optimal control problem, provided that there exist more inputs (control vectors) in the given discrete controlled system. These inputs are governed by rational individuals (players). Each of them wants to influence the system, using his appropriate control, in accordance with his aim, which is expressed as a certain cost functional (costs). Then an N-player nonzero-sum multistage game may be formulated in the following way.

It is assumed that the discrete dynamic system is described by a vector difference equation (upper index will denote the corresponding player)

$$(3.1) \qquad x_{k+1} = f_k(x_k, u_k^1 \ldots, u_k^N), \quad k = 0, 1, \ldots, K - 1, \quad x_0 \text{ given},$$

where K is a given positive integer (number of stages), $k = 0, 1, \ldots, K$ denotes the current stage of the system, $x_k \in E^n$ (n-dimensional Eulidean space) denotes stage of the system at the stage k, $u_k^i \in E^{m_i}$ is control (input) of player i at the stage k, and finally $f_k : E^n \times E^{m_1} \times \ldots \times E^{m_N} \to E^n$. If not otherwise stated, the all vectors are supposed to be column-vectors.

The aim of each player is to choose his control sequence $u^i = (u_0^i, u_1^i, \ldots, u_{K-1}^i)$, $i = 1, \ldots, N$ and a corresponding state trajectory $x = (x_0, x_1, \ldots, x_K)$, determined by (3.1), such that his cost functional

$$(3.2) \qquad J_i = g^i(x_K) + \sum_{k=0}^{K-1} h_k^i(x_k, u_k^1, \ldots, u_k^N), \quad i = 1, \ldots, N$$

is minimized. Here $g^i : E^n \to E^1$, $h_k^i : E^n \times E^{m_1} \times \ldots \times E^{m_N} \to E^1$.

An admissible control vector u_k^i of player i at the stage k, $i = 1, \ldots, N$, $k = 0, 1, \ldots$ $\ldots, K - 1$ is supposed to satisfy the state-dependent constraints

$$(3.3) \qquad u_k^i \in U_k^i(x_k), \quad i = 1, \ldots, N, \quad k = 0, 1, \ldots, K - 1,$$

where

$$U_k^i(x) = \{u^i \mid Q_k^i(x, u^i) = 0\,, \quad q_k^i(x, u^i) \leqq 0\}\,,$$

and $Q_k^i : E^n \times E^{m_i} \to E^{r_k{}^i}$, $q_k^i : E^n \times E^{m_i} \to E^{s_k{}^i}$. The inequality sign for vectors is used in the following sense: Let $a \in E^p$; then $a \leqq 0 \Leftrightarrow a_j \leqq 0, j = 1, \ldots, p$.

Definition 5. The triple

$$(3.4) \qquad [N, (U_k^i(x),\ i = 1, \ldots, N,\ k = 0, 1, \ldots, K - 1)\,,\ \ (J_i,\ i = 1, \ldots, N)]$$

subject to (3.1) is called the N-player nonzero-sum multistage game.

Through this paper we consider two types of strategy classes defined further.

Definition 6. Any sequence of points

$$u^i = \{u_0^i, u_1^i, \ldots, u_{K-1}^i \mid u_k^i \in U_k^i(x),\ k = 0, 1, \ldots, K - 1\}$$

is denoted as an admissible open-loop strategy of the i-th player, $i = 1, \ldots, N$.

Definition 7. In analogous was, any sequence of functions

$$\varphi^i(x) = \{\varphi_0^i(x), \varphi_1^i(x), \ldots, \varphi_{K-1}^i(x) \mid \varphi_k^i(x) \in U_k^i(x),\ k = 0, 1, \ldots, K - 1\}$$

where $\varphi_k^i : E^n \to E^{m_i}$, is denoted as an admissible closed-loop strategy of the i-th player, $i = 1, \ldots, N$.

To have the multistage game just stated well-posed, it is always necessary to specify for which strategy class (open-loop or closed-loop) and which solution type the necessary optimality conditions should be derived.

In the next sections we often refer to [18] in order to derive necessary optimality conditions for various solution types. Therefore it is assumed that the reader is to a certain degree acquainted with [18]. This primarily concerns the discussion of various concepts and assumptions, which is not repeated here.

Assumption 1. All functions appearing in the relations (3.1)–(3.3) are continuously differentiable in their domains of definition.

Such assumption is a natural one when dealing with necessary optimality conditions. Further we need the so-called directional convexity concept, which is a basic one the derivation of necessary optimality conditions in the form of discrete maximum principle – see [18].

Definition 8. Let R be a closed convex cone in E^n with vertex in the origin. A set $\Gamma \in E^n$ is said to be R-directionally convex if, for every vector $\tilde{z}$ in the convex hull of Γ, there exists a vector $z_R \in R$ such that $\tilde{z} + z_R \in \Gamma$.

For this purpose, let us denote

$$\tilde{h}^i_k(x, u^1, \ldots, u^N) = h^i_k(x, u^1, \ldots, u^N), \quad i = 1, \ldots, N, \quad k = 0, 1, \ldots, K-2, \tag{3.5}$$

$$\tilde{h}^i_{K-1}(x, u^1, \ldots, u^N) = h^i_{K-1}(x, u^1, \ldots, u^N) + g^i(f_{K-1}(x, u^1, \ldots, u^N)),$$
$$i = 1, \ldots, N,$$

and consider in E^{n+1} sets

$$V^i_k(x, u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^N) = \{(a, v) \mid a \in E^1, v \in E^n,$$
$$a = \tilde{h}^i_k(x, u^1, \ldots, u^N), v = f_k(x, u^1, \ldots, u^N), u^i \in U^i_k(x)\},$$
$$i = 1, \ldots, N, \quad k = 0, 1, \ldots, K-1, \tag{3.6}$$

and a closed convex cone with vertex in the origin

$$R = \{r \mid r \in E^{n+1}, \ r = (\varrho, 0, \ldots, 0), \ \varrho \leqq 0\}. \tag{3.7}$$

Assumption 2. For $i = 1, \ldots, N$, the sets $V^i_k(x, u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^N)$, $k = 0, 1, \ldots, K-1$ are R-directionally convex for any $x \in E^n$, $u^j \in E^{m_j}$, $j = 1, \ldots, N$, $j \neq i$.

Finally, we need certain "regularity" properties of the state-dependent constraints (3.3). Consider therefore the constraints

$$U(x) = \{u \in E^m \mid Q(x, u) = 0, \ q(x, u) \leqq 0\}, \tag{3.8}$$

where the functions $Q : E^n \times E^m \to E^r$, $q : E^n \times E^m \to E^s$ are continuously differentiable. For a given pair (x, u) denote by

$$I[q(x, u)] = \{l \in \{1, \ldots, s\} \mid q^l(x, u) = 0\} \tag{3.9}$$

the so-called active index set of the constraining function q, where $l = 1, \ldots, s$ denotes single component of q.

Assumption 3. For $i = 1, \ldots, N$ and $k = 0, 1, \ldots, K-1$, the vectors (l, m denote the single components of constraints in (3.3))

$$\frac{\partial}{\partial u^i} Q^{il}_k(x, u^i), \quad l = 1, \ldots, r^i_k, \quad \frac{\partial}{\partial u^i} q^{im}_k(x, u^i), \ m \in I[q^i_k(x, u^i)] \tag{3.10}$$

are linearly independent for any $x \in E^n$, $u^i \in U^i_k(x)$.

It is easy to see that in this setting it is not possible to handle multistage games with explicite state constraints. Namely, the Assumption 3 requires the explicite dependence on u^i for functions Q^i_k, q^i_k, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K-1$. On the other hand, if we try to incorporate into the formulation of a multistage game also state constraints, as it is usually done in discrete optimal control problems, the obtained necessary optimality conditions for equilibrium solution contain certain number of reduntant multipliers. Therefore such conditions would be of little practical importance – see [15]. However, in the case of vector optimization problems (noninferior solution) we are able to include also state constraints by this approach, as it is shown in [15] and [17].

4. NECESSARY OPTIMALITY CONDITIONS FOR EQUILIBRIUM SOLUTION

First, let us consider the equilibrium solution over the class of open-loop strategies. From Definition 2 we can conclude, that each player solves only a discrete optimization problem with a state-dependent region of admissible controls, provided that the other players use corresponding equilibrium strategies. Then we can immediately apply the general results from [18, Theorem 5] to this optimization problem.

Through this and also next sections the Hamiltonian notation from [18] will be used for simplicity. As usual, let us introduce the Hamiltonian of the i-th player, $i = 1, \ldots, N$ at the stage k, $k = 0, 1, \ldots, K - 1$ by formula

$$H_{k+1}^i(x, u^1, \ldots, u^N) = -h_k^i(x, u^1, \ldots, u^N) + \lambda_{k+1}^i f_k(x, u^1, \ldots, u^N), \tag{4.1}$$

where row-vectors $\lambda_{k+1} \in E^n$ will be defined later.

Theorem 1. Let the multistage game satisfy Assumptions 1–3. Suppose that the admissible strategy N-tuple $(u^{*1}, \ldots, u^{*N})$ is an equilibrium solution of the game in question on the class of open-loop strategies. The corresponding state trajectory is denoted by $x_0^*, x_1^*, \ldots, x_K^*$.

Then for each $i = 1, \ldots, N$ there exist row-vectors

$$\lambda_k^i \in E^n,\ k = 1, \ldots, K, \quad \zeta_k^i \in E^{r_k{}^i},\ \xi_k^i \in E^{s_k{}^i},\ k = 0, 1, \ldots, K - 1$$

such that the following conditions (a)–(c) are satisfied:

(a)
$$\lambda_k^i = \frac{\partial}{\partial x} H_{k+1}^i(x_k^*, u_k^{*1}, \ldots, u_k^{*N}) + \zeta_k^i \frac{\partial}{\partial x} Q_k^i(x_k^*, u_k^{*i}) + \xi_k^i \frac{\partial}{\partial x} q_k^i(x_k^*, u_k^{*i}),$$
$$k = 1, \ldots, K - 1,$$

with

$$\lambda_K^i = -\frac{\partial}{\partial x} g^i(x_K^*);$$

(b)
$$\frac{\partial}{\partial u^i} H_{k+1}^i(x_k^*, u_k^{*1}, \ldots, u_k^{*N}) + \zeta_k^i \frac{\partial}{\partial u^i} Q_k^i(x_k^*, u_k^{*i}) + \xi_k^i \frac{\partial}{\partial u^i} q_k^i(x_k^*, u_k^{*i}) = 0,$$
$$k = 0, 1, \ldots, K - 1;$$

(c)
$$\xi_k^i \leqq 0, \quad \xi_k^i q_k^i(x_k^*, u_k^{*i}) = 0, \quad k = 0, 1, \ldots, K - 1.$$

This theorem can be also denoted as an N-sided discrete maximum principle, i.e. we have N discrete optimization problems coupled through (3.1) and (3.2). To formulate the analogous conditions for the closed-loop strategy class, we must take into the account function dependence given in Definition 7. Thus. after the obvious changes in the formulation of Theorem 1, we obtain:

Theorem 2. Consider again a multistage game (3.4) satisfying the Assumptions 1–3. Let the strategy N-tuple $(\varphi^{*1}(x), \ldots, \varphi^{*N}(x))$ be its equilibrium solution on the class of closed-loop strategies, and let $x_0^*, x_1^*, \ldots, x_K^*$ be the corresponding state trajectory. Finally, let us suppose that the functions $\varphi_k^{*i}(x)$, $i = 1, \ldots, N$, $k = 0, 1, \ldots \ldots, K-1$ are continuously differentiable in the neighbourhood of this trajectory, i.e. in the neighbourhood of the points x_k^*, $k = 0, 1, \ldots, K$.

Then for each $i = 1, \ldots, N$ there exist row-vectors

$$\lambda_k^i \in E^n,\ k = 1, \ldots, K-1\,,\quad \zeta_k^i \in E^{r_k{}^i},\ \xi_k^i \in E^{s_k{}^i},\ k = 0, 1, \ldots, K-1$$

such that the following conditions (a)–(c) are satisfied:

$$\text{(a)}\qquad \lambda_k^i = \frac{\partial}{\partial x} H_{k+1}^i(x_k^*, \varphi_k^{*1}(x_k^*), \ldots, \varphi_k^{*N}(x_k^*)) + \zeta_k^i \frac{\partial}{\partial x} Q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) +$$

$$+ \xi_k^i \frac{\partial}{\partial x} q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) + \sum_{\substack{j=1\\ j\neq i}}^{N} \left[\frac{\partial}{\partial u^j} H_{k+1}^i(x_k^*, \varphi_k^{*1}(x_k^*), \ldots, \varphi_k^{*N}(x_k^*))\right]\left[\frac{\partial}{\partial x} \varphi_k^{*j}(x_k^*)\right],$$

$$k = 1, \ldots, K-1\,,$$

with

$$\lambda_K^i = -\frac{\partial}{\partial x} g^i(x_K^*)\,;$$

$$\text{(b)}\qquad \frac{\partial}{\partial u^i} H_{k+1}^i(x_k^*, \varphi_k^{*1}(x_k^*), \ldots, \varphi_k^{*N}(x_k^*)) + \zeta_k^i \frac{\partial}{\partial u^i} Q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) +$$

$$+ \xi_k^i \frac{\partial}{\partial u^i} q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) = 0\,,\quad k = 0, 1, \ldots, K-1\,;$$

$$\text{(c)}\qquad \xi_k^i \leqq 0\,,\quad \xi_k^i\, q_k^i(x_k^*, \varphi_k^{*i}(x_k^*)) = 0\,,\quad k = 0, 1, \ldots, K-1\,.$$

On comparing Theorems 1 and 2 we see, that the equilibrium costs (outcomes) differ, in general, as the result of the summation term in condition (a) of Theorem 2. We can, therefore, expect that for the same multistage game, the equilibrium solutions on the open-loop and closed-loop strategy classes will not be generally identical. The situation is thus the same as in the case of differential games, as reported by Starr and Ho in [1] and [2]. However, Sandell in his paper [22] explored this question for a general case of the so-called "feedback games" of Witsenhausen [23]. He came to the conclusion that any open-loop equilibrium strategy N-tuple would be also a closed-loop one. On the one hand, it seems reasonable to treat the open-loop strategy class to be a special case (with a trivial state-dependence) of the closed-loop one. But on the other hand, further research in this direction will be needed to clear up this matter.

Maybe, it would be more appropriate (from practical point of view) to consider only the constant regions of admissible controls for each player in (3.3) when dealing with the open-loop strategy class. Otherwise it can easily happen, that the deviation from his open-loop strategy by player i will result into the inadmissible control sequence of all remaining players, which hold fast their equilibrium strategies. Such conclusion is directly obtained, if we realize the given state-dependence of control constraints in (3.3), i.e. a certain change of the existing state can convert the existing admissible open-loop strategies into the inadmissible ones.

Let us also remark that if in Theorem 2 (closed-loop case) we assume $N = 2$ and $J_1 = -J_2$, i.e. a two-player zero-sum multistage game, our results are almost identical with those of [9]. The results in [9] were derived applying a geometric method, while we used the general theory of discrete systems (discrete maximum principle) derived in [18] using the mathematical programming theory.

5. NECESSARY OPTIMALITY CONDITIONS FOR OTHER SOLUTION TYPES

From the definition of the minimax solution we see that, in fact, only two-player zero-sum multistage game must be solved, in which as a theoretical opponent of the i-th player (cost functional J_i) act all remaining players. For such game the both theorems stated in the previous section are identical, provided that the admissible control regions (3.3) are constant, i.e. functions Q_k^i, q_k^i do not depend on x – see also the pertinent discussion in Section 4. Then the closed-loop solution (saddle-point) will be a synthesis of open-loop problems. This is also a reason, that we prefer to study the more general case of a closed-loop minimax solution for the multistage game (3.4).

Theorem 3. Let two-player zero-sum multistage game, in which results the minimax problem of player i, satisfy Assumptions 1–3. Suppose that the admissible strategy N-tuple $(\varphi'^1(x), \ldots, \varphi'^{i-1}(x), \bar{\varphi}^i(x), \varphi'^{i+1}(x), \ldots, \varphi'^N(x))$ is a saddle-point (minimax solution) of this game. The corresponding state trajectory denote by $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_K$.

Then there exist row-vectors

$$\bar{\lambda}_k^i \in E^n\,, \quad k = 1, \ldots, K\,, \quad \bar{\zeta}_k^j \in E^{r_k j}\,, \quad \bar{\xi}_k^j \in E^{s_k j}\,, \quad k = 0, 1, \ldots, K - 1\,, \\ j = 1, \ldots, N$$

such that the following conditions (a)–(e) are satisfied:

(a) $$\bar{\lambda}_k^i = \frac{\partial}{\partial x} \bar{H}_{k+1}^i(\bar{x}_k, u_k'^1, \ldots, \bar{u}_k^i, \ldots, u_k'^N) + \bar{\zeta}_k^i \frac{\partial}{\partial x} Q_k^i(\bar{x}_k, \bar{u}_k^i) + \bar{\xi}_k^i \frac{\partial}{\partial x} q_k^i(\bar{x}_k, \bar{u}_k^i) -$$
$$- \sum_{\substack{j=1 \\ j \neq i}}^{N} \left[\bar{\zeta}_k^j \frac{\partial}{\partial x} Q_k^j(\bar{x}_k, u_k'^j) + \bar{\xi}_k^j \frac{\partial}{\partial x} q_k^j(\bar{x}_k, u_k'^j) \right], \quad k = 1, \ldots, K - 1,$$

with

$$\bar{\lambda}_K^i = - \frac{\partial}{\partial x} g^i(\bar{x}_K),$$

where we for simplicity denoted

$$\bar{u}_k^i = \bar{\varphi}_k^i(\bar{x}), \quad u_k'^j = \varphi_k'^j(\bar{x}_k), \quad j = 1, \ldots, N, \quad j \neq i, \quad k = 0, 1, \ldots, K - 1,$$

and where

$$\bar{H}_{k+1}^i(x, u^1, \ldots, u^N) = -h_k^i(x, u^1, \ldots, u^N) + \bar{\lambda}_{k+1}^i f_k(x, u^1, \ldots, u^N)$$
$$k = 0, 1, \ldots, K - 1;$$

(b) $$\frac{\partial}{\partial u^i} \bar{H}_{k+1}^i(\bar{x}_k, u_k'^1, \ldots, \bar{u}_k^i, \ldots, u_k'^N) + \bar{\zeta}_k^i \frac{\partial}{\partial u^i} Q_k^i(\bar{x}_k, \bar{u}_k^i) + \bar{\xi}_k^i \frac{\partial}{\partial u^i} q_k^i(\bar{x}_k, \bar{u}_k^i) = 0,$$
$$k = 0, 1, \ldots, K - 1;$$

(c) $$\frac{\partial}{\partial u^j} \bar{H}_{k+1}^i(\bar{x}_k, u_k'^1, \ldots, \bar{u}_k^i, \ldots, u_k'^N) + \bar{\zeta}_k^j \frac{\partial}{\partial u^j} Q_k^j(\bar{x}_k, u_k'^j) + \bar{\xi}_k^j \frac{\partial}{\partial u^j} q_k^j(\bar{x}_k, u_k'^j) = 0,$$
$$j = 1, \ldots, N, \quad j \neq i, \quad k = 0, 1, \ldots, K - 1;$$

(d) $$\bar{\xi}_k^i \leqq 0, \quad \bar{\xi}_k^i q_k^i(\bar{x}_k, \bar{u}_k^i) = 0, \quad k = 0, 1, \ldots, K - 1;$$

(e) $$\bar{\xi}_k^j \leqq 0, \quad \bar{\xi}_k^j q_k^j(\bar{x}_k, u_k'^j) = 0, \quad j = 1, \ldots, N, \quad j \neq i, \quad k = 0, 1, \ldots, K - 1.$$

If we want to formulate analogous results also for an open-loop strategy class, we must be aware of the following difficulty. Namely, the assumed state-dependent constraints (3.3) prevent us to obtain from Theorem 1, for a two-player zero-sum case, the "symmetric" optimality conditions, as we have in a closed-loop case. This problem was studied in [15]. Here we make only the obvious final conclusion. We have to assume the fact mentioned earlier, i.e. the admissible control regions (3.3) are not state-dependent. Then Theorem 3 will be valid also for the open-loop minimax solution, if in the formulation of this theorem all terms containing partial derivatives of the constraining functions (3.3) with respect to x are neglected. This result further implies that, if the regions of admissible controls (3.3) do not depend on x, the closed-loop minimax solution can be obtained as a synthesis of the open-loop minimax problems.

The minimax solution is extremely pessimistic, and thus little probable, as we briefly discussed in Section 2. Therefore it can be advisable to construct the solution from different reasons, e.g. see the hierarchical solution concept for multistage games in [21].

As the last case let us study the noninferior solutions of a multistage game. From the definition of a noninferior set it follows that in this case only a discrete optimal control problem with vector-valued cost functional must be solved. As controls we can choose any admissible strategy N-tuple. These optimization problems are treated in [15] and [17] in detail. It is further evident, that for both, open-loop and closed-loop strategy classes we obtain the same set of necessary optimality conditions for a noninferior solution.

First, let us reformulate the Assumption 3. Instead of (3.6) we now define in E^{n+N} sets – cf. also (3.5):

$$\begin{aligned} (5.1) \qquad \hat{V}_k(x) = \{(a, v) \mid a \in E^N,\ v \in E^n,\ a_j = \tilde{h}_k^j(x, u^1, \ldots, u^N),\ \ j = 1 \ldots, N, \\ v = f_k(x, u^1, \ldots, u^N),\ u^i \in U_k^i(x),\ i = 1, \ldots, N\}, \\ k = 0, 1, \ldots, K-1 \end{aligned}$$

and

$$(5.2) \qquad \hat{R} = \{r \mid r \in E^{n+N},\ r = (\varrho_1, \ldots, \varrho_N, 0, \ldots, 0), \varrho_j \leqq 0,\ \ j = 1, \ldots, N\}.$$

Assumption 2a. The sets $\hat{V}_k(x)$, $k = 0, 1, \ldots, K-1$ are $\hat{R}$-directionally convex for any $x \in E^n$.

Theorem 4. Consider discrete optimal control problem with a vector-valued cost functional $J = (J_1, \ldots, J_N)$, in which results problem of finding the noninferior solution set for multistage game (3.4). Let the Assumption 1, 2a and 3 be satisfied for this problem. Finally, suppose that the admissible strategy N-tuple $(\hat{u}^1, \ldots, \hat{u}^N)$ belongs to the noninferior set and denote by $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K$ the corresponding state trajectory.

Then there exist a vector $\mu \in E^N$, $\mu_i \geqq 0$, $i = 1, \ldots, N$, $\sum_{i=1}^{N} \mu_i = 1$, and row-vectors

$$\hat{\lambda}_k \in E^n,\quad k = 1, \ldots, K,\quad \hat{\zeta}_k^i \in E^{r_k{}^i},\quad \hat{\xi}_k^i \in E^{s_k{}^i},\quad k = 0, 1, \ldots, K-1$$

such that the following conditions (a)–(c) are satisfied:

$$\text{(a)} \qquad \hat{\lambda}_k = \frac{\partial}{\partial x} \hat{H}_{k+1}(\hat{x}_k, \hat{u}_k^1, \ldots, \hat{u}_k^N) + \sum_{i=1}^{N} \left[\hat{\zeta}_k^i \frac{\partial}{\partial x} Q_k^i(\hat{x}_k, \hat{u}_k^i) + \hat{\xi}_k^i \frac{\partial}{\partial x} q_k^i(\hat{x}_k, \hat{u}_k^i)\right],$$

$$k = 1, \ldots, K-1,$$

with

$$\hat{\lambda}_K = -\sum_{i=1}^{N} \mu_i \frac{\partial}{\partial x} g^i(\hat{x}_K),$$

and where

$$\hat{H}_{k+1}(x, u^1, \ldots, u^N) = - \sum_{i=1}^{N} \mu_i h_k^i(x, u^1, \ldots, u^N) + \hat{\lambda}_{k+1} f_k(x, u^1, \ldots, u^N),$$
$$k = 0, 1, \ldots, K - 1 ;$$

(b) $$\frac{\partial}{\partial u^i} \hat{H}_{k+1}(\hat{x}_k, \hat{u}_k^1, \ldots, \hat{u}_k^N) + \hat{\zeta}_k^i \frac{\partial}{\partial u^i} Q_k^i(\hat{x}_k, \hat{u}_k^i) + \hat{\xi}_k^i \frac{\partial}{\partial u^i} q_k^i(\hat{x}_k, \hat{u}_k^i) = 0,$$
$$i = 1, \ldots, N, \quad k = 0, 1, \ldots, K - 1 ;$$

(c) $$\hat{\xi}_k^i \leqq 0, \quad \hat{\xi}_k^i q_k^i(\hat{x}_k, \hat{u}_k^i) = 0, \quad i = 1, \ldots, N, \quad k = 0, 1, \ldots, K - 1 .$$

Practically, we have obtained in this way the discrete maximum principle for discrete optimal control problems with a state-dependent region of admissible controls and with a vector-valued (N-dimensional) cost functional. It is easy to prove that taking

(5.3) $$J = \sum_{i=1}^{N} \mu_i J_i, \quad \mu_i > 0, \quad i = 1, \ldots, N, \quad \sum_{i=1}^{N} \mu_i = 1$$

as a scalar cost functional and solving the corresponding discrete optimal control problem, the resulting strategy N-tuple will lie in the noninferior set. Thus, when $(N - 1)$-parameter family of discrete optimal control problems with scalar cost functional (5.3) is solved, the desired noninferior set is obtained except, maybe, of those points, which are computed with some $\mu_i = 0$. In general, such points are not necessarily noninferior. Also certain convexity assumptions must hold for such conclusion – see [24]. In a concrete game the just mentioned points must be treated separately in order to obtain the whole noninferior set.

From a practical point of view we can such noninferior solutions (with some $\mu_i = 0$) simply neglect, because they totally ignore the cost functional of some players. So it is hardly probable, that these players will take part in cooperation in such case.

If we additionally assume that $U_k^i \neq U_k^i(x)$, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K - 1$, i.e. the constant regions of admissible controls, then, except of Theorem 2, the all remaining theorems are also valid if in Assumption 1 only the continuity of f_k, h_k^i, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K - 1$ with respect to u^i $(i = 1, \ldots, N)$ is required. Of course, then only a maximum condition can be always used – see [18, Theorem 4].

6. LINEAR MULTISTAGE GAMES WITH QUADRATIC COST FUNCTIONALS

Through this section we study a special class of N-player nonzero-sum multistage games. Namely, we assume that the system equations (3.1) are linear in state variable x and control variables u^i, $i = 1, \ldots, N$, and that the cost functionals (3.2) are quadratic functions of the same variables. The main advantage of this simplification of the original nonlinear case is the fact, that such case is amenable to the analytical treatment under the relatively mild, rather technical assumptions. Moreover, the class of linear multistage games with quadratic cost functionals can sometimes serve as the first approximation when working with more complicated, nonlinear cases. Therefore we assume that

$$(6.1) \qquad x_{k+1} = A_k x_k + \sum_{j=1}^{N} B_k^j u_k^j + e_k\,, \quad k = 0, 1, \ldots, K-1\,, \quad x_0 \text{ given}\,.$$

Here A_k is $(n \times n)$-matrix, B_k^j are $(n \times m_j)$-matrices, $j = 1, \ldots, N$ and the column-vector e_k represents a constant forcing term. As in Section 3, $x \in E^n$ and $u^j \in E^{m_j}$, $j = 1, \ldots, N$.

The cost functionals are supposed to be quadratic, i.e.

$$(6.2) \qquad J_i = \tfrac{1}{2}\sum_{k=0}^{K} (z_k^i - C_k^i x_k)^T Q_k^i (z_k^i - C_k^i x_k) + \tfrac{1}{2}\sum_{k=0}^{K-1}\sum_{j=1}^{N} (u_k^j)^T R_k^{ij} u_k^j\,,$$
$$i = 1, \ldots, N$$

Here Q_k^i are $(t_i \times t_i)$-matrices with the positive integer t_i denoting the dimension of the so called output vector $C_k^i x_k$, C_k^i are $(t_i \times n)$-matrices and R_k^{ij} are $(m_j \times m_j)$-matrices. Finally, z_k^i are prescribed constant column-vectors in E^{t_i}. The range of various indices i, j, k is always uniquely determined by (6.1) and (6.2). We assume that each player is currently informed about the realized state x_k of the game. Without any loss of generality we also suppose that the matrices Q_k^i, $i = 1, \ldots, N$, $k = 0, 1, \ldots$ $\ldots, K$ and R_k^{ij}, $i, j = 1, \ldots, N$, $k = 0, 1, \ldots, K-1$ are symmetric. Additionally, let the matrices Q_k^i, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K$ be positive semidefinite and matrices R_k^{ii}, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K-1$ positive definite.

For the multistage game (6.1) and (6.2) we have the following practical interpretation. Such game can describe a simple, discrete-time model of an economy governed by N players: x_k is the state of economic system, the $C_k^i x_k$ are the outputs and z_k^i are the output schedules. Thus each player would like to keep up with his prescribed output schedule as close as possible. The cost functional J_i consists of three parts: (i) the actual control effort $\sum (u_k^i)^T R^{ii} u_k^i$, (ii) a penalty for not keeping up with the schedule, and (iii) direct costs resulting from the control effort of the other players.

First, let us study *the open-loop equilibrium solution*. Due to our assumptions on the various matrices it is easy to see that the convexity assumption (Assumption 2) is satisfied. The remaining assumptions are satisfied in a trivial way, because there are no constraints and all functions appearing in (6.1) and (6.2) are either linear, or quadratic, i.e. continuously differentiable.

Now we are able to apply Theorem 1 to this multistage game. Looking carefully through the conditions (a) and (b) of this theorem we can, in general, always determine u_k^{*i} explicitely in terms of λ_{k+1}, as indicated in condition (b). Inserting this value of u_k^{*i} into the corresponding condition (a) and (6.1), we obtain a discrete linear boundary-value problem. This problem is then possible to solve by standart methods. There exist two equivalent ways to determine open-loop equilibrium strategies.

Method A. Suppose that $(u^{*1}, \ldots, u^{*N})$ is the desired open-loop equilibrium solution. From the point of view of player i, he only solves an optimal control problem with parameters u^{*j}, $j = 1, \ldots, N$, $j \neq i$. From Theorem 1 we obtain for player i, $i = 1, \ldots, N$ (T denotes transposition):

$$(6.3) \qquad u_k^{*i} = (R_k^{ii})^{-1} (B_k^i)^T [P_{k+1}^i (W_k^i x_k + w_k^i) + (p_{k+1}^i)^T], \quad k = 0, 1, \ldots, K-1,$$

where the symmetric matrices P_{k+1}^i and row-vectors p_{k+1}^i are determined by the equations

$$(6.4) \quad P_k^i = -(C_k^i)^T Q_k^i C_k^i + A_k^T [(P_{k+1}^i)^{-1} - B_k^i (R_k^{ii})^{-1} (B_k^i)^T]^{-1} A_k, \quad k = 1, \ldots, K-1,$$
$$P_K^i = -(C_K^i)^T Q_K^i C_K^i,$$

$$(6.5) \qquad p_k^i = (z_k^i)^T Q_k^i C_k^i + [(w_k^i)^T P_{k+1}^i + p_{k+1}^i] A_k, \quad k = 1, \ldots, K-1,$$
$$p_K^i = (z_K^i)^T Q_K^i C_K^i.$$

We used the notation (E is the unit matrix of appropriate dimension)

$$(6.6) \qquad W_k^i = [E - B_k^i (R_k^{ii})^{-1} (B_k^i)^T P_{k+1}^i]^{-1} A_k, \quad k = 0, 1, \ldots, K-1,$$
$$w_k^i = [E - B_k^i (R_k^{ii})^{-1} (B_k^i)^T P_{k+1}^i]^{-1} [\sum_{\substack{j=1 \\ j \neq i}}^{N} B_k^j u_k^{*j} +$$
$$+ B_k^i (R_k^{ii})^{-1} (B_k^i)^T (p_{k+1}^i)^T + e_k], \quad k = 0, 1, \ldots, K-1.$$

At this place it is assumed that the inversions indicated in (6.6) exist.

In fact, we have constructed directly a synthesis of the open-loop strategies for player i. The word "synthesis" is used here in the same sense as it has in the theory of optimal control. Eliminating now x_k from (6.3) using repeatedly substitution via (6.1), and performing such procedure for every $i = 1, \ldots, N$, we see that a system of linear algebraic equations for u_k^{*i}, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K-1$ is finally reached. These equations are not explicitely stated here, because of their rather great complexity in a general case.

If we additionally consider (6.4) and (6.5) also for $k = 0$, we can write for open-loop equilibrium costs of player i:

$$(6.7) \qquad J_i^* = -(\tfrac{1}{2} x_0^T P_0^i x_0 + p_0^i x_0 + q_0)$$

 where q_0 is evaluated from the following equation

$$(6.8)\qquad q_k^i = q_{k+1}^i + \tfrac{1}{2}(w_k^i)^T P_{k+1}^i w_k^i + p_{k+1}^i w_k^i - \tfrac{1}{2}(z_k^i)^T Q_k^i z_k^i -$$
$$- \tfrac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{N} (u_k^{*j})^T R_k^{ij} u_k^{*j} - \tfrac{1}{2}[(w_k^i)^T P_{k+1}^i + p_{k+1}^i] \,.$$
$$. \, B_k^i (R_k^{ii})^{-1} (B_k^i)^T [P_{k+1}^i w_k^i + (p_k^i)^T] \,, \quad k = 0, 1, \ldots, K-1 \,,$$
$$q_K = \tfrac{1}{2}(z_K^i)^T Q_K^i z_K^i \,.$$

Method B. Suppose now that each player uses the open-loop synthesis (6.3). In a quite analogous way as above we obtain:

$$(6.9)\qquad u_k^{*i} = (R_k^{ii})^{-1} (B_k^i)^T [(\Pi_{k+1}^i)^T (\Omega_k x_k + \omega_k) + (\pi_{k+1}^i)^T] \,,$$
$$i = 1, \ldots, N \,, \quad k = 0, 1, \ldots, K-1 \,,$$

where generally non-symmetric matrices Π_{k+1}^i and row-vectors π_{k+1}^i are determined by the equations

$$(6.10)\quad \Pi_k^i = -(C_k^i)^T Q_k^i C_k^i + \Omega_k^T \Pi_{k+1}^i A_k \,, \quad i = 1, \ldots, N \,, \quad k = 1, \ldots, K-1 \,,$$
$$\Pi_K^i = -(C_K^i)\, Q_K^i C_K^i \,, \quad i = 1, \ldots, N \,,$$
$$(6.11)\quad \pi_k^i = (z_k^i)^T Q_k^i C_k^i + [\omega_k^T \Pi_{k+1}^i + \pi_{k+1}^i] A_k \,, \quad i = 1, \ldots, N \,, \quad k = 1, \ldots, K-1$$
$$\pi_K^i = (z_K^i)^T Q_K^i C_K^i \,, \quad i = 1, \ldots, N \,.$$

We denoted

$$(6.12)\qquad \Omega_k = [E - \sum_{j=1}^{N} B_k^j (R_k^{jj})^{-1} (B_k^j)^T (\Pi_{k+1}^j)^T]^{-1} A_k \,,$$
$$\omega_k = [E - \sum_{j=1}^{N} B_k^j (R_k^{jj})^{-1} (B_k^j)^T (\Pi_{k+1}^j)^T]^{-1} \,.$$
$$. \, [e_k + \sum_{j=1}^{N} B_k^j (R_k^{jj})^{-1} (B_k^j)^T (\pi_{k+1}^j)^T] \,, \quad k = 0, 1, \ldots, K-1 \,.$$

Again it is assumed, that the inversions in (6.12) exist. In this approach we do not find an analogy of (6.7). Although the expression (6.9) can be somewhat misleading, because of its "closed-loop" form, we point out once again, that only an open-loop equilibrium solution is obtained applying the above stated relations.

Now let us turn our attention to *the closed-loop equilibrium solution*. Similarly as in the previous case we see that all assumptions of Theorem 2 are satisfied, and that we again have to solve a certain discrete linear boundary-value problem. Thus we obtain that

$$(6.13)\qquad \varphi_k^{*i}(x_k) = (R_k^{ii})^{-1} (B_k^i)^T [\mathbf{P}_{k+1}^i (\mathbf{W}_k x_k + \mathbf{w}_k) + (\mathbf{p}_{k+1}^i)^T] \,,$$
$$i = 1, \ldots, N \,, \quad k = 0, 1, \ldots, K-1 \,,$$

where the symmetric matrices $\boldsymbol{P}_{k+1}$ represent a solution of coupled, discrete, matrix Riccati-type equations

(6.14)
$$\boldsymbol{P}_k^i = -(C_k^i)^T Q_k^i C_k^i + \boldsymbol{W}_k^T[\boldsymbol{P}_{k+1}^i - $$
$$- \sum_{j=1}^{N} \boldsymbol{P}_{k+1}^j B_k^j (R_k^{jj})^{-1} R_k^{ij} (R_k^{jj})^{-1} (B_k^j)^T \boldsymbol{P}_{k+1}^j] \boldsymbol{W}_k ,$$
$$i = 1, \ldots, N, \quad k = 1, \ldots, K - 1,$$
$$\boldsymbol{P}_K^i = -(C_K^i) Q_K^i C_K^i , \quad i = 1, \ldots, N ,$$

and the row-vectors $\boldsymbol{p}_{k+1}^i$ are given as

(6.15)
$$\boldsymbol{p}_k^i = (z_k^i)^T Q_k^i C_k^i + [\boldsymbol{w}_k^T \boldsymbol{P}_{k+1}^i + \boldsymbol{p}_{k+1}^i] \boldsymbol{W}_k -$$
$$- \sum_{j=1}^{N} [\boldsymbol{w}_k^T \boldsymbol{P}_{k+1}^j + \boldsymbol{p}_{k+1}^j] B_k^j (R_k^{jj})^{-1} R_k^{ij} (R_k^{jj})^{-1} (B_k^j)^T \boldsymbol{P}_{k+1}^j \boldsymbol{W}_k ,$$
$$i = 1, \ldots, N , \quad k = 1, \ldots, K - 1 ,$$
$$\boldsymbol{p}_K^i = (z_K^i)^T Q_K^i C_K^i , \quad i = 1, \ldots, N .$$

We denoted

(6.16)
$$\boldsymbol{W}_k = [E - \sum_{j=1}^{N} B_k^j (R_k^{jj})^{-1} (B_k^j)^T \boldsymbol{P}_{k+1}^j]^{-1} A_k ,$$
$$\boldsymbol{w}_k = [E - \sum_{j=1}^{N} B_k^j (R_k^{jj})^{-1} (B_k^j)^T \boldsymbol{P}_{k+1}^j]^{-1} [e_k + \sum_{j=1}^{N} B_k^j (R_k^{jj})^{-1} (B_k^j)^T (\boldsymbol{p}_{k+1}^j)^T] ,$$
$$k = 0, 1, \ldots, K - 1 .$$

Also here we cannot, at least not in a simple way, guarantee the existence of inversions in (6.16), and we are thus forced to assume this fact.

If we formally consider (6.14) and (6.15) also for $k = 0$, we can write the closed-loop equilibrium costs for all players

(6.17)
$$\boldsymbol{J}_i^* = -(\tfrac{1}{2} x_0^T \boldsymbol{P}_0^i x_0 + \boldsymbol{p}_0^i x_0 + \boldsymbol{q}_0^i) , \quad i = 1, \ldots, N ,$$

where $\boldsymbol{q}_0^i$ is evaluated from the following equation:

(6.18)
$$\boldsymbol{q}_k^i = \boldsymbol{q}_{k+1}^i + \tfrac{1}{2} \boldsymbol{w}_k^T \boldsymbol{P}_{k+1}^i \boldsymbol{w}_k + \boldsymbol{p}_{k+1}^i \boldsymbol{w}_k - \tfrac{1}{2} (z_k^i)^T Q_k^i z_k^i -$$
$$- \tfrac{1}{2} \sum_{j=1}^{N} [\boldsymbol{w}_k^T \boldsymbol{P}_{k+1}^j + \boldsymbol{p}_{k+1}^j] B_k^j (R_k^{jj})^{-1} R_k^{ij} (R_k^{jj})^{-1} (B_k^j)^T \, .$$
$$. \, [\boldsymbol{P}_{k+1}^j \boldsymbol{w}_k + (\boldsymbol{p}_{k+1}^j)^T] , \quad i = 1, \ldots, N , \quad k = 0, 1, \ldots, K - 1 ,$$
$$\boldsymbol{q}_K^i = -\tfrac{1}{2} (z_K^i)^T Q_K^i z_K^i , \quad i = 1, \ldots, N .$$

Further we briefly discuss *the minimax solution* for player i. Let us additionally assume that the matrices Q_k^i $k = 0, 1, \ldots, K$ are positive definite and matrices R_k^{ij},

$j = 1, \ldots, N$, $j \neq i$, $k = 0, 1, \ldots, K - 1$ negative definite. Then we can evidently apply Theorem 3 to this case. It is possible to seek directly the closed-loop minimax strategy for player i. Thus we have

$$\bar{u}_k^i = \bar{\varphi}^i(x_k) = (R_k^{ii})^{-1} (B_k^i)^T [\bar{P}_{k+1}^i(\bar{W}_k^i x_k + \bar{w}_k^i) + (\bar{p}_{k+1}^i)^T], \quad k = 0, 1, \ldots, K - 1, \tag{6.19}$$

and the corresponding strategies of the other players

$$u_k^{\prime j} = \varphi_k^{\prime j}(x_k) = -(R_k^{ij})^{-1} (B_k^j)^T [\bar{P}_{k+1}^i(\bar{W}_k^i x_k + \bar{w}_k^i) + \bar{p}_{k+1}^i)^T], \quad j = 1, \ldots, N, \quad j \neq i, \quad k = 0, 1, \ldots, K - 1, \tag{6.20}$$

where the symmetric matrices $\bar{P}_{k+1}^i$ and row-vectors $\bar{p}_{k+1}^i$ are determined by the equations

$$\bar{P}_k^i = -(C_k^i) Q_k^i C_k^i + A_k^T[(\bar{P}_{k+1}^i)^{-1} - \sum_{j=1}^N B_k^j(R_k^{ij})^{-1} (B_k^j)^T]^{-1} A_k, \quad k = 1, \ldots, K - 1, \quad \bar{P}_K^i = -(C_K^i) Q_K^i C_K^i, \tag{6.21}$$

$$\bar{p}_k^i = (z_k^i)^T Q_k^i C_k^i + [(\bar{w}_k^i)^T P_{k+1}^i + \bar{p}_{k+1}^i] \bar{W}_k^i - [(\bar{w}_k^i)^T \bar{P}_{k+1}^i + \bar{p}_{k+1}^i] \cdot [\sum_{j=1}^N B_k^j(R_k^{ij})^{-1} (B_k^j)^T] \bar{P}_{k+1}^i \bar{W}_k^i, \quad k = 1, \ldots, K - 1, \quad \bar{p}_K^i = (z_K^i)^T Q_K^i C_K^i. \tag{6.22}$$

We denoted

$$\bar{W}_k^i = (\bar{P}_{k+1}^i)^{-1} [(\bar{P}_{k+1}^i)^{-1} - \sum_{j=1}^N B_k^j(R_k^{ij})^{-1} (B_k^j)^T]^{-1} A_k, \quad \bar{w}_k^i = (\bar{P}_{k+1}^i)^{-1} [(\bar{P}_{k+1}^i)^{-1} - \sum_{j=1}^N B_k^j(R_k^{ij})^{-1} (B_k^j)^T]^{-1} \cdot [e_k + \sum_{j=1}^N B_k^j(R_k^{ij})^{-1} (B_k^j)^T (\bar{p}_{k+1}^i)^T], \quad k = 0, 1, \ldots, K - 1\,k. \tag{6.23}$$

In this case we see that the inversions in (6.23) are a priori guaranteed. If we further consider the equation

$$\bar{q}_k^i = \bar{q}_{k+1}^i + \tfrac{1}{2}(\bar{w}_k^i)^T \bar{P}_{k+1}^i \bar{w}_k^i + \bar{p}_{k+1}^i \bar{w}_k^i - \tfrac{1}{2}(z_k^i)^T Q_k^i C_k^i - [(\bar{w}_k^i)^T \bar{P}_{k+1}^i + \bar{p}_{k+1}^i] [\sum_{j=1}^N B_k^j(R_k^{ij})^{-1} (B_k^j)^T] [\bar{P}_{k+1}^i \bar{w}_k^i + (\bar{p}_{k+1}^i)^T], \quad k = 0, 1, \ldots, K - 1, \quad \bar{q}_K^i = -\tfrac{1}{2}(z_K^i)^T Q_K^i z_K^i, \tag{6.24}$$

and if in (6.21) and (6.22) also $k = 0$ is admitted, the minimax costs of player i can be written

$$\bar{J}_i = -(\tfrac{1}{2}x_0^T\bar{P}_0^i x_0 + \bar{p}_0^i x_0 + \bar{q}_0^i) . \tag{6.25}$$

Finally let us study *the noninferior solution set*. We assume that also the matrices Q_k^i, $i = 1, \ldots, N$, $k = 0, 1, \ldots, K$ and R_k^{ij}, $i, j = 1, \ldots, N$, $i \neq j$, $k = 0, 1, \ldots, K - 1$ are positive definite. Hence we conclude, that all studied solution types cannot, in general, exist simultaneously for linear multistage game with quadratic cost functionals, e.g. see the assumption needed for the derivation of a minimax solution. Under just stated assumptions it is possible to obtain the noninferior solution set, having in mind the exceptions discussed in Section 5, as the solution of the parametric, discrete optimal control problem with the cost functional

$$\hat{J}(\mu) = \sum_{i=1}^{N} \mu_i J_i , \quad \mu_i > 0, \; i = 1, \ldots, N , \quad \sum_{i=1}^{N} \mu_i = 1 . \tag{6.26}$$

We have denoted $\mu = (\mu_1, \ldots, \mu_N)$. The noninferior strategies are then obtained as functions of the parameter μ. Namely,

$$\begin{aligned} \hat{u}_k^i(\mu) = \hat{\varphi}_k^i(x_k, \mu) = {} & [\sum_{j=1}^{N} \mu_j R_k^{ji}]^{-1} (B_k^i)^T [\hat{P}_{k+1}(\mu)(\hat{W}_k(\mu)\, x_k + \hat{w}_k(\mu)) + \\ & + \hat{p}_{k+1}^T(\mu)] , \quad i = 1, \ldots, N , \quad k = 0, 1, \ldots, K - 1 , \end{aligned} \tag{6.27}$$

where the symmetric matrices $\hat{P}_{k+1}(\mu)$ are the solution of the parametric, discrete, matrix Riccati-type equation

$$\begin{aligned} \hat{P}_k(\mu) = {} & -\sum_{j=1}^{N} \mu_j (C_k^j)^T Q_k^j C_k^j + A_k^T [\hat{P}_{k+1}^{-1}(\mu) - \sum_{i=1}^{N} B_k^i [\sum_{j=1}^{N} \mu_j R_k^{ji}]^{-1} . \\ & . (B_k^i)^T]^{-1} A_k , \quad k = 1, \ldots, K - 1 , \\ \hat{P}_K(\mu) = {} & -\sum_{j=1}^{N} \mu_j (C_K^j)\, Q_K^j C_K^j , \end{aligned} \tag{6.28}$$

and for the row-vectors $\hat{p}_{k+1}(\mu)$ we have

$$\begin{aligned} \hat{p}_k(\mu) = {} & \sum_{j=1}^{N} \mu_j (z_k^j)^T Q_k^j C_k^j + [\hat{w}_k^T(\mu)\, \hat{P}_{k+1}(\mu) + \hat{p}_{k+1}(\mu)]\, \hat{W}_k(\mu) - \\ & - [\hat{w}_k^T(\mu)\, \hat{P}_{k+1}(\mu) + \hat{p}_{k+1}(\mu)] [\sum_{i=1}^{N} B_k^i [\sum_{j=1}^{N} \mu_j R_k^{ji}]^{-1} (B_k^i)^T]\, \hat{P}_{k+1}(\mu)\, \hat{W}_k(\mu) , \\ & k = 1, \ldots, K - 1 , \\ \hat{p}_K(\mu) = {} & \sum_{j=1}^{N} \mu_j (z_K^j)^T Q_K^j C_K^j . \end{aligned} \tag{6.29}$$

We used notation

$$(6.30) \qquad \hat{W}_k = \hat{P}_{k+1}^{-1}(\mu)\,[\hat{P}_{k+1}^{-1}(\mu) - \sum_{i=1}^{N} B_k^i[\sum_{j=1}^{N} \mu_j R_k^{ji}]^{-1}\,(B_k^i)^T]^{-1}\,A_k\,,$$

$$k = 0, 1, \ldots, K-1\,,$$

$$(6.31) \qquad \hat{w}_k = \hat{P}_{k+1}^{-1}(\mu)\,[\hat{P}_{k+1}^{-1}(\mu) - \sum_{i=1}^{N} B_k^i[\sum_{j=1}^{N} \mu_j R_k^{ji}]^{-1}\,(B_k^i)^T]^{-1}\,.$$

$$.\,[e_k + \sum_{i=1}^{N} B_k^i[\sum_{j=1}^{N} \mu_j R^{ji}]^{-1}\,(B_k^i)^T\,\hat{p}_{k+1}^T(\mu)]\,,$$

$$k = 0, 1, \ldots, K-1\,.$$

Again, the necessary inversions in (6.30) and (6.31) exist due to the stated assumptions. If we consider (6.28) and (6.29) also for $k = 0$, and if we use the equation

$$(6.32) \qquad \hat{q}_k(\mu) = \hat{q}_{k+1}(\mu) + \tfrac{1}{2}\hat{w}_k^T(\mu)\,\hat{P}_{k+1}(\mu)\,\hat{w}_k(\mu) + \hat{p}_{k+1}(\mu)\,\hat{w}_k(\mu) -$$

$$- \tfrac{1}{2}\sum_{j=1}^{N} \mu_j (z_k^j)^T\,Q_k^j z_k^j - [\hat{w}_k^T(\mu)\,\hat{P}_{k+1}(\mu) + \hat{p}_{k+1}(\mu)]\,.$$

$$.\,[\sum_{i=1}^{N} B_k^i[\sum_{j=1}^{N} \mu_j R_k^{ji}]^{-1}\,(B_k^i)^T]\,[\hat{P}_{k+1}(\mu)\,\hat{w}_k(\mu) + \hat{p}_{k+1}^T(\mu)]\,,$$

$$k = 0, 1, \ldots, K-1\,,$$

$$\hat{q}_K(\mu) = -\tfrac{1}{2}\sum_{j=1}^{N} \mu_j (z_K^j)^T\,Q_K^j z_K^j\,,$$

we can express the total costs of all players (6.26) as a function of parameter μ:

$$(6.33) \qquad \hat{J}(\mu) = -[\tfrac{1}{2}x_0^T\,\hat{P}_0(\mu)\,x_0 + \hat{p}_0(\mu)\,x_0 + \hat{q}_0(\mu)]\,.$$

Each player can compute his own costs as a function of μ using (6.2) and (6.27). As mentioned earlier, the cases with some $\mu_i = 0$, $i = 1, \ldots, N$ must be explored separately in order to determine the whole noninferior set.

7. ILLUSTRATIVE EXAMPLES

This section is devoted to the solution of three simple examples of two-player nonzero-sum multistage games in order to illustrate and clarify certain aspects of the presented theory. First two examples are simple linear multistage games with quadratic cost functionals. The last example has additional control constraints for both players. The computational procedure is always only briefly sketched; the detailed description and some additional results can be found in [15]. We also remark that all variables will be scalars through this section.

Example 1. Suppose that the dynamic system is described by the equation

(7.1) $$x_{k+1} = x_k + u_k^1 + u_k^2\,, \quad k = 0, 1, \ldots, K - 1\,, \quad x_0 \text{ given}\,.$$

The cost functionals have the form

(7.2) $$J_1 = \tfrac{1}{2}(x_K)^2 + \tfrac{1}{2}\sum_{k=0}^{K-1}(u_k^1)^2\,, \quad J_2 = \tfrac{1}{2}(x_K)^2 + \tfrac{1}{2}\sum_{k=0}^{K-1}(u_k^2)^2\,.$$

In this simple game both players try to cooperate in minimizing the final distance from the origin, but each player is penalized for such effort.

Let us compute the *open-loop equilibrium strategies* using the Method B from the previous section. We obtain the following results (transposition is neglected in this scalar case):

(7.3) $$u_k^{*i} = \Pi_{k+1}^i \Omega_k x_k\,, \quad k = 0, 1, \ldots, K - 1\,, \quad i = 1, 2\,,$$

(7.4) $$\Pi_k^i = \Omega_k \Pi_{k+1}^i\,, \quad k = 1, \ldots, K - 1\,, \quad \Pi_K^i = -1\,, \quad i = 1, 2\,,$$

(7.5) $$\Omega_k = (1 - \Pi_{k+1}^1 - \Pi_{k+1}^2)^{-1}\,, \quad k = 0, 1, \ldots, K - 1\,.$$

All other variables, e.g. π_k, ω_k are zero in this case. From (7.4) and (7.5) we have

$$\Pi_k^1 = \Pi_k^2 = -\frac{1}{2(K - k) + 1}\,, \quad k = 1, \ldots, K\,,$$

and according to (7.3)

$$u_k^{*1} = u_k^{*2} = -\frac{1}{2(K - k) + 1} x_k\,, \quad k = 0, 1, \ldots, K - 1\,,$$

from which, taking into the account (7.1), it follows

(7.6) $$u_k^{*1} = u_k^{*2} = -\frac{1}{2K + 1} x_0\,, \quad k = 0, 1, \ldots, K - 1\,.$$

From (7.2) we compute the corresponding costs

(7.7) $$J_1^* = J_2^* = \frac{1}{2}\frac{K + 1}{(2K + 1)^2}(x_0)^2\,.$$

The same results can be obtained also by Method A. However, such approach is more tedious also in this simple case. By the way, the result (7.6) can be easily checked, if we convert the multistage game in question to a static one, substituting repeatedly for x_k in (7.2) according to (7.1). Then it is sufficient to apply necessary conditions for extreme of function.

For *the closed-loop equilibrium solution* evidently hold the relations

$$(7.8) \qquad \varphi_k^{*i}(x_k) = \boldsymbol{P}_k^i \boldsymbol{W}_k x_k\,, \quad k = 0, 1, \ldots, K-1\,, \quad i = 1, 2\,,$$

$$(7.9) \qquad \boldsymbol{P}_k^i = \boldsymbol{W}_k[\boldsymbol{P}_{k+1}^i - (\boldsymbol{P}_{k+1}^i)^2]\,, \quad k = 0, 1, \ldots, K-1\,,$$

$$\boldsymbol{P}_K^i = -1\,, \quad i = 1, 2,$$

$$\boldsymbol{W}_k = (1 - \boldsymbol{P}_{k+1}^1 - \boldsymbol{P}_{k+1}^2)^{-1}\,, \quad k = 0, 1, \ldots, K-1\,.$$

Now it is not possible to solve (7.9) for a general K. Therefore we have chosen $K = 3$. We get

$$(7.10) \qquad \boldsymbol{P}_3^1 = \boldsymbol{P}_3^2 = -1\,, \quad \boldsymbol{P}_2^1 = \boldsymbol{P}_2^2 = -\tfrac{2}{9}\,, \quad \boldsymbol{P}_1^1 = \boldsymbol{P}_1^2 = -\tfrac{22}{169}\,,$$

$$\boldsymbol{P}_0^1 = \boldsymbol{P}_0^2 = -\tfrac{4202}{213^2}\,.$$

Using (7.8) we can write

$$(7.11) \qquad \varphi_0^{*1}(x_0) = \varphi_0^{*2}(x_0) = -\tfrac{22}{213}x_0\,,$$

$$\varphi_1^{*1}(x_1) = \varphi_1^{*2}(x_1) = -\tfrac{9}{13}x_1 = -\tfrac{26}{213}x_0\,,$$

$$\varphi_2^{*1}(x_2) = \varphi_2^{*2}(x_2) = -\tfrac{1}{3}x_2 = -\tfrac{39}{213}x_0\,.$$

We can conclude that the absolute effort of both players increases during the course of the game. The corresponding closed-loop equilibrium costs are

$$(7.12) \qquad \boldsymbol{J}_1^* = \boldsymbol{J}_2^* = -\tfrac{1}{2}\boldsymbol{P}_0^1(x_0)^2 = -\tfrac{1}{2}\boldsymbol{P}_0^2(x_0)^2 = \tfrac{2101}{213^2}(x_0)^2 = 0{\cdot}0463(x_0)^2\,.$$

For $K = 3$ we have from (7.7) that

$$J_1^* = J_2^* = \tfrac{2}{49}(x_0)^2 = 0{\cdot}0408(x_0)^2\,.$$

Hence, in this case the open-loop equilibrium is more attractive for the both players than the closed-loop one. On the other hand, the open-loop equilibrium solution is clearly more sensitive to sudden changes in the system (disturbances, opposing player, etc.).

The minimax solutions of this example cannot exist, while $R_k^{12} = R_k^{21} = 0$, and thus the security level is $+\infty$ for both players.

Therefore, let us try to find *the noninferior solution set*. In this case we optimize the cost functional

$$\hat{J}(\mu) = \mu J_1 + (1-\mu)\,J_2\,, \quad 0 < \mu < 1\,.$$

From (6.27)–(6.31) we have

$$(7.13) \quad \hat{P}_k(\mu) = \frac{\mu(1-\mu)\,\hat{P}_{k+1}(\mu)}{\mu(1-\mu) - \hat{P}_{k+1}(\mu)}\,, \quad k = 0, 1, \ldots, K-1\,, \quad \hat{P}_K(\mu) = -1\,,$$

$$\hat{W}_k(\mu) = \frac{\mu(1-\mu)}{\mu(1-\mu) - \hat{P}_{k+1}(\mu)}\,, \quad k = 0, 1, \ldots, K-1\,.$$

Hence,

$$(7.14)\qquad \hat{u}_k^1 = \hat{\varphi}_k^1(x_k) = \frac{1}{\mu}\hat{P}_{k+1}(\mu)\,\hat{W}_k(\mu) = -\frac{1-\mu}{\mu(1-\mu)+K-k}\,x_k\,,$$

$$k = 0, 1, \ldots, K-1\,,$$

$$\hat{u}_k^2 = \hat{\varphi}_k^2(x_k) = \frac{1}{1-\mu}\hat{P}_{k+1}(\mu)\,\hat{W}_k(\mu) = -\frac{\mu}{\mu(1-\mu)+K-k}\,x_k\,,$$

$$k = 0, 1, \ldots, K-1\,,$$

and

$$(7.15)\qquad \left.\begin{aligned} \hat{u}_k^1 &= -\frac{1-\mu}{\mu(1-\mu)+K}\,x_0\,,\\ \hat{u}_k^2 &= -\frac{\mu}{\mu(1-\mu)+K}\,x_0\,, \end{aligned}\right\} k = 0, 1, \ldots, K-1\,,$$

i.e. the noninferior strategies of both players are constant during the course of the game. The cost functionals have the corresponding value

$$(7.16)\qquad \left.\begin{aligned} \hat{J}_1(\mu) &= \frac{1}{2}\frac{(1-\mu)^2(\mu^2+K)}{[\mu(1-\mu)+K]^2}(x_0)^2\,,\\ \hat{J}_2(\mu) &= \frac{1}{2}\frac{\mu^2[(1-\mu)^2+K]}{[\mu(1-\mu)+K]^2}(x_0)^2\,, \end{aligned}\right\} 0 \leqq \mu \leqq 1\,.$$

We have included in (7.16) also the values $\mu = 0$ and $\mu = 1$, because these values of parameter μ define noninferior solutions in this case. If we, for example, choose

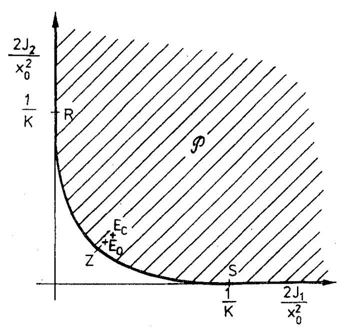

Fig. 1. Set of attainable outcomes for Example 1.

 $\mu = 0{\cdot}5$, we have

$$\hat{J}_1(0{\cdot}5) = \hat{J}_2(0{\cdot}5) = \frac{1}{2}\frac{1}{4K+1}(x_0)^2 = 0{\cdot}0382(x_0)^2 \quad \text{for} \quad K = 3\,.$$

Therefore, if cooperating, both players can benefit, i.e. they achieve lower costs than for equilibrium solutions. We see that also in this "academic" example we encounter the "prisoners' dilemma" situation.

In Fig. 1 we schematically depicted (not in scale) the set of attainable outcomes $\mathscr{P}$ for this game. As E_0, resp. E_c, we denoted open-loop, resp. closed-loop, equilibrium solution. Curve RZS denotes the noninferior set, e.g. point R corresponds to $\mu = 1$, Z to $\mu = 0{\cdot}5$ and S to $\mu = 0$.

Example 2. Consider again the dynamical system (7.1) and the cost functionals

$$(7.17) \qquad J_1 = \tfrac{1}{2}(x_K)^2 + \tfrac{1}{2}\sum_{k=0}^{K-1}\left[(u_k^1)^2 - (u_k^2)^2\right],$$

$$J_2 = \tfrac{1}{2}(x_K)^2 + \tfrac{1}{2}\sum_{k=0}^{K-1}\left[-(u_k^1)^2 + (u_k^2)^2\right].$$

The open-loop equilibrium solution is the same as in previous example, i.e. it is given by (7.6). However, the equilibrium outcome differs from (7.7):

$$J_1^* = J_2^* = \frac{1}{2}\frac{1}{(2K+1)^2}(x_0)^2 = 0{\cdot}0102(x_0)^2 \quad \text{for} \quad K = 3\,.$$

The closed-loop equilibrium solution for $K = 3$ is given:

$$(7.18) \qquad \varphi_0^{*1}(x_0) = \varphi_0^{*2}(x_0) = -\tfrac{9}{139}x_0\,,$$

$$\varphi_1^{*1}(x_1) = \varphi_1^{*2}(x_1) = -\tfrac{1}{11}x_1 = -\tfrac{11}{139}x_0\,,$$

$$\varphi_2^{*1}(x_2) = \varphi_2^{*2}(x_2) = -\tfrac{1}{3}x_2 = -\tfrac{33}{139}x_0\,,$$

and the corresponding costs

$$(7.19) \qquad J_1^* = J_2^* = \tfrac{1}{2}\left(\tfrac{33}{139}\right)^2(x_0)^2 = 0{\cdot}0282(x_0)^2$$

Similarly as in Example 1, the effort of both players increases during the course of the game and also the open-loop equilibrium solution gives lower costs to both players than the closed-loop one.

We have constructed this example to illustrate the concept of *minimax solution.* Now we have $R_k^{12} = R_k^{21} = -1$, $k = 0, 1, \ldots, K-1$, i.e. the minimax solution exists. In a simple way we have for this case (because of the symmetry of the game in question, only the minimax solution for player 1 is considered):

$$(7.20) \qquad \left.\begin{aligned} \bar{u}_k^1 &= \bar{\varphi}_k^1(x_k) = -x_k = -x_0 \\ u_k'^2 &= \varphi_k'^2(x_k) = \quad x_k = x_0 \end{aligned}\right\} k = 0, 1, \ldots, K - 1 .$$

The minimax costs for player 1 and the corresponding costs of player 2 are:

$$(7.21) \qquad \bar{J}_1 = \tfrac{1}{2}(x_0)^2 , \quad J_2' = \tfrac{1}{2}(x_0)^2 .$$

In this example we do not study *the noninferior set*, because it can be shown that the necessary convexity assumptions methioned in Section 5 are not satisfied.

Example 3. Also in this last example the dynamic system is defined by (7.1) with the following changes. Namely, we assume that the initial state $x_0 = 0$. Further, the control of each player is constrained:

$$(7.22) \qquad 0 \leqq u_k^i \leqq 1 , \quad i = 1, 2, \quad k = 0, 1, \ldots, K - 1 .$$

Each player strives to have terminal state x_K as large as possible and each is penalized for his control effort to enlarge x_k. Then the cost functionals can have a form (normalized with respect to the number of stages K):

$$(7.23) \qquad J_i = \frac{1}{K} [-x_K + \sum_{k=0}^{K-1} (u_k^i)^2] , \quad i = 1, 2 .$$

To solve this problem we must apply the general theory of Sections 4 and 5. For the sake of simplicity we shall consider *the open-loop strategy class. The equilibrium solution* is obtained from Theorem 1, because all assumptions of this theorem are satisfied, as can be readily verified. Thus we have

$$(7.24) \qquad u_k^{*i} = \tfrac{1}{2}, \quad i = 1, 2, \quad k = 0, 1, \ldots, K - 1 ,$$

i.e. the constraints (7.22) are not active for the equilibrium solution. The corresponding equilibrium costs:

$$J_1^* = J_2^* = -\tfrac{3}{4} .$$

For *the minimax solution* for player 1 we obtain from Theorem 3 that

$$(7.25) \qquad \left.\begin{aligned} \bar{u}_k^1 &= \tfrac{1}{2} \\ u_k'^2 &= 0 \end{aligned}\right\} k = 0, 1, \ldots, K - 1 .$$

The minimax costs (security level) for player 1 and the corresponding costs of player 2 are

$$\bar{J}_1 = -\tfrac{1}{4} , \quad J_2' = -\tfrac{1}{2} ,$$

i.e. both players have higher costs during the minimax play in comparison with equilibrium solution. Interchanging the roles of both players we obtain also the minimax solution for player 2, because the game is fully symmetric.

The noninferior set is determined by Theorem 4. As in Example 1, we solve one-parameter family of scalar optimization problems, i.e.

$$\hat{J}(\mu) = \mu J_1 + (1 - \mu) J_2\,, \quad 0 \leqq \mu \leqq 1\,. \tag{7.26}$$

It will be clear from the subsequent construction, that we are allowed to incorporate into (7.26) also the values $\mu = 0$ and $\mu = 1$. The condition (b) of Theorem 4 implies that

$$\left.\begin{aligned} &\hat{u}_k^1 = 1\,, \quad \hat{u}_k^2 = \frac{1}{2(1-\mu)}\,, \quad 0 \leqq \mu \leqq \tfrac{1}{2} \\ &\hat{u}_k^1 = \frac{1}{2\mu}\,, \quad \hat{u}_k^2 = 1\,, \quad \tfrac{1}{2} \leqq \mu \leqq 1 \end{aligned}\right\} k = 0, 1, \ldots, K - 1\,. \tag{7.27}$$

After inserting these values in (7.23) we get

$$\hat{J}_1(\mu) = \begin{cases} -\dfrac{1}{2(1-\mu)}\,, & 0 \leqq \mu \leqq \tfrac{1}{2}\,, \\ -1 - \dfrac{1}{2\mu} + \dfrac{1}{4\mu^2}\,, & \tfrac{1}{2} \leqq \mu \leqq 1\,, \end{cases} \tag{7.28}$$

$$\hat{J}_2(\mu) = \begin{cases} -1 - \dfrac{1}{2(1-\mu)} + \dfrac{1}{4(1-\mu)^2}\,, & 0 \leqq \mu \leqq \tfrac{1}{2}\,, \\ -\dfrac{1}{2\mu} & \tfrac{1}{2} \leqq \mu \leqq 1\,. \end{cases}$$

Then it clearly holds that

$$\hat{J}_1(\mu) = \hat{J}_2(1 - \mu)\,, \quad 0 \leqq \mu \leqq 1\,,$$

as we expected, because of the existing symmetry of the game in question. Further it follows from (7.28) that

$$\begin{aligned} \hat{J}_1(\mu) &= (\hat{J}_2(\mu))^2 + \hat{J}_2(\mu) - 1\,, \quad \tfrac{1}{2} \leqq \mu \leqq 1\,, \\ J_2(\mu) &= (\hat{J}_1(\mu))^2 + \hat{J}_1(\mu) - 1\,, \quad 0 \leqq \mu \leqq \tfrac{1}{2}\,. \end{aligned} \tag{7.29}$$

Thus, the noninferior set is given by two parts of parabolas, which are parametrically defined by (7.29). Moreover, it is not very hard to show that if we fix one cost functional, say $J_1 = c$, $-1 \leqq c \leqq 0$, we see that the other cost functional J_2 can vary, using the admissible controls, in range from zero to its minimal value $J_2 =$

$= c^2 + c - 1$, and vice versa. This shows that the set of attainable outcomes $\mathscr{P}$ is bounded by coordinate axes and the parts of parabolas

$$\left.\begin{array}{ll} J_1 = c\,, & J_2 = c^2 + c - 1 \\ J_1 = c^2 + c - 1 & J_2 = c \end{array}\right\} \quad -1 \leqq c \leqq 0\,. \tag{7.30}$$

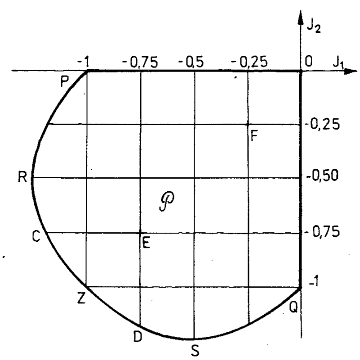

Fig. 2. Set of attainable outcomes for Example 3.

The obtained set of attainable outcomes $\mathscr{P}$ is indicated in Fig. 2. Point F denotes minimax (security) levels for both players and point E the equilibrium solution. Curve RZS is the noninferior set – see (7.29). Finally CZD denotes the so called dominant negotiation set, i.e. the set of such noninferior solutions, which dominate equilibrium outcome E. The mentioned parts of parabolas in (7.30) are given by QSZ and PRZ.

8. CONCLUDING REMARKS

In the presented paper an attempt has been made to study certain well-known solution types for a general class of N-player nonzero-sum multistage games with state-dependent regions of admissible controls. In this way necessary optimality conditions for equilibrium, minimax and noninferior solution type have been obtained applying the general results from [18]. These optimality conditions enabled us, for the class of linear multistage games with quadratic cost functionals, to compute the analytic form of all considered solution types. To be more concrete, it was always necessary to solve a discrete linear boundary-value problem.

As we could expect, when studying open-loop and closed-loop equilibrium solutions, we encountered here the same difficulties as in the case of N-player nonzero--sum differential games. This problem should be explored in more detailed way in the future, especially in connection with some new approaches to this subject – see [22] and [23]. Other interesting and still open problem is the incorporation of state constraints into the formulation of multistage game.

A question may arise, concerning possible applications of the developed theory. The primal area, where the theory of multistage games can be successfully applied, seems to be the field of economic analysis. It is mainly the theory of nonzero-sum dynamic games, either differential or multistage, that makes it possible to analyse the behaviour of economic systems. For instance, within the framework of dynamic nonzero-sum games it is possible to study the dividend policies of firms operating in an imperfect competitive market [1], or to analyse a macroeconomic system as illustrated on the balance-of-payment adjustment problem between two countries in [25].

In fact, the problem studied by Myoken in [25] is nothing else than a linear two-player nonzero-sum multistage game with quadratic cost functionals. In [25] the problem of interest is solved using the pseudoinversion. In this respect it is felt, that the deeper results concerning the multistage games presented in this paper can brought more light into certain problems of economic analysis. Moreover, a nonzero-sum multistage game can serve as a basic tool in analysing economic competition between the individuals, companies, or countries.

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